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## ON A MEAN METHOD OF SUMMABILITY

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ABSTRACT. Let  $p(x)$  be a nondecreasing real-valued continuous function on  $R_+ := [0, \infty)$  such that  $p(0) = 0$  and  $p(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Given a real or complex-valued integrable function  $f$  in Lebesgue's sense on every bounded interval  $(0, x)$  for  $x > 0$ , in symbol  $f \in L^1_{loc}(R_+)$ , we set

$$s(x) = \int_0^x f(u)du$$

and

$$\sigma_p(s(x)) = \frac{1}{p(x)} \int_0^x s(u)dp(u), \quad x > 0$$

provided that  $p(x) > 0$ .

A function  $s(x)$  is said to be summable to  $l$  by the weighted mean method determined by the function  $p(x)$ , in short,  $(\bar{N}, p)$  summable to  $l$ , if

$$\lim_{x \rightarrow \infty} \sigma_p(s(x)) = l.$$

If the limit  $\lim_{x \rightarrow \infty} s(x) = l$  exists, then  $\lim_{x \rightarrow \infty} \sigma_p(s(x)) = l$  also exists. However, the converse is not true in general. In this paper, we give an alternative proof a Tauberian theorem stating that convergence follows from summability by weighted mean method on  $R_+ := [0, \infty)$  and a Tauberian condition of slowly decreasing type with respect to the weight function due to Karamata. These Tauberian conditions are one-sided or two-sided if  $f(x)$  is a real or complex-valued function, respectively. Alternative proofs of some well-known Tauberian theorems given for several important summability methods can be obtained by choosing some particular weight functions.

## 1. INTRODUCTION

Let  $p(x)$  be a nondecreasing real-valued continuous function on  $R_+ := [0, \infty)$ . Throughout this paper, we assume that  $p(0) = 0$  and  $p(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . Given a real-valued integrable function  $f$  in Lebesgue's sense on every bounded interval  $(0, x)$  for  $x > 0$ , in symbol  $f \in L^1_{loc}(R_+)$ , we set

$$s(x) = \int_0^x f(u)du \tag{1.1}$$

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and

$$\sigma_p(s(x)) = \frac{1}{p(x)} \int_0^x s(u) dp(u), \quad x > 0$$

provided that  $p(x) > 0$ .

A function  $s(x)$  is said to be summable to  $l$  by the weighted mean method determined by the function  $p(x)$ , in short,  $(\bar{N}, p)$  summable to  $l$ , if

$$\lim_{x \rightarrow \infty} \sigma_p(s(x)) = l. \quad (1.2)$$

Clearly, if the ordinary limit

$$\lim_{x \rightarrow \infty} s(x) = l \quad (1.3)$$

exists, then (1.2) holds. However, the converse implication is not true in general. We may get the converse implication by adding some assumption(s) on  $s(x)$ , which is so-called Tauberian condition(s). Any theorem which states that convergence of (1.3) follows from (1.2) and a Tauberian condition is said to be a Tauberian theorem for summability by the weighted mean method.

A real-valued function  $s(x)$  defined on  $R_+$  is said to be slowly decreasing with respect to  $p$  if

$$\lim_{\lambda \rightarrow 1^+} \liminf_{t \rightarrow \infty} \min_{t \leq x \leq T} (s(x) - s(t)) \geq 0, \quad (1.4)$$

where

$$T := p^{-1}(\lambda p(t)), \quad t > 0. \quad (1.5)$$

Note that the concept of slow decrease with respect to  $p$  is due to Karamata [4].

It is easy to see that a real-valued function  $s(x)$  is slowly decreasing with respect to  $p$  if and only if for every  $\epsilon > 0$  there exist  $t_0 = t_0(\epsilon) > 0$  and  $\lambda = \lambda(\epsilon) > 1$  such that  $s(x) - s(t) \geq -\epsilon$  whenever  $t_0 \leq t \leq x \leq T$ .

An equivalent reformulation of (1.4) can be given as follows (see Fekete and Moricz [1]):

$$\lim_{\lambda \rightarrow 1^-} \liminf_{t \rightarrow \infty} \min_{T \leq x \leq t} (s(t) - s(x)) \geq 0, \quad (1.6)$$

where  $T$  is defined in (1.5). It is easy to see that a real valued function  $s(x)$  is slowly decreasing with respect to  $p$  if and only if for every  $\epsilon > 0$  there exist  $t_1 = t_1(\epsilon) > 0$  and  $\lambda = \lambda(\epsilon)$  with  $0 < \lambda < 1$  such that  $s(t) - s(x) \geq -\epsilon$  whenever  $t_1 \leq T \leq x \leq t$ .

A real-valued function  $s(x)$  defined on  $R_+$  is said to be slowly decreasing if (1.4) holds, where  $p(x) = x$  for all  $x > 0$ . Recall that the term "slow decrease" is introduced by Schmidt [7] for sequences of real numbers.

In [3], we obtained an alternative proof of Theorem 2.1 below when a Tauberian condition is of slowly decreasing type.

In this paper, we give an alternative proof a Tauberian Theorem stating that convergence follows from summability by weighted mean method over  $R_+$  and a Tauberian condition of slowly decreasing type with respect to the weight function, due to Karamata [4].

Alternative proofs of some well-known Tauberian theorems given for several important summability methods can be obtained by choosing some particular weight functions.

## 2. MAIN RESULTS

By using proving techniques in [6], we give an alternative proof of the following Tauberian theorem [5] for the weighted mean summability of integrals of real-valued functions over  $R_+$ .

**Theorem 2.1.** *Let  $p(x)$  be a nondecreasing real-valued continuous function on  $R_+$  such that  $p(0) = 0$  and  $p(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If a real-valued function  $f \in L_{loc}^1(R_+)$  is such that (1.2) holds and its integral function  $s(x)$  is slowly decreasing with respect to  $p$ , then (1.3) holds.*

*Proof.* By the regularity of the summability method by the weighted mean, without loss of generalization, we assume that  $l = 0$ . Assume that  $s(x)$  does not converge to 0 as  $x \rightarrow \infty$ .

Then, we have either  $\limsup_{x \rightarrow \infty} s(x) > 0$  or  $\liminf_{x \rightarrow \infty} s(x) < 0$ .

First, we assume that  $\limsup_{x \rightarrow \infty} s(x) > 0$ . Then, there exist  $\alpha > 0$  and a sequence  $(n_i)$  such that  $s(n_i) \geq \alpha$  for all nonnegative integers  $i$ . Choosing  $\epsilon = \frac{\alpha}{2}$  in the equivalent form of (1.4), we find  $\lambda > 1$  and  $t_0 \geq 0$  such that  $s(x) \geq s(n_i) - \frac{\alpha}{2} \geq \frac{\alpha}{2}$  for  $t_0 \leq n_i < x \leq m_i = p^{-1}(\lambda p(n_i))$ .

Since

$$\begin{aligned} \sigma_p(s(m_i)) - \frac{p(n_i)}{p(m_i)} \sigma_p(s(n_i)) &= \sigma_p(s(m_i)) - \frac{1}{\lambda} \sigma_p(s(n_i)) \\ &= \frac{1}{p(m_i)} \int_{n_i}^{m_i} s(u) dp(u), \end{aligned}$$

we have

$$\begin{aligned} \sigma_p(s(m_i)) - \frac{p(n_i)}{p(m_i)} \sigma_p(s(n_i)) &\geq \frac{\alpha}{2p(m_i)} \int_{n_i}^{m_i} dp(u) \\ &= \frac{\alpha}{2} \left(1 - \frac{1}{\lambda}\right) \end{aligned} \quad (2.1)$$

for  $t_0 \leq n_i < x \leq m_i = p^{-1}(\lambda p(n_i))$ . We conclude by (2.1) that  $0 \geq \frac{\alpha}{2} (1 - \frac{1}{\lambda})$ . This contradicts our assumption that  $\limsup_{x \rightarrow \infty} s(x) > 0$ . Then, we have

$$\limsup_{x \rightarrow \infty} s(x) \leq 0. \quad (2.2)$$

Next, we assume that  $\liminf_{x \rightarrow \infty} s(x) < 0$ . Then, there exist  $\beta < 0$  and a sequence  $(n_i)$  such that  $s(n_i) \leq \beta < 0$  for all nonnegative integers  $i$ . Choosing  $\epsilon = -\frac{\beta}{2}$  in the equivalent form of (1.4), we find  $0 < \lambda < 1$  and  $t_1 = t_1(\epsilon)$  such that  $s(x) \leq s(n_i) - \frac{\beta}{2} \leq \frac{\beta}{2}$  for  $t_1 \leq m_i = p^{-1}(\lambda p(n_i)) < x \leq n_i$ .

Since

$$\begin{aligned} \sigma_p(s(n_i)) - \frac{p(m_i)}{p(n_i)} \sigma_p(s(m_i)) &= \sigma_p(s(n_i)) - \lambda \sigma_p(s(m_i)) \\ &= \frac{1}{p(n_i)} \int_{m_i}^{n_i} s(u) dp(u), \end{aligned}$$

we have

$$\begin{aligned} \sigma_p(s(n_i)) - \frac{p(m_i)}{p(n_i)} \sigma_p(s(m_i)) &\leq \frac{\beta}{2p(n_i)} \int_{m_i}^{n_i} dp(u) \\ &= \frac{\beta}{2} (1 - \lambda) \end{aligned} \quad (2.3)$$

for  $t_1 \leq m_i = p^{-1}(\lambda p(n_i)) \leq x \leq n_i$ . We conclude by (2.3) that  $0 \leq \frac{\beta}{2}(1 - \lambda)$ . This contradicts our assumption that  $\liminf_{x \rightarrow \infty} s(x) < 0$ . Then, we have

$$\liminf_{x \rightarrow \infty} s(x) \geq 0. \quad (2.4)$$

Combining (2.2) and (2.4) gives convergence of  $s(x)$  to 0 as  $x \rightarrow \infty$ .  $\square$

A real-valued function  $s(x)$  defined on  $\mathbf{R}_+$  is said to be slowly increasing with respect to  $p$  if  $-s$  is slowly decreasing with respect to  $p$ .

**Remark.** Theorem 2.1 remains true if slow decrease of  $s(x)$  with respect to  $p$  is replaced by slow increase of  $s(x)$  with respect to  $p$ .

For a complex-valued integrable function  $f$  in Lebesgue's sense on every bounded interval  $(0, x)$  for  $0 < x < \infty$ , we have the following Tauberian theorem.

**Theorem 2.2.** *Let  $p(x)$  be a nondecreasing real-valued continuous function on  $R_+$  such that  $p(0) = 0$  and  $p(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . If a complex-valued function  $f \in L_{loc}^1(R_+)$  is such that (1.2) holds and its integral function  $s(x)$  is slowly oscillating with respect to  $p$ , then (1.3) holds.*

The proving technique in Theorem 2.1 is also valid for the proof of Theorem 2.2.

We remind the reader that a complex-valued function  $s(x)$  defined on  $R_+$  is said to be slowly oscillating with respect to  $p$  ([4]) if

$$\lim_{\lambda \rightarrow 1^+} \limsup_{t \rightarrow \infty} \max_{t \leq x \leq T} |s(x) - s(t)| = 0, \quad (2.5)$$

where  $T$  is defined as (1.5).

It is easy to see that a real-valued function  $s(x)$  is slowly oscillating with respect to  $p$  if and only if for every  $\epsilon > 0$  there exist  $t_0 = t_0(\epsilon) > 0$  and  $\lambda = \lambda(\epsilon) > 1$  such that  $|s(x) - s(t)| \leq \epsilon$  whenever  $t_0 \leq t \leq x \leq T$ .

An equivalent reformulation of (2.5) can be given as follows (see Fekete and Moricz [1]):

$$\lim_{\lambda \rightarrow 1^-} \limsup_{t \rightarrow \infty} \max_{T \leq x \leq t} |s(t) - s(x)| = 0, \quad (2.6)$$

where  $T$  is defined in (1.5). It is easy to see that a real valued function  $s(x)$  is slowly decreasing with respect to  $p$  if and only if for every  $\epsilon > 0$  there exist  $t_1 = t_1(\epsilon) > 0$  and  $\lambda = \lambda(\epsilon)$  with  $0 < \lambda < 1$  such that  $|s(t) - s(x)| \leq \epsilon$  whenever  $t_1 \leq T \leq x \leq t$ .

A complex-valued function  $s(x)$  defined on  $R_+$  is said to be slowly oscillating if (2.5) holds, where  $p(x) = x$  for all  $x > 0$ .

Recall that the concept of slow oscillation was introduced by Hardy [2] for sequences of real numbers.

### 3. PARTICULAR WEIGHTS

Some particular choices of weight functions can lead to alternative proofs of some well-known Tauberian theorems given for several important summability methods. If  $p(x) = x$  for all  $x > 0$ , then weighted mean method  $(\overline{N}, p)$  reduces to the Cesàro summability method. If  $p(x) = \ln x$  for all  $x \in [1, \infty)$  and zero for all  $x \in [0, 1)$ , then the weighted mean method  $(\overline{N}, p)$  reduces the harmonic mean method of first order. For other particular choices of the weight function  $p$ , we obtain the harmonic mean method of higher order. Our main Theorem 2.1 applies to all of these methods.

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