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ON THE (DELTA,f)-LACUNARY STATISTICAL CONVERGENCE OF THE FUNCTIONS

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Abstract. In this paper, we introduce the concept of Δ_f-lacunary statistical convergence for a Δ-measurable real-valued function defined on time scale, where f is an unbounded modulus. Our motivation here is that this definition includes many well-known concepts which already exist in the literature. We also define strong Δ_f-lacunary Cesáro summability on a time scale and give some results related to these new concepts. Furthermore, we obtain necessary and sufficient conditions for the equivalence of Δ_f-convergence and Δ_f-lacunary statistical convergence on a time scale.

1. Introduction

The idea of statistical convergence for sequences of real and complex numbers, which was introduced by Fast [1] and Steinhaus [2] independently, is a generalization of ordinary convergence. This concept depends on density of subset of natural numbers $\mathbb{N}$. The natural (or asymptotic density) of a set $K \subset \mathbb{N}$ defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leqslant n : k \in K\}\right|,$$

if the limit exist, where $|A|$ indicates the cardinality of any set A. A sequence $x = (x_k)$ is said to be statistically convergent to L, if for every $\varepsilon > 0$, the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - L| \geqslant \varepsilon\}$ has zero natural density, i.e., for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}\right| = 0,$$

and written as $st - \lim x = L$. The set of all statistical convergent sequences is denoted by S. Over the years, statistical convergence and related notions have been studied by many researchers [3–15].

The idea of a modulus function was introduced by Nakano [16]. Later, Ruckle [17], Maddox [18] and many authors used this concept to construct some sequence spaces. A function $f : [0, \infty) \to [0, \infty)$ is called modulus function, or simply modulus, if

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i) $f(x)=0$ if and only if $x=0$,
ii) $f(x+y)\leqslant f(x)+f(y)$ for $x\geqslant 0$, $y\geqslant 0$,
iii) f is increasing,
iv) f is continuous from right at 0.

From the above properties (ii) and (iv), it is clear that a modulus function f is continuous everywhere on $[0,\infty)$. A modulus function may be bounded or unbounded. For example, $f(x)=\frac{x}{1+x}$ is bounded, but $f(x)=x^p$, where $0<p\leqslant 1$, is unbounded.

Aizpuru et al. [9] defined a new concept of density by using an unbounded modulus function, and also with this way, they defined f-statistical convergence for sequences as follows:

A sequence $x=(x_k)$ is said to be f-statistically convergent to L, if for each $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{f(n)}f\left(\left|\left\{k\leqslant n:|x_k-L|\geqslant\varepsilon\right\}\right|\right)=0,$$

where f is an unbounded modulus function, and one writes it as $st^f-\lim x_k=L$.

A time scale is any arbitrary nonempty closed subset of real numbers $\mathbb{R}$ and is denoted by $\mathbb{T}$. The time scales calculus was first introduced by Hilger [20]. This new theory allows one to unify discrete and continuous analysis as it has the differentiation and integration of independent domain used. Because of this feature, it has received much attention and its applications have been studied in many areas of science [21–24]. In addition, the first studies related to the statistical convergence and summability theory on time scales were done in [25] and [26], independently. In the following years, as a continuation and generalization of these studies, many researchers have moved well known some topics in summability theory for sequences or Lebesque measurable functions to time scale calculus [27–33]. Before giving these definitions, we shall mention some basic concepts of the time scale calculus that we will use in later sections.

The forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ can be defined by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},$$

for $t\in\mathbb{T}$. In this definition we put $\inf\emptyset=\sup\mathbb{T}$, where $\emptyset$ is an empty set. The graininess function $\mu:\mathbb{T}\to[0,\infty)$ can be defined by $\mu(t)=\sigma(t)-t$. A closed interval in a time scale $\mathbb{T}$ is given by $[a,b]_{\mathbb{T}}=\{t\in\mathbb{T}:a\leqslant t\leqslant b\}$. Open intervals or half-open intervals are defined similarly.

Let F denote the family of all left closed and right open intervals on $\mathbb{T}$ of the form $[a,b)_{\mathbb{T}}$. Let $m:F\to[0,\infty)$ be a set function on F such that $m([a,b)_{\mathbb{T}})=b-a$. Then, the set function m is a countably additive measure on F. Now, the Caratheodory extension of the set function m associated with the family F is said to be the Lebesgue Δ-measure on $\mathbb{T}$ and this is denoted by μ_Δ. In this case, it is known that if $a\in\mathbb{T}\backslash\{\max\mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(\{a\})=\sigma(a)-a$. If $a,b\in\mathbb{T}$ and $a\leqslant b$, then $\mu_\Delta((a,b)_{\mathbb{T}})=b-\sigma(a)$. If $a,b\in\mathbb{T}\backslash\{\max\mathbb{T}\}$ and $a\leqslant b$, then $\mu_\Delta((a,b]_{\mathbb{T}})=\sigma(b)-\sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}})=\sigma(b)-a$ (see [23]).

Now, we recall some basic concepts related to summability theory on time scales. We should note that throughout the paper, we consider that $\mathbb{T}$ is a time scale satisfying $\inf\mathbb{T}=t_0>0$ and $\sup\mathbb{T}=\infty$.

Definition 1.1. *[26] Let $g:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. We say that g is statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon>0$,*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\{s\in[t_0,t]_\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left([t_0,t]_\mathbb{T}\right)}=0,$$

which is denoted by $st_\mathbb{T}-\lim_{t\to\infty}g(t)=L$.

Let $\theta=(k_r)$ is an increasing sequence of non-negative integers with $k_0=0$ and $\sigma(k_r)-\sigma(k_{r-1})\to\infty$ as $r\to\infty$. Then θ is called a lacunary sequence with respect to $\mathbb{T}$ [27].

Definition 1.2. *[27] Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$. A Δ-measurable function $g:\mathbb{T}\to\mathbb{R}$ is said to be lacunary statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon>0$,*

$$\lim_{r\to\infty}\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)}=0,$$

which is denoted by $st_{\theta-\mathbb{T}}-\lim_{t\to\infty}g(t)=L$.

Definition 1.3. *[27] Let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$ and let $g:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. Then g is strongly lacunary Cesáro summable to L on $\mathbb{T}$, if there exists an $L\in\mathbb{R}$ such that*

$$\lim_{r\to\infty}\frac{1}{\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)}\int\limits_{(k_{r-1},k_r]_\mathbb{T}}|g(s)-L|\,\Delta s=0.$$

The set of all strongly lacunary Cesáro summable functions on $\mathbb{T}$ is denoted by $N_{\theta-\mathbb{T}}$.

Definition 1.4. *[32] Let f be a modulus function and $g:\mathbb{T}\to\mathbb{R}$ be a Δ-measurable function. We say that g is Δ_f-convergent to a number L on $\mathbb{T}$, if for every $\varepsilon>0$,*

$$\lim_{t\to\infty}\frac{1}{f\left(\mu_\Delta\left([t_0,t]_\mathbb{T}\right)\right)}f\left(\mu_\Delta\left(\{s\in[t_0,t]_\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}\right)\right)=0,$$

and we write it as $\Delta_f-\lim_{t\to\infty}g(t)=L$. Also, we denote the set of all Δ_f-convergent functions on $\mathbb{T}$ by $S_\mathbb{T}^f$.

Our aim here is to introduce the concepts of Δ_f-lacunary statistical convergence and strong Δ_f-lacunary Cesáro summability on a time scale $\mathbb{T}$ with respect to a modulus function f, by continuing of [32]. We also present several results related to these new concepts.

2. Δ_f-Lacunary Statistical Convergence on Time Scale

We start this section by defining the concept of Δ_f-lacunary statistical convergence on a time scale.

Definition 2.1. *Let f be an unbounded modulus function and let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$. Then a Δ-measurable function $g:\mathbb{T}\to\mathbb{R}$ is Δ_f-lacunary statistically convergent to a number L on $\mathbb{T}$, if for every $\varepsilon>0$,*

$$\lim_{r\to\infty}\frac{1}{f\left(\mu_\Delta\left((k_{r-1},k_r]_\mathbb{T}\right)\right)}f\left(\mu_\Delta\left(\{s\in(k_{r-1},k_r]_\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}\right)\right)=0,$$

and we write it as $st_{\theta-\mathbb{T}}^{f}-\lim_{t\to\infty} g(t)=L$*. The set off all* Δ_f*-lacunary statistically convergent functions on* $\mathbb{T}$ *is denoted by* $S_{\theta-\mathbb{T}}^{f}$.

This definition includes some special cases.

Remark. ***i)*** *If we choose* $f(x)=x$ *in Definition 2.1, then* Δ_f*-lacunary statistical convergence is reduced to lacunary statistical convergence on a time scale introduced in [27].*

ii) *If we take* $\mathbb{T}=\mathbb{N}$*, then Definition 2.1 gives us the concept of* f*-lacunary statistical convergence which is defined in [10].*

Theorem 2.1. *Let* f *be an unbounded modulus function and let* $\theta=(k_r)$ *be a lacunary sequence on* $\mathbb{T}$*. For any a* Δ*-measurable function* $g:\mathbb{T}\to\mathbb{R}$, $\lim_{t\to\infty} g(t)=L$ *implies* $st_{\theta-\mathbb{T}}^{f}-\lim_{t\to\infty} g(t)=L$.

Proof. Suppose that $\lim_{t\to\infty} g(t)=L$. Then, for each $\varepsilon>0$, the set $\{s\in\mathbb{T}\colon |g(s)-L|\geqslant\varepsilon\}$ is bounded. Since

$$\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\subseteq\{s\in\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}$$

and modulus function f is increasing, therefore

$$\frac{f\left(\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\right)\right)}{f\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}\leqslant\frac{f\left(\mu_\Delta\left(\{s\in\mathbb{T}:|g(s)-L|\geqslant\varepsilon\}\right)\right)}{f\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}.$$

Taking limit as $r\to\infty$ on both sides, we get

$$\lim_{r\to\infty}\frac{f\left(\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\right)\right)}{f\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)}=0,$$

which means that $st_{\theta-\mathbb{T}}^{f}-\lim_{t\to\infty} g(t)=L$. □

Theorem 2.2. *Let* f *be an unbounded modulus function and let* $\theta=(k_r)$ *be a lacunary sequence on* $\mathbb{T}$*. For any a* Δ*-measurable function* $g:\mathbb{T}\to\mathbb{R}$, $st_{\theta-\mathbb{T}}^{f}-\lim_{t\to\infty} g(t)=L$ *implies* $st_{\theta-\mathbb{T}}-\lim_{t\to\infty} g(t)=L$.

Proof. Suppose that $st_{\theta-\mathbb{T}}^{f}-\lim_{t\to\infty} g(t)=L$. Then, using the definition of limit and also using the properties of modulus function f, for every $p\in\mathbb{N}$, for sufficiently large $t\in\mathbb{T}$, we have

$$\begin{aligned}
f\left(\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\right)\right) &\leqslant\frac{1}{p}f\left(\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)\right)\\
&\leqslant\frac{1}{p}pf\left(\frac{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}{p}\right)\\
&=f\left(\frac{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}{p}\right)
\end{aligned}$$

and since f is increasing, we get

$$\frac{\mu_\Delta\left(\{s\in(k_{r-1},k_r]_{\mathbb{T}}:|g(s)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}\leqslant\frac{1}{p},$$

which means that $st_{\theta-\mathbb{T}}-\lim\limits_{t\to\infty} g(t)=L$. Hence, the proof is completed. □

Corollary 2.3. *Let f be an unbounded modulus function and let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$. For any a Δ-measurable function $g:\mathbb{T}\to\mathbb{R}$, we have*

$$\lim_{t\to\infty} g(t)=L \Rightarrow st^{f}_{\theta-\mathbb{T}}-\lim_{t\to\infty} g(t)=L \Rightarrow st_{\theta-\mathbb{T}}-\lim_{t\to\infty} g(t)=L.$$

3. Strong Δ_f-Lacunary Cesáro Summability and Δ_f-Lacunary Statistical Convergence on Time Scale

Now, we first introduce strong Δ_f-lacunary Cesáro summability of a Δ-measurable function defined on a time scale. We also investigate the relationship between the strong Δ_f-lacunary Cesáro summability and strong lacunary Cesáro summability on a time scale.

Definition 3.1. *Let f be a modulus function and let $\theta=(k_r)$ be a lacunary sequence on $\mathbb{T}$. Then a Δ-measurable function $g:\mathbb{T}\to\mathbb{R}$ is said to be strongly Δ_f-lacunary Cesáro summable to a number L on $\mathbb{T}$ if*

$$\lim_{r\to\infty}\frac{1}{\mu_\Delta\left((k_{r-1},k_r]_{\mathbb{T}}\right)}\int\limits_{(k_{r-1},k_r]_{\mathbb{T}}} f\left(|g(s)-L|\right)\Delta s=0.$$

The set off all strongly Δ_f-lacunary Cesáro summable functions on $\mathbb{T}$ is denoted by $N^{f}_{\theta-\mathbb{T}}$.

We now give some lemmas we will use next theorem.

Lemma 3.1. *[18] Let f be any modulus function and let $0<\delta<1$. Then, for each $x\geqslant\delta$, we have $f(x)\leqslant 2f(1)\delta^{-1}x$.*

Lemma 3.2. *[19] Let f be any modulus function. Then $\lim\limits_{t\to\infty}\frac{f(t)}{t}$ exists.*

Theorem 3.3. ***i)*** *For any modulus function f, we have $N_{\theta-\mathbb{T}}\subset N^{f}_{\theta-\mathbb{T}}$.*

ii) *Let f be any modulus function. If $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$, then we have $N^{f}_{\theta-\mathbb{T}}\subset N_{\theta-\mathbb{T}}$.*

Proof. It is easy to see using Lemma 3.1 and Lemma 3.2. □

As a corollary we have

Corollary 3.4. *Let f be any modulus function. If $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$, then we have $N^{f}_{\theta-\mathbb{T}}=N_{\theta-\mathbb{T}}$.*

Now, we give the relationship between the Δ_f-lacunary statistical convergence and strong Δ_f-lacunary Cesáro summability on time scale. Before doing this, we remind Jensen's inequality on time scale.

Lemma 3.5. *[22] Let $a,b\in\mathbb{T}$ and $c,d\in\mathbb{R}$. If $\phi:[a,b]\to(c,d)$ is rd-continuous and $F:(c,d)\to\mathbb{R}$ is convex, then*

$$F\left(\frac{\int_a^b \phi(t)\Delta t}{b-a}\right)\leqslant\frac{\int_a^b F(\phi(t))\Delta t}{b-a}.$$

Lemma 3.6. *[18] There is a modulus function f for which there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$.*

Theorem 3.7. *Let $g : \mathbb{T} \to \mathbb{R}$ be a Δ-measurable function and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, we have*

i) *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$. If g is strongly Δ_f-lacunary Cesáro summable to L, then $st^f_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$, but not conversely.*

ii) *If $st^f_{\theta-\mathbb{T}} - \lim_{t\to\infty} g(t) = L$ and g is a bounded function, then g is strongly Δ_f-lacunary Cesáro summable to L, for any unbounded modulus function f.*

Proof. It can be proved by considering similar way with in Theorem 14 of [10] and Theorem 1, Theorem 2 of [27] and also using Lemma 3.5. □

Now, under certain restrictions on $\theta = (k_r)$ and modulus function f, we investigate necessary and sufficient conditions for the equivalence of Δ_f-convergence and Δ_f-lacunary statistical convergence on a time scale.

Theorem 3.8. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$. Then, we have*

$$S^f_{\mathbb{T}} \subset S^f_{\theta-\mathbb{T}} \quad \textit{if and only if} \quad \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} > 1.$$

Proof. The proof can be done easily by combining the ideas in Lemma 3.1 of [28] and Lemma 17 of [10]. Hence, we omit it. □

Theorem 3.9. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant cf(x) f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \leqslant Mt$ for some $M \geqslant 0$ and for all $t \in \mathbb{T}$. Then, we have*

$$S^f_{\theta-\mathbb{T}} \subset S^f_{\mathbb{T}} \quad \textit{if and only if} \quad \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

Proof. The proof can be done easily in view of Lemma 3.2 of [28] and Lemma 19 of [10]. Hence, we omit it. □

We here note that all the restrictions apart from $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ on the modulus function f in Theorem 3.8 and Theorem 3.9 are needed only in the necessity part of these theorems.

Combining Theorem 3.8 and Theorem 3.9, we obtain the following result.

Corollary 3.10. *Let f be an unbounded convex modulus function for which $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exists and there exists a positive constant c such that $f(xy) \geqslant$*

$cf(x)f(y)$ for all $x \geqslant 0$, $y \geqslant 0$ and let $\theta = (k_r)$ be a lacunary sequence on $\mathbb{T}$ such that $\mu(t) \leqslant Mt$ for some $M \geqslant 0$ and for all $t \in \mathbb{T}$. Then, we have

$$S^f_{\theta-\mathbb{T}} = S^f_{\mathbb{T}} \quad \textit{if and only if} \quad 1 < \liminf_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} \leqslant \limsup_{r\to\infty} \frac{\sigma(k_r)}{\sigma(k_{r-1})} < \infty.$$

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References

[1] H. Fast, Sur la convergence statistique, Colloq. Math. **2** (1951), 241-244.
[2] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. **2** (1) (1951), 73-74.
[3] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly **66** (1959), 361-375.
[4] J. A. Fridy, On statistical convergence, Analysis **5** (1985), 301-313.
[5] J. A. Fridy and C. Orhan, Lacunary statistical convergence, Pacific J. Math. **160** (1993), 43-51.
[6] J. S. Connor, The statistical and strong p-Cesáro convergence of sequences, Analysis **8** (1988), 47-63.
[7] J. S. Connor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. **32** (1989), 194-198.
[8] F. Moricz, Statistical limits of measurable functions, Analysis **24** (1) (2004), 1-18.
[9] A. Aizpuru, M. C. Listan-Garcia and F. Rambla-Barreno, Density by moduli and statistical convergence, Quaest. Math. **37** (4) (2014), 525-530.
[10] V. K. Bhardwaj and S. Dhawan, Density by moduli and lacunary statistical convergence, Abstr. Appl. Anal. (2016).
[11] M. Et and H. Sengul, On (Δ^m, I)-lacunary statistical convergence of order α, J. Math. Anal. **7** (5) (2016), 78-84.
[12] H. Sengul and M. Et, f-lacunary statistical convergence and strong f-lacunary summability of order α, Filomat **32** (13) (2018), 4513-4521.
[13] H. Cakalli, A new approach to statistically quasi Cauchy sequences, Maltepe Journal of Mathematics **1** (1) (2019), 1-8.
[14] S. Yildiz, Lacunary statistical p-quasi Cauchy sequences, Maltepe Journal of Mathematics **1** (1) (2019), 9-17.
[15] I. Taylan, Abel statistical delta quasi Cauchy sequences of real numbers, Maltepe Journal of Mathematics **1** (1) (2019), 18-23.
[16] H. Nakano, Concave modulars, J. Math. Soc. Japan. **5** (1953), 29-49.
[17] W. H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, Can. J. Math. **25** (1973), 973-978.
[18] I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. **100** (1) (1986), 161-166.
[19] I. J. Maddox, Inclusions between FK spaces and Kuttner's theorem, Math. Proc. Cambridge Philos. Soc. **101** (3) (1987), 523-527.
[20] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, Results Math. **18** (1-2) (1990), 18-56.
[21] M. Bohner and A. Peterson, Dynamic equations on time scales: An introduction with applications, Birkhäuser, Boston, 2001.
[22] R. Agarwal, M. Bohner and A. Peterson, Inequalities on time scales: a survey, Math. Inequal. Appl. **4** (4) (2001), 535-558.
[23] G. S. Guseinov, Integration on time scales, J. Math. Anal. Appl. **285** (1) (2003), 107-127.

[24] A. Cabada and D. R. Vivero, Expression of the Lebesgue $\Delta-$integral on time scales as a usual Lebesgue integral: Application to the calculus of $\Delta-$antiderivatives, Math. Comput. Model. **43** (1-2) (2006), 194-207.

[25] M. S. Seyyidoglu and N. O. Tan, A note on statistical convergence on time scale, J. Inequal. Appl. (219) (2012).

[26] C. Turan and O. Duman, Statistical convergence on time scales and its characterizations, Springer Proc. Math. Stat. **41** (2013), 57-71.

[27] C. Turan and O. Duman, Convergence methods on time scales, AIP Conf. Proc. **1558** (1) (2013), 1120-1123.

[28] C. Turan and O. Duman, Fundamental properties of statistical convergence and lacunary statistical convergence on time scales, Filomat **31** (14) (2017), 4455-4467.

[29] Y. Altin, H. Koyunbakan and E. Yilmaz, Uniform statistical convergence on time scales, J. Appl. Math. (2014).

[30] B. Sozbir and S. Altundag, Weighted statistical convergence on time scale, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **26** (2019), 137-143.

[31] B. Sozbir and S. Altundag, $\alpha\beta$-statistical convergence on time scales, Facta Univ. Ser. Math. Inform. **35** (1) (2020), 141-150.

[32] N. Turan and M. Basarir, On the Δ_g-statistical convergence of the function defined time scale, AIP Conference Proceedings **2183** (040017) (2019). https://doi.org/10.1063/1.5136137.

[33] M. Basarir, A note on the (θ, φ)-statistical convergence of the product time scale, Konuralp Journal of Mathematics, **8** (1) (2020), 192-196.

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