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ON THE EXISTENCE AND MULTIPLICITY OF POSITIVE RADIAL SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATION ON BOUNDED ANNULAR DOMAINS VIA FIXED POINT INDEX

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Abstract. In this paper, we study the existence and multiplicity of positive radial solutions for a class of local elliptic boundary value problem defined on bounded annular domains. The existence and multiplicity of positive radial solutions are obtained by means of fixed point index theory. We include an example to illustrate our results.

1. Introduction

In this paper, we are interested in the existence of radial positive solutions to the following boundary value problem (BVP)

$$\begin{cases} -\triangle u(x) = f(|x|, u(x)), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega = \left\{x \in \mathbb{R}^N : R_0 < |x| < R_1,\ N \geq 3\right\}$ with $0 < R_0 < R_1$ is an annulus in $\mathbb{R}^N$ and $f \in C([0,1] \times [0,\infty), [0,\infty))$.

The study of such problems is motivated by a lot of physical applications starting from the well-known Poisson-Boltzmann equation (see [2, 26, 34]), also they serve as models for some phenomena which arise in fluid mechanics, such as the exothermic chemical reactions or autocatalytic reactions (see [31], Section 5.11.1). The nonlinearity f in applications always has a special form and here we assume only the continuity of f and some inequalities at some points for the values of this function. However, we know that in the integrand should stay a superposition of u with a given function (usually the exponent of u in applications) instead of u alone, but we treat this paper as the first step in this direction. The method we use is typical for local BVP. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative function in an appropriate Banach space. The most popular fixed point theorem in a cone is the cone-compression

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and cone-expansion theorem due to M. Krasnosel'skii [25] which we use in the form taken from [12], [19]. We also point out the fact that problems of type (1.1) when equation does not contain parameter λ, are connected with the classical boundary value theory of Bernstein [1] (see also the studies of Granas, Gunther and Lee [17] for some extensions to nonlinear problems).

The existence and uniqueness of positive radial solutions for equations of type (1.1) when equation does not contain parameter λ, were obtained in [5], [27], [36]. Wang [36] proved that if $f:(0,\infty)\rightarrow(0,\infty)$ satisfies $\underset{z\rightarrow 0}{lim}\frac{f(z)}{z}=\infty$ and $\underset{z\rightarrow\infty}{lim}\frac{f(z)}{z}=0$ then problem (1.1) when equation does not contain parameter λ, has a positive radial solution in $\Omega=\left\{x\in\mathbb{R}^N,\,N>2\right\}$. That result was extended for the systems of elliptic equations by Ma [24]. We quote also the research of Ovono el al. [32] where the diffusion at each point depends on all the values of the solutions in a neighborhood of this point and Chipot et al. [13] considred the solvability of a class of nonlocal problems which admit a formulation in term of quasi-variational inequalities. There is a wide literature that deals with existence multiplicity results for various second-order, fourth-order and higher-order boundary value problems by different approaches, see [8, 9, 10, 11, 12, 14, 29, 30].

In 2011, Bohneure et al. [6] studied the existence of positive increasing radial solutions for superlinear Neumann problem in the unit ball B in $\mathbb{R}^N,\ N\geq 2$,

$$\begin{cases} -\Delta u+u=a\left(|x|\right)f\left(u\right), \ in\ \ B, \\ u>0, \qquad\qquad\qquad\qquad in\, B, \\ \partial_t u=0, \qquad\qquad\qquad on\ \partial B, \end{cases}$$

where $a\in C^1\left([0,1],\mathbb{R}\right),\ a\left(0\right)>0$ is nondecreasing, $f\in C^1\left([0,1],\mathbb{R}\right),\ f\left(0\right)=0,\ \underset{s\rightarrow 0^+}{lim}\frac{f(s)}{s}=0$ and $\underset{s\rightarrow+\infty}{lim}\frac{f(s)}{s}>\frac{1}{a(0)}$.

In 2011, Hakimi and Zertiti [22] studied the nonexistence of radial positive solutions for a nonpositone problem when the nonliearity is superlinear and has more than one zero,

$$\begin{cases} -\triangle u\left(x\right)=\lambda f\left(u\left(x\right)\right), \ \ x\in\mathit{\Omega}, \\ u\left(x\right)=0, \qquad\qquad\quad x\in\partial\mathit{\Omega}, \end{cases}$$

where $f\in C\left([0,+\infty),\mathbb{R}\right)$.

In 2014, Sfecci [35] obtained the existence result by introduced the $lim\,sup$ type of nonresonance condition with respect to the first positive eigenvalue λ_1 provided $\underset{|u|\rightarrow\infty}{lim}\ sup\frac{2F(u)}{u^2}<\lambda_1$ with a double $lim\,inf$ condition like the following one $\underset{u\rightarrow-\infty}{lim}\ sup\frac{2F(u)}{u^2}<\frac{\pi^2}{4\rho^2}$ *and* $\underset{u\rightarrow+\infty}{lim}\ inf\frac{2F(u)}{u^2}<\frac{\pi^2}{4\rho^2}$ for the following Neumann problems defined on the ball $B_R=\left\{x\in\mathbb{R}^N,\ |x|<R\right\}$,

$$\begin{cases} -\triangle u\left(x\right)=f\left(u\left(x\right)\right)+e\left(|x|\right), \ in\ B_R, \\ u\left(x\right)=0, \qquad\qquad\qquad\qquad on\ \partial B_R, \end{cases}$$

where $f\in C\left(\mathbb{R},\mathbb{R}\right)$, $e\in C\left([0,R],\mathbb{R}\right)$, F is a primitive of f and $\Omega=(-2\rho,2\rho)\subset\mathbb{R}$.

In 2014, Butler et. al, [7] studied the positive radial solutions to the BVP

$$\begin{cases} -\Delta u+u=\lambda a\left(|x|\right)f\left(u\right), \ x\in\Omega, \\ \frac{\partial u}{\partial\eta}+\overline{c}\left(u\right)u=0, \qquad |x|=r_0, \\ u\left(x\right)\rightarrow 0, \qquad\qquad |x|\rightarrow\infty, \end{cases}$$

where $f \in C([0,\infty),\mathbb{R})$, $\Omega = \left\{x \in \mathbb{R}^N : N > 2, |x| > r_0 \, with \, r_0 > 0\right\}$, λ is a positive parameter, $a \in C([r_0,\infty),\mathbb{R}^+)$ such that $\underset{r\to\infty}{lim}\, a(r) = 0$, $\frac{\partial}{\partial u}$ is the outward normal derivative and $\overline{c} \in C([0,\infty),(0,\infty))$.

In 2003, Stanzy [34], by using the norm-type cone expansion and compression theorem proved that problem (1.1) has at least one positive radial solution under the following conditions
(B_1) for any $M > 0$ there exist a function $p_M \in C((1,+\infty),\mathbb{R}^+)$ with

$$\int_1^\infty s\left(1 - s^{2n}\right) p_M(s)\, ds < \infty,$$

such that

$$0 \leq f(s,u) \leq p_M(s)\,, for\ any\ (s,u) \in (1,\infty) \times [0,M]\,,$$

(B_2) there exist a set $B \in ((1,+\infty),\mathbb{R}^+)$ of positive measure such that

$$\underset{u\to+\infty}{lim} \frac{f(s,u)}{u} = +\infty,\ uniformly\, with\, respect\, to\, s \in B,$$

(B_3) there exist a function $p \in C((1,+\infty),\mathbb{R}^+)$ with $\int_1^\infty s\left(1-s^{2n}\right) p(s)\, ds < \infty$ such that

$$\underset{u\to 0^+}{lim} \frac{f(s,u)}{up(s)} = 0,\ uniformly\, with\, respect\, to\, s \in B.$$

In 2006, Han [21], replacing the conditions listed above (B_1), (B_2) and (B_3) by the weaker ones

$$\underset{u\to 0^+}{lim}\, inf \underset{s\in[c,d]}{min} \frac{f(s,u)}{u} > \xi, \quad \underset{u\to 0^+}{lim}\, sup \frac{f(s,u)}{up(s)} < \eta,$$

uniformly with respect to $s \in (1,+\infty)$ for suitable positive numbers ξ and η, the authors proved that problem (1.1) still has at least one positive radial solution.

In 2014, Wu [37], studied problem (1.1) under some conditions concerning the first eigenvalues corresponding to the relevant linear operators, they obtained several existence theorems on multiple positive radial solutions of (1.1) in an exterior domain.

Inspired and motivated by the works mentioned above, we deal with existence and multiplicity of radial positive solutions to the BVP (1.1), our approach is based on fixed point index theory. The paper is organized as follows. In Section 2, we changes problem (1.1) into a sigular two-point boundary value problem and we will state all the lemmas which will be used to prove our main results in the later section. Setion 3 is devoted to the existence and multiplicity of positive solutions and positive radial solutions for BVP (1.1) and we give an example to illustrate our results.

2. Preliminaries

We shall consider the Banach space $E = C[0,1]$ equipped with sup norm $\|u\| = \underset{0\leq t\leq 1}{max} |u(t)|$ and $C^+[0,1]$ is the cone of nonnegative functions in $C[0,1]$.

Definition 2.1. *Anonempty closed and convex set $P \subset E$ is called a cone of E if it satisfies*

(i) $u \in P,\ r > 0$ implies $ru \in P$,
(ii) $u \in P,\ -u \in P$ implies $u = \theta$, where θ denotes the zero element of E.

Definition 2.2. *A cone P is said to be normal if there exists a positive number N called the normal constant of P, such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\|$.*

We are interested in finding radial solutions for problem (1.1). We proceed as in introduction. Since we are looking for the existence of nonnegative radial solutions $u(x) = z(|x|)$ of the problem (1.1), where $z : \mathbb{R}^+ \to \mathbb{R}$, one can substitute $v(t) = z\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}$ for $t \in [0,1]$, $n \geq 3$, thus reducing the BVP (1.1) to the following singular two-point BVP

$$\begin{cases} -v''(t) = g(t, v(t)),\ t \in (0,1), \\ \ v(0) = v(1) = 0, \end{cases} \tag{2.1}$$

where

$$g(t,v) = \phi(t) f\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, v\right), \tag{2.2}$$

$$A = \frac{(R_0R_1)^{n-2}}{R_1^{n-2} - R_0^{n-2}} \text{ and } B = \frac{R_1^{n-2}}{R_1^{n-2} - R_0^{n-2}}, \tag{2.3}$$

and

$$\phi(t) = \left(\frac{R_1^{-(n-2)} - R_0^{-(n-2)}}{n-2}\right)^2 \left(R_1^{-(n-2)} - \left(R_1^{-(n-2)} - R_0^{-(n-2)}\right)t\right),\ n \geq 3. \tag{2.4}$$

we can reformulate g as

$$g(t,v) = \phi(t) f\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}, v\right),$$

where

$$\phi(t) = \left(\frac{R_1^{-(n-2)} - R_0^{-(n-2)}}{n-2}\right)^2 \left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2} - R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right]\left[\frac{A}{B-t}\right]^{\frac{2(n-1)}{n-2}}$$

We observe that the existence of radial positive solutions of (1.1) is equivalent to the existence of positive solutions of the problem (2.1).

In arriving our results, we need the following six preliminary lemmas. The first one is well known.

Lemma 2.1. *Let $y(\cdot) \in C[0,1]$. If $u \in C^2[0,1]$, then the BVP (2.1) has a unique solution*

$$v(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{2.5}$$

where

$$G(t,s)=\begin{cases} s(1-t), & 0\le t\le s\le 1, \\ t(1-s), & 0\le s\le t\le 1. \end{cases} \tag{2.6}$$

Lemma 2.2. *For any* $(t,s)\in[0,1]\times[0,1]$, *we have*

$$0<G(t,s)\le G(s,s)=s(1-s).$$

Proof. The proof is evident, we omit it. $\square$

Lemma 2.3. *(see* [23]*) For* $y(t)\in C^+[0,1]$. *Then the unique solution* $u(t)$ *of* BVP (2.1) *is nonnegative and satisfies*

$$\underset{R_0\le t\le R_1}{min} v(t)\ge c\|v\|,$$

where $c=min\{R_0,1-R_1\}$ *and* $[R_0,R_1]\subset(0,1)$.

If we let

$$P=\left\{v\in C^+[0,1]:\quad v(t)\ge 0,\ for\, t\in[0,1]\right\},$$

and

$$Q=\left\{v\in C^+[0,1]:\quad \underset{R_0\le t\le R_1}{min} v(t)\ge c\|v\|\right\},$$

then it is easy see that P and Q are cones in $E=C[0,1]$.
Let $\Omega_r=\{u\in E:\|u\|<r\}$ be the open ball of radius r in E and the operator $A:E\to E$ define by

$$(Av)(t)=\int_0^1 G(t,s)\,g(s,v(s))\,ds,\ t\in[0,1]. \tag{2.7}$$

Define a set H by

$$H=\left\{h\in C\left((0,1),\mathbb{R}^+\right) : h\neq 0, \int_0^1 t(1-t)\,h(t)\,dt<+\infty\right\}. \tag{2.8}$$

Now, we define an integral operators $T_h:E\to E$ for $h\in H$ by

$$(T_hv)(t)=\int_0^1 G(t,s)\,h(s)\,v(s)\,ds,\ for\, v\in E. \tag{2.9}$$

We have the following lemma.

Lemma 2.4. *For any* $h\in H$ *we have*
(i) T_h *is a completely continuous linear operator and the specteral radius* $r(T_h)\neq 0$ *and* T_h *has a positive eigenfunction* φ_{1h} *corresponding to its first eigenvalue* $\lambda_{1h}=(r(T_h))^{-1}$,
(ii) $T_h(P)\subset Q$,
(iii) there exist $\delta_1,\ \delta_2>0$, *such that*

$$\delta_1 G(t,s)\le\varphi_{1h}(s)\le\delta_2 G(s,s),\ t,\ s\in[0,1], \tag{2.10}$$

(iv) define a functional J_h *by* $J_h(v)=\int_0^1 h(t)\,\varphi_{1h}(t)\,v(t)\,dt$ *for* $v\in E$. *Then* $J_h(T_hv)=\lambda_{1h}^{-1}J_h(v)$ *for* $v\in E$,

(v) let

$$P_0 = \left\{ v \in P : J_h(v) \geq \lambda_{1h}^{-1} \delta_1 \|v\| \right\}, \tag{2.11}$$

then P_0 is a cone in E and $T_h(P) \subset P_0$ where δ_1 is defined by (2.10).

To prove Lemma 2.4, we need the following lemmas.

Lemma 2.5. *(see* [24]*) Suppose that E is a Banach space, $T_n : E \to E$, $n \in \mathbb{N}^*$ are completely continuous operators, $T : E \to E$ and*

$$\lim_{n \to +\infty} \max_{\|u\| < r} \|T_n u - Tu\| = 0, \quad \forall r > 0, \tag{2.12}$$

then T is completely continuous operator.

Lemma 2.6. *(see* [25]*) Suppose that E is a Banach space, $T : E \to E$ is completely continuous linear operators and $T(P) \subset P$. If there exist $\psi \in E \setminus (-P)$ and a constant $\mu > 0$ such that $\mu T\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction corresponding to its first eigenvalue $\lambda_1 (r(T))^{-1}$.*

Proof. Proof of Lemma 2.4. It follows from the definition of H that for any $v \in E$

$$|(T_h v)(t)| \leq \int_0^1 G(t,s) h(s) |v(s)| \, ds,$$

$$\leq \|v\| \int_0^1 G(t,s) h(s) \, ds < +\infty. \tag{2.13}$$

Obviously, $T_h(P) \subset P$ and $T_h : E \to E$ is a positive linear operators.
We will show tha $T_h : E \to E$ is completely continuous. For any natural number $n \geq 2$, let

$$h_n(t) = \begin{cases} \inf\limits_{t \leq s \leq \frac{1}{n}} h(s), & 0 \leq t \leq \frac{1}{n}, \\ h(t), & \frac{1}{n} \leq t \leq \frac{n-1}{n}, \\ \inf\limits_{\frac{n-1}{n} \leq s \leq 1} h(s), & \frac{n-1}{n} \leq t \leq 1. \end{cases} \tag{2.14}$$

Then $h_n : [0,1] \to [0,\infty)$ is continuous and $h_n(t) \leq h(t)$ for all $t \in (0,1)$.
Let

$$(T_{h_n} v)(t) = \int_0^1 G(t,s) h_n(s) v(s) \, ds. \tag{2.15}$$

Now, we show that $T_{h_n} : E \to E$ is completely continuous. For any $r > 0$ and $v \in \Omega_r$, according to (2.14), (2.15) and the absolute continuity of integral, we have

$$\lim_{n \to +\infty} \|T_{h_n} v - Tv\| = \lim_{n \to +\infty} \max_{t \in [0,1]} \left| \int_0^1 G(t,s) (h_n(s) - h(s)) v(s) \, ds \right|$$

$$\leq \|v\| \lim_{n \to +\infty} \left| \int_0^1 G(s,s) (h_n(s) - h(s)) \, ds \right|$$

$$\leq \|v\| \lim_{n \to +\infty} \int_{e(n)} G(s,s) (h(s) - h_n(s)) \, ds$$

$$\leq \|v\| \lim_{n\to+\infty} \int\limits_{e(n)} G(s,s)\, h(s)\, ds = 0, \tag{2.16}$$

where $e(n) = \left[0, \frac{1}{n}\right] \cup \left[\frac{n-1}{n}, 1\right]$.
Therefore, by Lemma 2.5, $T_{h_n} : E \to E$ is a completely continuous operator. It is obvious that there exists $t_1 \in (0,1)$ such that $G(t_1,t_1)\, h(t_1) > 0$. Thus there is $[a_1, b_1] \subset (0,1)$ such that $t_1 \in (a_1, b_1)$ and $G(t,s)\, h(s) > 0$ for all t, $s \in [a_1, b_1]$. Take $\zeta \in P$ such that $\zeta(t_1) > 0$ and $\zeta(t) = 0$ for all $t \notin [a_1, b_1]$. Then, for $t \in [a_1, b_1]$

$$\begin{aligned} (T_h\zeta)(t) &= \int\limits_0^1 G(t,s)\, h(s)\, \zeta(s)\, ds \\ &\geq \int\limits_{a_1}^{b_1} G(t,s)\, h(s)\, \zeta(s)\, ds > 0. \end{aligned} \tag{2.17}$$

So, there exist a constant $\mu > 0$ such that $\mu\,(T_h\zeta)(t) \geq \zeta(t)$ for all $t \in [0,1]$. From Lemma 2.6, we have that the spectral radius $r(T_h) \neq 0$ and T_h has a positive eigenfunction corresponding to its first eigenvalue $\lambda_{1h}\,(r(T_h))^{-1}$.
(ii) To prove $T_h(P) \subset Q$, we only need to show

$$\min_{t\in[R_0,R_1]} (T_hv)(t) \geq \min\{R_0, 1-R_1\}\, \|T_hv\| \quad \text{for} \quad v \in P. \tag{2.18}$$

In fact, for every $v \in P$, from $0 < G(t,s) \leq G(s,s) = s(1-s)$ for t, $s \in [0,1]$, we have

$$\begin{aligned} (T_hv)(t) &= \int\limits_0^1 G(t,s)\, h(s)\, v(s)\, ds \\ &\leq \int\limits_0^1 s(1-s)\, h(s)\, v(s)\, ds, \end{aligned}$$

so, for any $v \in P$, we have

$$\|T_hv\| \leq \int\limits_0^1 s(1-s)\, h(s)\, v(s)\, ds. \tag{2.19}$$

Notice that, for $t \in [R_0, R_1]$,

$$G(t,s) = \begin{cases} s(1-t) \geq s(1-R_1), & s \leq t, \\ t(1-s) \geq R_0(1-s), & t \leq s. \end{cases} \tag{2.20}$$

Thus, for $(t,s) \in [R_0,R_1] \times [0,1]$, we have

$$G(t,s) \geq \min\{R_0, 1-R_1\}\, s(1-s). \tag{2.21}$$

It follows, from (2.19) and (2.21) that for all $v \in P$

$$(T_hv)(t) = \int\limits_0^1 G(t,s)\, h(s)\, v(s)\, ds$$

$$\geq min\left\{R_0, 1 - R_1\right\} \int_0^1 s\left(1 - s\right) h\left(s\right) v\left(s\right) ds$$

$$\geq min\left\{R_0, 1 - R_1\right\} \left\|T_h v\right\|, \ t \in \left[R_0, R_1\right]. \tag{2.22}$$

So, (2.18) holds. Thus, T_h maps P into Q.
(iii) Since φ_{1h} is a positive eigenfunction of T_h, we know from the maximum principle (see [18]) that $\varphi_{1h}\left(t\right) > 0$ for all $t \in (0, 1)$.
Note that $G\left(0, s\right) = G\left(1, s\right) = 0$ for $s \in (0, 1)$, we have $\varphi_{1h}\left(0\right) = \varphi_{1h}\left(1\right) = 0$.
This impleis that $\varphi'_{1h}\left(0\right) > 0$ and $\varphi'_{1h}\left(1\right) < 0$ (see [18]).
Define a function Φ_h on $[0, 1]$ by

$$\Phi_h\left(s\right) = \begin{cases} \varphi'_{1h}\left(0\right), & s = 0, \\ \frac{\varphi_{1h}(s)}{s(1-s)}, & s \in \left(0, 1\right), \\ -\varphi'_{1h}\left(1\right), & s = 1. \end{cases} \tag{2.23}$$

Then, it is easy to see that Φ_h continuous on $[0, 1]$ and $\Phi_h\left(s\right) > 0$ for all $s \in [0, 1]$.
So, there exist $\delta_1, \ \delta_2 > 0$, such that

$$\delta_1 G\left(t, s\right) \leq \delta_1 s\left(1 - s\right) \leq \varphi_{1h}\left(s\right) \leq \delta_2 s\left(1 - s\right) \leq \delta_2 G\left(s, s\right), \tag{2.24}$$

for all $t, \ s \in [0, 1]$.
(iv) From (2.10), for all $v \in E$, we have

$$J_h\left(v\right) = \int_0^1 h\left(t\right) \varphi_{1h}\left(t\right) v\left(t\right) dt$$

$$\leq \delta_2 \int_0^1 t\left(1 - t\right) h\left(t\right) v\left(t\right) dt < +\infty.$$

So, $J : E \to \mathbb{R}$ is well defined.
For all $v \in E$, we have

$$J_h\left(T_h v\right) = \int_0^1 h\left(t\right) \varphi_{1h}\left(t\right) \left(\int_0^1 G\left(t, s\right) h\left(s\right) v\left(s\right) ds\right) dt$$

$$= \int_0^1 h\left(s\right) v\left(s\right) \left(\int_0^1 G\left(s, t\right) h\left(t\right) \varphi_{1h}\left(t\right) dt\right) ds$$

$$= \int_0^1 h\left(s\right) v\left(s\right) \left(r_{1h} \varphi_{1h}\left(s\right)\right) ds$$

$$= \lambda_{1h}^{-1} J_h\left(v\right), \tag{2.25}$$

for $v \in E$. Then $J_h\left(T_h v\right) = \lambda^{-1} J_h\left(v\right)$ for $v \in E$.
(v) It is easy to verify that P_0 is a cone in E. It follows from (2.10) and (2.25) that

$$J_h\left(T_h v\right) = \lambda_{1h}^{-1} \int_0^1 h\left(s\right) \varphi_{1h}\left(s\right) v\left(s\right) ds$$

$$\geq \delta_1 \lambda_{1h}^{-1} \int_0^1 h(s)\, G(t,s)\, v(s)\, ds$$
$$= \delta_1 \lambda_{1h}^{-1} (T_h v)(t)\,,\ for\ v \in P. \tag{2.26}$$

The proof is completed. □

3. Existence results

3.1. **Positive solutions of singular two-point boundary value problems.** The following Lemma is a well-known result of the fixed point index theory, which will play an important role in the proof of our main results.

Lemma 3.1. (*see* [18])*Let Ω be a bounded open set in E with $\theta \in \Omega, A : P \cap \overline{\Omega} \to P$ a completely continuous operator, where θ denotes the null element of E. Assume that A has no fixed point on $P \cap \partial\Omega$.*

(i) (Homotopy invariance) If $u \neq \mu A u$ for all $\mu \in [0,1]$ and $u \in P \cap \partial\Omega$, then the fixed point index $i(A, P \cap \Omega, P) = 1$,

(ii) (omitting a direction) if there exists an element $\psi_0 \in P \setminus \{\theta\}$ such that $u \neq Au + \mu\psi_0$ for all $u \in P \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, P \cap \Omega, P) = 0$,

(iii) (cone expansion) if $\|Au\| \geq \|u\|$ for all $u \in P \cap \partial\Omega$, then $i(A, P \cap \Omega, P) = 0$,

(iv) (additivity) suppose Ω_1 is an open subset of Ω with $\theta \in \Omega_1$ and $u \neq Au$ for $u \in P \cap \partial\Omega_1$, then

$$i(A, P \cap \Omega, P) = i(A, P \cap \Omega_1, P) + i\left(A, P \cap \left(\Omega \setminus \overline{\Omega}\right), P\right),$$

(v) $i(A, P \cap \Omega, P) \neq 0$, then A has at least one fixed point in $P \cap \Omega$.

Denote

$$M_1 = \left(\underset{t\in[R_0,R_1]}{min} \int_{R_0}^{R_1} G(t,s)\, ds \right)^{-1},\ \eta = \left(\underset{t\in[0,1]}{max} \int_{R_0}^{R_1} G(t,s)\, ds \right)^{-1}. \tag{3.1}$$

The following conditions holds.

(H_1) $g \in C((0,1) \times \mathbb{R}^+, \mathbb{R}^+)$ and for any $M > 0$ there exists a function $h_M \in H$ such that

$$g(t,v) \leq h_M(t)\,,\ \forall (t,v) \in (0,1) \times [0,M]\,, \tag{3.2}$$

(H_2) there exists a function $h \in H$ such that

$$\underset{v\to 0^+}{lim}\ sup \frac{g(t,v)}{h(t)\, v} < \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1)\,, \tag{3.3}$$

(H_3) there exists a function $h \in H$ such that

$$\underset{v\to +\infty}{lim}\ sup \frac{g(t,v)}{h(t)\, v} < \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1)\,, \tag{3.4}$$

(H_4) $\underset{v\to 0^+}{lim}\ inf \underset{t\in[R_0,R_1]}{min} \frac{g(t,v)}{v} > M_1$,

(H_5) $\underset{v\to +\infty}{lim}\ inf \underset{t\in[R_0,R_1]}{min} \frac{g(t,v)}{v} > M_1$,

(H_6) there exists a number $l > 0$ such that

$$g(t,v) > \eta l, \quad for\ (t,v) \in [R_0, R_1] \times [min\{R_0, 1-R_1\} l, l], \tag{3.5}$$

where η defined in (3.1),
(H_7) there exists a function $h \in H$ such that

$$\lim_{v\to 0^+} inf \frac{g(t,v)}{h(t)v} > \lambda_{1h},\ uniformly\ with\ respect\ to\ t \in (0,1), \tag{3.6}$$

(H_8) there exists a function $h \in H$ with $h(t) \neq 0$ for $t \in [R_0, R_1]$ and $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$g(t,v) \geq h(t)q(v), \quad \forall (t,v) \in (0,1) \times \mathbb{R}^+, \tag{3.7}$$

$$\lim_{v\to\infty} inf \frac{q(v)}{v} > \lambda_{1h}. \tag{3.8}$$

Lemma 3.2. *Assume (H_1) holds. Then $A : Q \to Q$ is a completely continuous operator.*

Proof. The proof is similar to that of Lemma 3.1 in [21]. □

Lemma 3.3. *assume (H_1) holds.*

(i) If (H_2) holds. Then $i(A, Q \cap \Omega_r, Q) = 1$ for sufficiently small positive number r.

(ii) If (H_3) holds. Then $i(A, Q \cap \Omega_R, Q) = 1$ for sufficiently large positive number R.

(iii) If (H_4) holds. Then $i(A, Q \cap \Omega_r, Q) = 0$ for sufficiently small positive number r.

(iv) If (H_5) holds. Then $i(A, Q \cap \Omega_R, Q) = 0$ for sufficiently large positive number R.

(v) If (H_6) holds. Then $i(A, Q \cap \Omega_l, Q) = 0$.

(vi) If (H_7) holds. Then $i(A, Q \cap \Omega_r, Q) = 0$ for sufficiently small positive number r.

(ii) If (H_8) holds. Then $i(A, Q \cap \Omega_R, Q) = 0$ for sufficiently large positive number R.

Proof. (i) By (H_2) there exists $r > 0$ such that

$$g(t,v) \leq \lambda_{1h} h(t) v,\ \forall (t,v) \in (0,1) \times [0,r]. \tag{3.9}$$

Define $S_h v = \lambda_{1h} T_h v$ for $v \in E$, then $S_h : E \to E$ is a bounded linear operator with $S_h(P) \subset Q$ and the spectral radial $r(S_h) = 1$. For every $v \in Q \cap \partial\Omega_r$, it follows from (3.9) that for $t \in [0,1]$,

$$(Av)(t) = \int_0^1 G(t,s)\, g(s, v(s))\, ds$$

$$\leq \lambda_{1h}\int_0^1 G(t,s)\,h(s)\,v(s)\,ds$$

$$\leq \lambda_{1h}(T_h v)(t) = (S_h v)(t)\,. \tag{3.10}$$

So,

$$Av \leq S_h v, \quad \forall v \in Q \cap \partial\Omega_r. \tag{3.11}$$

If there exist $v_1 \in Q \cap \partial\Omega_r$ and $\mu_1 \in [0,1]$ such that $v_1 = \mu_1 A v_1$, then it is easy to see that $\mu_1 \in (0,1)$.

Thus $\tau_1 = \mu_1^{-1} > 1$ and $\tau_1 v_1 = Av_1 \leq S_h v_1$. By induction, we have $\tau_1^n v_1 = Av_1 \leq S_h^n v_1$, $n = 1,2,\ldots$. Then $\tau_1^n v_1 = S_h^n v_1 \leq \|S_h\|\,\|v_1\|$ and taking the sepremum on $[0,1]$ gives $\tau_1^n \leq \|S_h^n\|$. By the spectral radius formula, we have

$$r(S_h) = \lim_{n\to+\infty} \sqrt[n]{\|S_h^n\|} \geq \tau_1 > 1, \tag{3.12}$$

which is contradiction.

According to the homotopy property invarience of fixed point index, we have $i(A, Q\cap\Omega_r, Q) = 1$.

(ii) By (H_3) there exists $\sigma > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$g(t,v) \leq \varepsilon_0 \lambda_{1h} h(t)\,v,\ \forall (t,v) \in (0,1)\times[\sigma,+\infty)\,. \tag{3.13}$$

From (H_1) there is $h_\sigma \in H$ such that $g(t,v) \leq h_\sigma(t)$ for all $(t,v) \in (0,1)\times[0,\sigma]$. Hence

$$g(t,v) \leq \varepsilon_0\lambda_{1h} h(t)\,v + h_\sigma(t)\,,\ \forall (t,v) \in (0,1)\times[0,+\infty)\,. \tag{3.14}$$

Define $S_h v = \varepsilon_0 \lambda_{1h} T_h v$, for $v \in E$, then $S_h : E \to E$ is a bounded linear operator with $S_h(P) \subset Q$. Let $C_1 = \int_0^1 t(1-t)\,h_\sigma(t)\,dt < +\infty$. Set

$$W = \{v \in Q \,:\, v = \rho A v,\ \rho \in [0,1]\}\,. \tag{3.15}$$

Next, we prove that W is bounded. For any $v \in W$. From (3.14), we have

$$\begin{aligned} v(t) &= \rho(Av)(t) \leq (Av)(t) \\ &= \int_0^1 G(t,s)\,g(s,v(s))\,v(s)\,ds \\ &\leq \varepsilon_0\lambda_{1h}\int_0^1 G(t,s)\,h(s)\,v(s)ds + \int_0^1 G(t,s)\,h_\sigma(s)\,ds \\ &\leq \varepsilon_0\lambda_{1h}(T_h v)(t) + C_1 \\ &= (S_h v)(t) + C_1,\ t \in [0,1]\,. \end{aligned}$$

Thus

$$((I - S_h)\,v)(t) \leq C_1, \quad \forall v \in W,\ t \in [0,1]\,. \tag{3.16}$$

Since λ_{1h} is the first eigenvalue of S_h, $r(S_h)^{-1} > 1$. therefore, the inverce operator $(I - S_h)^{-1}$ exists and

$$(I - S_h)^{-1} = I + S_h + S_h^2 + \cdots + S_h^n + \cdots \tag{3.17}$$

It follows from $T_h(P) \subset Q$ that $(I - S_h)^{-1}(P) \subset Q$. Hence, we have from (3.16) that

$$v(t) \leq (I - S_h)^{-1} C_1,\ \forall v \in W,\ t \in [0,1] \tag{3.18}$$

that is W is bounded. Choose $R > \{\rho, supW\}$, then $v \neq \sigma Av$ for all $\sigma \in [0,1]$ and $v \in Q \cap \Omega_R$. By the homotopy property invarience of fixed point index, we have $i\left(A, Q \cap \Omega_R, Q\right) = 1$.
$(iii) - (v)$ have been proved in [21], so we skip it.
(vi) By (H_7) there exist $r > 0$ such that

$$g\left(t,v\right) \geq \lambda_{1h} h\left(t\right) v,\ \ \forall \left(t,v\right) \in (0,1) \times [0,r]\,. \tag{3.19}$$

For any $v \in Q \cap \Omega_r$, we have

$$\begin{aligned}(Av)(t) &= \int\limits_0^1 G\left(t,s\right) g\left(s, v\left(s\right)\right) ds \\ &\geq \lambda_{1h} \int\limits_0^1 G\left(t,s\right) h\left(s\right) v\left(s\right) ds \\ &= \lambda_{1h} \left(T_h v\right)(t)\,,\ t \in [0,1]\,. \end{aligned}\tag{3.20}$$

Without loss of generality, we can suppose that A has no fixed point on $Q \cap \partial\Omega_r$. Suppose that there exist $v_1 \in Q \cap \partial\Omega_r$ and $\mu_1 \geq 0$ such that $v_1 = Av_1 + \mu_1\varphi_{1h}$. Then $\mu_1 > 0$ and $v_1 = Av_1 + \mu_1\varphi_{1h} \geq \mu_1\varphi_{1h}$. Let

$$\mu^* = sup\left\{\rho > 0\ :\ v_1 \geq \rho\varphi_{1h}\right\}. \tag{3.21}$$

Then $\mu^* \geq \mu_1 > 0$ and $v_1 \geq \mu^*\varphi_{1h}$.
Since T_h is a positive linear operator, we have

$$\lambda_{1h} T_h v_1 \geq \mu^* \lambda_{1h} T_h \varphi_{1h}. \tag{3.22}$$

Hence, by (3.20) we have

$$v_1 = Av_1 + \mu_1\varphi_{1h} \geq \lambda_{1h} T_h v_1 + \mu_1\varphi_{1h} \geq \mu^*\varphi_{1h} + \mu_1\varphi_{1h}, \tag{3.23}$$

which is contradiction. Thus according to the homotopy property of omitting a direction for fixed point index, we have $i\left(A, Q \cap \Omega_r, Q\right) = 0$.
(vii) From (3.8) there exist there exists $\sigma > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$q\left(v\right) \geq \left(1 + \varepsilon_0\right) \lambda_{1h} v,\ \forall v \in [\sigma, +\infty)\,. \tag{3.24}$$

Since q is bounded on $[0,\sigma]$, there is a constant $C_2 > 0$ such that

$$q\left(v\right) \geq \left(1 + \varepsilon_0\right) \lambda_{1h} v - C_2,\ \forall v \in [0,\sigma]\,. \tag{3.25}$$

Thus

$$q\left(v\right) \geq \left(1 + \varepsilon_0\right) \lambda_{1h} v - C_2,\ \forall v \in [0,+\infty)\,.$$

Hence, by (3.7), we have

$$g\left(t,v\right) \geq \left(1 + \varepsilon_0\right) \lambda_{1h} v h\left(t\right) - C_2 h\left(t\right),\ \forall \left(t,v\right) \in (0,1) \times [0,+\infty)\,. \tag{3.26}$$

Let $C_3 = \int_0^1 h\left(t\right)\varphi_{1h}\left(t\right)\left(\int_0^1 G\left(t,s\right) h\left(s\right) ds\right) dt < +\infty$. Then $C_3 > 0$ is a finite constant. Take

$$R > C_3 \left(\varepsilon_0 min\left\{R_0, 1 - R_1\right\} \int\limits_{R_0}^{R_1} h\left(t\right)\varphi_{1h}\left(t\right) dt\right)^{-1}. \tag{3.27}$$

Suppose that there exsist $v_1 \in Q \cap \Omega_R$ and $\mu_1 \geq 0$ such that $v_1 = Av_1 + \mu_1 \varphi_{1h}$. Then

$$\begin{aligned} J_h(v_1) &= J(Av_1) + \mu_1 J(\varphi_{1h}) \\ &\geq J(Av_1) \\ &\geq \int_0^1 h(t)\varphi_{1h}(t)\left(\lambda_{1h}(1+\varepsilon_0)\int_0^1 G(t,s)h(s)v_1(s)\,ds - C_2 T_h(1)\right)dt \\ &= \lambda_{1h}(1+\varepsilon_0)J_h(T_h v_1) - C_3 \\ &= (1+\varepsilon_0)J_h(v_1) - C_3. \end{aligned} \tag{3.28}$$

Hence

$$J_h(v_1) \leq C_3 \varepsilon_0^{-1}.$$

On the other hand

$$\begin{aligned} J_h(v_1) &= \int_0^1 h(t)\varphi_{1h}v_1(t)\,dt \\ &\geq \int_{R_0}^{R_1} h(t)\varphi_{1h}v_1(t)\,dt \\ &\geq R\min\{R_0, 1-R_1\}\int_{R_0}^{R_1} h(t)\,.\varphi_{1h}dt. \end{aligned} \tag{3.29}$$

By the maximum principle, $\varphi_{1h}(t) > 0$ for all $t \in (0,1)$. By $h(t) \neq 0$ for $t \in [R_0, R_1]$, we have

$$\int_{R_0}^{R_1} h(t)\varphi_{1h}dt > 0.$$

Thus, from (3.28) and (3.29), we have

$$\begin{aligned} R &\leq \left(min\{R_0, 1-R_1\}\int_{R_0}^{R_1} h(t)\varphi_{1h}dt\right)^{-1} J_h(v_1) \\ &\leq C_3\left(min\{R_0, 1-R_1\}\int_{R_0}^{R_1} h(t)\varphi_{1h}dt\right)^{-1}. \end{aligned} \tag{3.30}$$

This is contradiction. So, by the property of omitting a direction for fixed point index, we have $i(A, Q \cap \Omega_R, Q) = 0$. The is completed. □

Now, we are in position to present our main results of this subsection.

Theorem 3.4. *Assume* $(H_1) - (H_3)$ *and* (H_6) *hold. Then the singular boundary value problem* (2.1) *has at least two positive solutions.*

Proof. According to Lemma 3.3, we can choose sufficiently small positive number r and sufficiently large positive number R satisfying $0 < r < l < R, i\,(A, P \cap \Omega_r, P) = 1,\ i\,(A, P \cap \Omega_R, P) = 1$. From $i\,(A, P \cap \Omega_l, P) = 0$ and additivity property of the fixed point index, we obtain

$$i\left(A, P \cap \left(\Omega_l \setminus \overline{\Omega_r}\right), P\right) = 0 - 1 = -1,$$

$$i\left(A, P \cap \left(\Omega_R \setminus \overline{\Omega_l}\right), P\right) = 1 - 0 = 1.$$

Hence, A has at least two fixed points, one in $\Omega_l \setminus \overline{\Omega_r}$ and another in $\Omega_R \setminus \overline{\Omega_l}$. That is the singular boundary value problem (2.1) has at least two positive solution. The proof is completed. □

Theorem 3.5. *If* (H_1) *and one of the following conditions are satisfied, then the singular boundary value problem* (2.1) *has at least one positive solution.*
(i) (H_2) *and* (H_5) *holds,*
(ii) (H_2) *and* (H_6) *holds,*
(iii) (H_2) *and* (H_8) *holds,*
(iv) (H_3) *and* (H_4) *holds,*
(v) (H_3) *and* (H_6) *holds,*
(vi) (H_3) *and* (H_7) *holds.*

Proof. By the property of the fixed point index, we only need to choose suitable positive numbers r and R. This completes the proof. □

We present an example to illustrate the applicability of the results shown before.

Example 3.1. *Let*

$$g\,(t,v) = \begin{cases} \frac{1}{t(t-1)}\left(\frac{cvl}{384}\right), & t \in (0,1)\,,\ v \in \left[0, \frac{1}{8}l\right], \\ \frac{1}{t(t-1)}\left(\frac{cvl}{192} \times \frac{l-4v}{l} + \frac{16l(8v-l)}{l}\right), & t \in (0,1)\,, v \in \left[\frac{1}{8}l, \frac{1}{4}l\right], \\ 16l, & t \in (0,1)\,,\ v \in \left[\frac{1}{4}l, l\right], \\ 16l + t\sqrt{v-l}, & t \in (0,1)\,,\ v \in [l, +\infty)\,, \end{cases}$$

where $c,\ l > 0$. *Obviously,* $g\,(t,v) \le h\,(t)\,\psi\,(v)$ *for all* $(t,v) \in (0,1) \times \mathbb{R}^+$*, where* $h\,(t) = \frac{1}{t(t-1)}$ *and*

$$\psi\,(v) = \begin{cases} \left(\frac{cvl}{384}\right), & t \in (0,1)\,, v \in \left[0, \frac{1}{8}l\right], \\ \left(\frac{cvl}{192} \times \frac{l-4v}{l} + \frac{16cvl(8v-l)}{l}\right), & t \in (0,1)\,, v \in \left[\frac{1}{8}l, \frac{1}{4}l\right], \\ 16cvl, & t \in (0,1)\,,\ v \in \left[\frac{1}{4}l, l\right], \\ 16cvl + t\sqrt{v-l}, & t \in (0,1)\,, v \in [l, +\infty)\,, \end{cases}$$

Since $\lambda = \frac{32}{3} < 16$, *if* $\lim\limits_{v \to 0^+} \frac{\psi(v)}{v} = \frac{cl}{348} < \lambda_{1h}$ *and* $\lim\limits_{v \to +\infty} \frac{\psi(v)}{v} = 16cl < \lambda_{1h}$*, then* g *satisfies all the conditions of Theorem 3.4, thus we infer that the singular boundary value problem* (2.1) *has at least two positive solutions.*

3.2. **Positive radial solutions of elliptic boundary value problems.**

Define a set

$$K = \left\{p \in C\left((R_0, R_1), \mathbb{R}^+\right)\ :\ p \neq 0,\right.$$

$$\left.\int_{R_0}^{R_1}\left(\frac{Bs^{n-2}-A}{s^{n-2}}\right)\left(1-\frac{Bs^{n-2}-A}{s^{n-2}}\right)\left(\frac{(n-2)\,As^{n-3}}{s^{2(n-2)}}\right)p\,(s)\,ds<+\infty\right\},$$

where A and B are defined above

Denote $c=\left(\frac{A}{B-R_0}\right)^{n-2}$ and $d=\left(\frac{A}{B-R_1}\right)^{n-2}$.

For $p\in K$, let

$$h\,(t)=\phi\,(s)\,p\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}\right),$$

we can reformulate h as

$$h\,(t)=\phi\,(t)\,p\left(\left(\frac{A}{B-t}\right)^{\frac{1}{n-2}}\right),$$

where

$$\phi\,(t)=\left(\frac{R_1^{-(n-2)}-R_0^{-(n-2)}}{n-2}\right)^2\left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2}-R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right]\left[\frac{A}{B-s}\right]^{2(n-1)}.$$

For convinience, we let

$$\Delta=\left(\frac{R_1^{-(n-2)}-R_0^{-(n-2)}}{n-2}\right)^2\left[\frac{1}{A^{\frac{2n-2}{n-2}}\left(R_1^{n-2}-R_0^{n-2}\right)^{\frac{2n-2}{n-2}}}\right].$$

Then $h\in H$. As in (2.9) and Lemma 2.4, h confirms an operator T_h and its first eigenvalue λ_{1h}. To emphasize their relation with p, we use the notations h_p, λ_{1h_p} and φ_{1h_p}.

According to (2.2), we formulate the following conditions which correspond to those in Section 3.1.

(C_1) $f\in C\left((R_0,R_1)\times\mathbb{R}^+,\mathbb{R}^+\right)$ and for any $M>0$ there exist a function $p_M\in K$ such that

$$f\,(s,u)\leq p_M\,(s)\,,\;\forall\,(s,u)\in(R_0,R_1)\times[0,M]\,,$$

(C_2) there exist a function $p\in K$ such that

$$\lim_{u\to0^+}sup\frac{f\,(s,u)}{p\,(s)\,u}<\lambda_{1h_p},\;uniformly\,with\,respect\,to\,t\in(R_0,R_1)\,,$$

(C_3) there exist a function $p\in K$ such that

$$\lim_{u\to+\infty}sup\frac{f\,(s,u)}{p\,(s)\,u}<\lambda_{1h_p},\;uniformly\,with\,respect\,to\,t\in(R_0,R_1)\,,$$

(C_4) $\lim\limits_{u\to0^+}inf\min\limits_{s\in[c,d]}\frac{f(s,u)}{u}>c^{2-2n}\Delta M_1$,

(C_5) $\lim\limits_{u\to+\infty}inf\min\limits_{s\in[c,d]}\frac{f(s,u)}{u}>c^{2-2n}\Delta M_1$,

(C_6) there exist a number $l>0$ such that

$$f\,(s,u)>\Delta\lambda l,\quad for\;\,(s,u)\in[c,d]\times[min\,\{R_0,1-R_1\}\,l,l]\,,$$

(C_7) there exist a function $p\in K$ such that

$$\lim_{u\to0^+}inf\frac{f\,(s,u)}{p\,(s)\,u}>\lambda_{1h_p},\;uniformly\,with\,respect\,to\,s\in(R_0,R_1)\,,$$

(C_8) there exist a function $p \in K$ with $p(s) \neq 0$ for $s \in (c,d)$ and $q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$f(s,u) \geq p(s)\, q(u), \quad \forall (s,u) \in (R_0, R_1) \times \mathbb{R}^+,$$

$$\lim_{u \to +\infty} inf \frac{q(u)}{u} > \lambda_{1h_p}.$$

Now, we are ready to state our main results for the elliptic BVP (1.1).

Theorem 3.6. *Assume* $(C_1) - (C_3)$ *and* (C_6) *hold. Then the singular boundary value problem* (1.1) *has at least two positive solution.*

Proof. The proof is similar to proof of Theorem 4.1 in [21] and from the proof of Theorem 3.4. □

Theorem 3.7. *If* (C_1) *and one of the following conditions are satisfied, then the singular boundary value problem* (1.1) *has at least one positive solution.*

(i) (C_2) *and* (C_5) *holds,*
(ii) (C_2) *and* (C_6) *holds,*
(iii) (C_2) *and* (C_8) *holds,*
(iv) (C_3) *and* (C_4) *holds,*
(v) (C_3) *and* (C_6) *holds,*
(vi) (C_3) *and* (C_7) *holds.*

Proof. The proof is similar to proof of Theorem 4.1 in [21] and from the proof of Theorem 3.5. □

Conclusion

In this contribution, we studied the existence and multiplicity of radial positive solutions for elliptic BVP (1.1) in the ball. The interest of such problem came from the lack of the existence of the multiple solutions by using bifurcation theory for shown that many local branches of solutions existe while, among them, only one is global and has no bifurcation point implies a considerable difficult to prove the existence of bifurcation point interior the ball. The main scope of these paper is the imposing some conditions on the nonlinearity f to prove the multiplicity of the solutions of problems (1.1) in smooth domains via fixed point index theory.

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