Maltepe Journal of Mathematics
ISSN:2667-7660, URL:http://dergipark.org.tr/tr/pub/mjm
Volume V Issue 2 (2023), Pages 24-31, Doi:https://doi.org/10.47087/mjm.1333727

DISTRIBUTION OF EIGENVALUES OF A PERTURBED DIFFERENTIATION OPERATOR ON THE INTERVAL

N.S. IMANBAEV
SOUTH KAZAKHSTAN STATE PEDAGOGICAL UNIVERSITY, A.BAITURSYNOV ST. 13, SHYMKENT, 160000, KAZAKHSTAN
INSTITUTE OF MATHEMATICS AND MATHEMATICAL MODELING, PUSHKIN ST. 125, ALMATY, 050010, KAZAKHSTAN

Abstract. In this paper, we construct a characteristic determinant of the spectral problem of a first-order differential equation on an interval with an integral perturbation in the boundary value condition, which is an entire analytic function of the spectral parameter. Based on the formula for the characteristic determinant, conclusions are drawn about the asymptotic behavior of the spectrum of the perturbed spectral problem depending on the modulus of continuity of the subinteral function.

1. Introduction

Works [1, 2, 3, 4] are devoted to studies of zeros of entire functions with an integral representation. Sometimes entire functions coincide with quasi-polynomials, zeros of which were investigated in papers [5, 6]. Connection between the zeros of quasi-polynomials and spectral problems is reflected in papers [7, 8, 9, 10, 11, 12]. Eigenvalue problems for some classes of differential operators on an interval are reduced to a similar problem. In particular, spectral problem for a first-order equation on an interval with a spectral parameter in a boundary-value condition with integral perturbation leads to the studied problem [13].

Asymptotic properties of entire functions with a given law of distribution of roots were deeply investigated in the doctoral dissertation of V.B. Sherstyukov, on its basis, the paper [14] was published.

The questions on location of the zeros of an entire function: on one ray, on a straight line, on several rays, in an angle or arbitrarily in the complex plane were studied in the works [1], [3], [9], [11] and [15].

Meramorphic functions of completely regular growth in the upper half-plane with respect to the growth function have been studied in one of the last works of K.G. Malyutin and M.V. Cabanco [16]. In the paper of Rabha W., Ibrahim,

2020 *Mathematics Subject Classification.* Primary: 34L10 ; Secondaries: 34B05.
Key words and phrases. differentiation operator, continuous functions, entire functions, boundary value conditions, integral perturbation, characteristic determinant, eigenvalues, zeros, asymptotics, continuity module, loaded differential equation.

Submitted on July 27th, 2023. Published on October 7th 2023.
Communicated by Huseyin CAKALLI and Charyyar ASHYRALYYEV.

Ibtisam Aldawish [17], a new symmetric differential operator associated with a special class of meromorphic - multivalent functions in a punctured unit disc is presented. This study explories some of its geometric properties. A new class of holomorphic functions related by a symmetric differential operator is considered.

The paper is devoted to construction of a characteristic determinant of the spectral problem for the differentiation operator on an interval with an integral perturbation in the boundary value condition, which is an entire holomorphic function, where the integrand function has continuity property. Based on the formula for the characteristic determinant, conclusions are established about asymptotics of the spectrum of the perturbed spectral problem depending on the continuity modulus of the integrand function.The considered problem belongs to the nonlocal type of spectral problems. Such problems have been studied many times before. Among the recent publications, we note works [18, 19, 20, 21]. The main fundamental feature of such problems is their non-self-adjointness. This causes the main difficulties in their study.

1.1. **Problem Statement .** In the space $W_2^1(-1,1)$ we consider the following problem on eigenvalues of the operator:

$$L_1 y \equiv y'(t) = \lambda y(t), \quad -1 \leq t \leq 1 \tag{1.1}$$

with boundary value condition

$$y(-1) - y(1) = \int_{-1}^{1} y(t) \cdot \Phi(t) dt, \tag{1.2}$$

where $\Phi(t)$ is a continuous function on the interval $[-1, 1]$ and $\Phi(-1) = \Phi(1) = 1$, λ is a complex number, spectral parameter.

It is required to find those complex value of λ, in which the operator equation (1.1) has a nonzero solution.

2. Main Results

We introduce the general solution of the equation (1.1) by the formula $y(t) = Ce^{\lambda t}$, $\forall C > 0$, and satisfing the boundary value condition (1.2), we obtain the characteristical determinant of the problem (1.1) – (1.2):

$$\Delta_1(\lambda) = e^{-\lambda} - e^{\lambda} - \int_{-1}^{1} e^{\lambda t} \cdot \Phi(t) dt, \tag{2.1}$$

which is an entire analytical function of the variable $\lambda = x+iy$, $Re\lambda = x$, $Im\lambda = y$, $i = \sqrt{-1}$.

If the function $\Phi(t) \equiv 0$, then we get that $\Delta_0(\lambda) = e^{-\lambda} - e^{\lambda}$ is a characteristical determinant of the following spectral problem:

$$L_0 y \equiv y'(t) = \lambda y(t), \quad -1 \leq t \leq 1, \quad y(-1) = y(1). \tag{2.2}$$

The numbers $\lambda_n^0 = in\pi$, $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$, are eigenvalues, moreover, $\forall C > 0$, $y_{n0}^0 = C \cdot e^{in\pi t}$ are eigen functions of the operator L_0, which forms a complete orthonormal system in $L_2(-1, 1)$, and forms a basis in $L_2(-1, 1)$.

In our case, the function $\Phi(t)$ is continuous on the interval $[-1,1]$. Due to well-known Rouche's theorem [22], we introduce the function in (2.1):

$$\Delta_1(\lambda)=\Delta_0(\lambda)-f(\lambda),\ \text{where}\ \Delta_0(\lambda)=e^{-\lambda}-e^{\lambda},$$

$$f(\lambda)=\int\limits_{-1}^{1}e^{\lambda t}\cdot\Phi(t)dt,$$

where all of these functions are entire analytical functions. We estimate the function $\Delta_0(\lambda)$ from below:

$$|\Delta_0(\lambda)|\geq e^{|\lambda|}-e^{-|\lambda|}\geq e^{x}-e^{-x}.$$

Distribution of zeros of the entire function $f(\lambda)$ is investigated separately. We split the interval $[-1,1]$ into $2m$ equal parts. Then the function $f(\lambda)$ takes the following form:

$$f(\lambda)=\int\limits_{-1}^{1}e^{\lambda t}\cdot\Phi(t)dt=\int\limits_{-1}^{\frac{-2(m-1)}{2m}}e^{\lambda t}\cdot\Phi(t)dt+\int\limits_{\frac{-2(m-1)}{2m}}^{\frac{2(2-m)}{2m}}e^{\lambda t}\cdot\Phi(t)dt$$

$$+\int\limits_{\frac{2(2-m)}{2m}}^{\frac{2(3-m)}{2m}}e^{\lambda t}\cdot\Phi(t)dt+\ldots+\int\limits_{\frac{2(m-1)}{2m}}^{1}e^{\lambda t}\cdot\Phi(t)dt=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(t)dt.$$

We transform the function $f(\lambda)$:

$$f(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(t)dt=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left[\Phi(t)-\Phi(\frac{p}{m})+\Phi(\frac{p}{m})\right]dt$$

$$=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(\frac{p}{m})dt+\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left[\Phi(t)-\Phi(\frac{p}{m})\right]dt.$$

Let us show that $f(\lambda)$ does not have zeros outside the domain ($|x|\leq nrw(\frac{1}{n})$, for some n). Due to the Rouche's theorem [22], we introduce the designation

$$h(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\Phi(\frac{p}{m})dt,$$

and

$$G(\lambda)=\sum_{p=-m+1}^{m}\int\limits_{\frac{p-1}{m}}^{\frac{p}{m}}e^{\lambda t}\cdot\left(\Phi(t)-\Phi(\frac{p}{m})\right)dt.$$

Let $Re\lambda > 0$. We compute the integrals included in the function $h(\lambda)$.

$$h(\lambda) = \sum_{p=-m+1}^{m} \Phi(\frac{p}{m})\frac{1}{\lambda}\left(e^{\lambda\frac{p}{m}} - e^{\lambda\frac{p-1}{m}}\right) = \frac{1}{\lambda}\left[\Phi(-1+\frac{1}{m})\left(e^{\lambda(-1+\frac{1}{m})} - e^{-\lambda}\right)\right.$$
$$+\Phi(-1+\frac{2}{m})\left(e^{\lambda(-1+\frac{2}{m})} - e^{\lambda(-1+\frac{1}{m})}\right) + \ldots + \Phi(1-\frac{2}{m})\left(e^{\lambda(1-\frac{2}{m})} - e^{\lambda(1-\frac{3}{m})}\right)$$
$$\left.+\Phi(1-\frac{1}{m})\left(e^{\lambda(1-\frac{1}{m})} - e^{\lambda(1-\frac{2}{m})}\right)\right] = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right)\right.$$
$$+e^{\lambda(-1+\frac{2}{m})}\left(\Phi(-1+\frac{2}{m}) - \Phi(-1+\frac{3}{m})\right)+e^{\lambda(-1+\frac{3}{m})}\left(\Phi(-1+\frac{3}{m}) - \Phi(-1+\frac{4}{m})\right)$$
$$\left.-\Phi(-1+\frac{1}{m})e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right].$$

Grouping the exponents in pairs, we have

$$h(\lambda) = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right) - \Phi(-1+\frac{1}{m})e^{-\lambda}\right.$$
$$\left.+\Phi(-1)e^{-\lambda} - \Phi(-1)e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right]$$
$$= \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right) + e^{-\lambda}\left(\Phi(-1) - \Phi(-1+\frac{1}{m})\right)\right.$$
$$\left.-\Phi(-1)e^{-\lambda} + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right) + \Phi(1)e^{\lambda}\right].$$

We denote

$$h_1(\lambda) = \frac{1}{\lambda}\left[\Phi(1)e^{\lambda} - \Phi(-1)e^{-\lambda}\right] = \frac{1}{\lambda}\left[e^{\lambda} - e^{-\lambda}\right],$$

and

$$g(\lambda) = \frac{1}{\lambda}\left[e^{\lambda(-1+\frac{1}{m})}\left(\Phi(-1+\frac{1}{m}) - \Phi(-1+\frac{2}{m})\right)\right.$$
$$\left.+e^{-\lambda}\left(\Phi(-1) - \Phi(-1+\frac{1}{m})\right) + \ldots + e^{\lambda(1-\frac{1}{m})}\left(\Phi(1-\frac{1}{m}) - \Phi(1)\right)\right]$$
$$= \frac{1}{\lambda}\sum_{p=-m+1}^{m} e^{\lambda(\frac{p-1}{m})}\left(\Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m})\right).$$

Then

$$\mu(\lambda) = G(\lambda) + g(\lambda) =$$
$$= \sum_{p=-m+1}^{m}\left(\int_{\frac{p-1}{m}}^{\frac{p}{m}} e^{\lambda t}\cdot\left(\Phi(t) - \Phi(\frac{p}{m})\right)dt + \frac{e^{\lambda(\frac{p-1}{m})}}{\lambda}\left(\Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m})\right)\right).$$

We estimate the function $h_1(\lambda)$ from below, at the same time as the remaining terms, that is, the function $\mu(\lambda)$ is estimated from above

$$|h_1(\lambda)| = \left|\frac{1}{\lambda}\left(e^{\lambda} - e^{-\lambda}\right)\right| \geq \frac{1}{|\lambda|}e^{\lambda} - \frac{1}{|\lambda|}\left|\underline{\underline{o}}(e^{-\lambda})\right|. \tag{2.3}$$

We estimate the function $\mu(\lambda)$ from above:

$$|\mu(\lambda)| \leq \sum_{p=-m+1}^{m} \left[\int_{\frac{p-1}{m}}^{\frac{p}{m}} |e^{\lambda t}| \cdot \left|\Phi(t) - \Phi(\frac{p}{m})\right| dt + \frac{e^{\lambda(\frac{p-1}{m})}}{|\lambda|} \left|\Phi(\frac{p-1}{m}) - \Phi(\frac{p}{m})\right| \right]$$

$$\leq \sum_{p=-m+1}^{m} \left[\int_{\frac{p-1}{m}}^{\frac{p}{m}} e^{xt} \sup_{\frac{p-1}{m} \leq t \leq \frac{p}{m}} \left|\Phi(t) - \Phi(\frac{p}{m})\right| dt + \frac{e^{x(\frac{p-1}{m})}}{|\lambda|} \sup_{|t-\tau| \leq \frac{1}{m}} |\Phi(t) - \Phi(\tau)| \right].$$

We introduce module of continuity of the function $\Phi(t)$ by the formula

$$w(\frac{1}{m}) = \sup_{|t-\tau| < \frac{1}{m}} |\Phi(t) - \Phi(\tau)|.$$

Then

$$|\mu(\lambda)| \leq \sum_{p=-m+1}^{m} \left[\int_{\frac{p-1}{m}}^{\frac{p}{m}} e^{xt} \cdot w(\frac{1}{m}) dt + \frac{e^{x(\frac{p-1}{m})}}{x} w(\frac{1}{m}) \right] \leq w(\frac{1}{m}) \frac{e^{x} - e^{-x}}{x}. \tag{2.4}$$

Therefore, due to (2.3), (2.4), we come to the estimation:

$$|f(\lambda)| \geq \frac{1}{|\lambda|} e^{x} - \frac{1}{|\lambda|} \left| \bar{\bar{o}}(\frac{1}{|\lambda}|) \right| - w(\frac{1}{m}) \left(\frac{e^{x} + e^{-x}}{x} \right).$$

Assuming that, $|\lambda| = r$, $m = [r]$, we have

$$|\lambda f(\lambda)| = |f_1(\lambda)| \geq e^{x} - \frac{e^{x} w(\frac{1}{r}) r}{x} - e^{-x} - \frac{e^{-x} r w(\frac{1}{r})}{x}. \tag{2.5}$$

For the final approval, we will choose n so that

$$\left| \frac{w(\frac{1}{r}) r}{x} \right| + e^{-2x} + e^{-2x} \cdot \frac{r w(\frac{1}{r})}{x} < \frac{1}{2}$$

as $x > nrw(\frac{1}{r})$. It is possible, since value of the left part of the last inequality is defined in the main first term.

By the condition of the Rouche's theorem [22], defining the main part of the function $\Delta_1(\lambda)$, due to lower estimation of the function $\Delta_0(\lambda)$ and $f_1(\lambda)$ in (2.5), i.e. $|\Delta_0(\lambda)| > |f_1(\lambda)|$, we come to the following theorem:

Theorem 2.1. *If the function $\Phi(t)$ is continuous on the interval $[-1, 1]$ and satisfies the condition $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the operator L_1 lie in the $|Re\lambda| < nrw(\frac{1}{r})$ at some n, where $\lambda = x+iy$, $Re\lambda = x$, and $w(\delta)$ is a continuity madule of the function $\Phi(t)$, $r = |\lambda|$.*

Remark. *If $\Phi(t)$ is continuous on $[-1, 1]$ and $\Phi(-1) = \Phi(1) = 1$, then all eigenvalues of the operator L_1 are lie in the $|Re\lambda| < nrw(\frac{1}{r})$ on the complex plane λ, which expands depending on properties of the continuity module $w(\delta)$ of the function $\Phi(t)$.*

Theorem 2.2. *Let $\Phi(t)$ be a continuous function on $[-1, 1]$ and $\Phi(-1) = \Phi(1) = 1$. Then set of zeros of the entire function $\Delta_1(\lambda)$ as $n \to \infty$, $\lambda_n = i\pi n + \underline{\underline{o}}(nw(\frac{1}{n}))$, where $w(h)$ is a continuity module of $\Phi(t)$.*

Proof. To prove Theorem 2.1 we used two functions $h_1(\lambda)$ and $\mu(\lambda)$, such that $f(\lambda) = h_1(\lambda) + \mu(\lambda)$. Zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ have the form $\lambda_n^0 = i\pi n$, $n = \pm 1, \pm 2, \ldots$. We consider a square T with sides 2ε and with a center at the point λ_n^0 on the complex plane λ. Assume that, sides of T are parallel to real and imaginary axes of the λ variable. Proof of Theorem 2.2 consists in choosing ε so that conditions of Rouche's Theorem [22] were satisfied for the functions $\Delta_0(\lambda)$, $h_1(\lambda)$ and $\mu(\lambda)$ on the sides of the square T. First, we consider the right half of the square T, that is, in the case $Re\lambda \geq 0$. Divide the side of the square T into two parts $0 \leq Re\lambda \leq C$ and $C \leq Re\lambda \leq \varepsilon$, where $C > 0$, which we will choose later.

2.1 Case. Let $0 \leq Re\lambda \leq C$. Since zeros of the functions $\Delta_0(\lambda)$ and $h_1(\lambda)$ are the same and these functions are equal to each other, therefore, it is enough to estimate the function $h_1(\lambda)$. Let's compare the modules of functions $h_1(\lambda)e^{-\lambda}$ and $\mu(\lambda)e^{-\lambda}$. Taking into account boundedness of the corresponding derivative, we obtain the following estimate:

$$|h_1(\lambda)e^{-\lambda}| = |h_1(\lambda)e^{-\lambda} - h_1(\lambda_n^0)e^{-\lambda_n^0}| = |\frac{d}{d\lambda}h_1(\lambda)e^{-\lambda}| \cdot |\lambda - \lambda_n^0| \geq \frac{C_1}{|\lambda|} \cdot \varepsilon.$$

Due to boundedness of modules of exponents, included in $\mu(\lambda)$, we write the inequalities

$$|\mu(\lambda)e^{-\lambda}| \leq C_2 w(\frac{1}{n}).$$

Therefore, to satisfy conditions of Rouche's Theorem it is enough to take ε from

$$\varepsilon = \underline{\underline{o}}(nw(\frac{1}{n})),$$

since module of λ behaves like $\lambda = n(1 + \overline{\overline{o}}(1))$.

2.2 Case. Let $C \leq Re\lambda \leq \varepsilon$. When $C > 0$, the module of $h_1(\lambda)$ is estimated by the module of one of the exponents included in $h_1(\lambda)$:

$$|h_1(\lambda)| = \left|\frac{e^{\lambda} - e^{-\lambda}}{\lambda}\right| \geq \frac{1}{2}\frac{e^x}{|\lambda|}.$$

We note that C must be chosen from the inequality $C > \ln\varphi$. Since modules of the exponents included in the function $\mu(\lambda)$ are bounded from above by the next exponent e^x, it is true that

$$|\mu(\lambda)| \leq e^x w(\frac{1}{n})C_3.$$

Hence, it follows that in order to satisfy the condition of Rouche's Theorem [22], it suffices to take ε from bound of the form

$$\varepsilon = \underline{\underline{o}}(nw(\frac{1}{n})).$$

Thus, Theorem 2.2 is completely proved. □

Remark. *One of the features of the considered problem is that the conjugate to (1.1) – (1.2) is the spectral problem for the loaded differential equation:*

$$L_1^* v = v'(t) + \Phi(t)v(1) = \overline{\lambda}v(t),$$

$$v(1) = v(-1).$$

In [23], eigenvalues of a loaded differential operator of the first order with general boundary value conditions on an interval were found, and in the papers [20], [24] and [25] questions on stability of basis properties of the root vectors of a loaded operator of multiple differentiation were studied in the space $L_2(0,1)$.

Acknowledgments. His research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP09260752).

References

[1] E.C. Titchmarsh, The zeros of certain integral function, Proceedings of the London Mathematical Society. **s2-25** (1926) 283 – 302.

[2] M.L. Cartwright, The zeros of certain integral function, The Quarterly Journal of Mathematics. **os-1** (1930) 38 –59.

[3] A.M. Sedletskii, On the zeros of the Fourier transform of finite measure, Mathematical Notes. **53** (1993) 77 – 84.

[4] N.S. Imanbaev, B.E. Kanguzhin, and B.T. Kalimbetov, On zeros the characteristic determinant of the spectral problem for a third-order differential operator on a segment with nonlocal boundary conditions, Advances in Difference Equations. **2013** (2013) 110.

[5] M.A. Naimark, *Linear differential operators*, Moscow: Fizmatlit. (2010).

[6] A.A. Shkalikov, The basis problem of the eigenfunctions of ordinary differential operators with integral boundary conditions, Moscow University Mathematics Bulletin. **37** (1982) 10 – 20.

[7] R. Bellman and K. Cook, *Differential-difference equations*, New York - London: Academic Press. (1963).

[8] O.H. Hald, Discontinuous Inverse eigenvalue problems, Communications on Pure Applied Mathematics. **37** (1984) 539 – 577.

[9] N.S. Imanbaev, Distribution of eigenvalues of a third-order differential operator with strongly regular nonlocal boundary conditions, AIP Conference Proceedings. **1997** (2018) 020027.

[10] B.Ja. Levin, *Distribution of zeros of entire functions*, Providence, R.I.:AMS. (1964).

[11] V.A. Lyubishkin, and Yu. Belabbasi, On regularized sums of root of an entire function of a certain class, Sov. Math. Dokl. **1980** (1980) 613 – 616.

[12] A.F. Leont'ev, *Tselye funktsii. Ryady eksponent (Entire functions. Exponential series)*, Moscow: Nauka (1983).

[13] N.S. Imanbaev, On nonlocal perturbation of the problem on eigenvalues of differentiation operator on a segment, Vestnik Udmurtskogo Universiteta. Math. Mekh. Komp. Nauki. **31** (2021) 186 – 193.

[14] V.B. Sherstyukov, Asymptotic properties of entire functions with given laws of distribution of zeros, Complex Analysis. Entire Functions and Their Applications, Itogi Nauki i Tekhniki. Ser. Sovrem. Math. Pril. Temat.Obz. **161** (2019) 104 – 129.

[15] G.G. Braichev, Sharp estimates of types of entire functions with zeros on rays, Mathematical Notes. **97** (2015) 510 – 520.

[16] K.G. Malyutin, and M.V. Kabanko, The meromorphic functions of completely regular growth on the upper half-plane, Vestnik Udmurtskogo Universiteta. Math., Mekh., Komp.Nauki. **30** (2020) 596 – 409.

[17] R.W. Ibrahim, and I. Aldawish, Difference formula defined by a new differential symmetric operator for a class of meromorphically multivalent functions, Advances in Difference Equations. **2021** (2021) 281.

[18] Kh.K. Ishkin, and R.I. Marvanov, On localization conditions for spectrum of model operator for Orr-Sommerfeld equation, Ufa Mathematical Journal. **12** (2020) 64 – 77.

[19] A.A. Lunyov ,and M.M. Malamud, On the completeness and Riesz basis property of root subspaces of boundary value problems for first-order systems and applications, Journal of Spectral Theory. **5** (2015) 17 – 70.

[20] M.A. Sadybekov and N.S. Imanbaev, Characteristic determinant of a boundary value problem, which does not have the basis property, Eurasian Mathematical Journal. **8** (2017) 40 – 46.

[21] A.M. Sarsenbi, Criteria for the Riesz basis property of systems of eigen-and associated functions for higner-order differential operators on an interval, Dokl. Akad. Nauk. . **77** (2008) 290 – 292.

[22] B.V. Shabat, *An introduction to complex analysis. In 2 parts. Part 1. Functions of one variable*, Moscow: URSS. (2015).

[23] N. Imanbaev, B. Kalimbetov, Zh. Khabibullaev, To the eigenvalue problems of a special-loaded first-order differential operator, International Journal Mathematical Analysis. **45** (2014) 2247 – 2254.

[24] N. Imanbaev, and M. Sadybekov, Stability of basis property of a periodic problem with nonlocal perturbation of boundary conditions, AIP Conference Proceedings. **1759** (2016) 020080.

[25] N.S. Imanbaev and M.A. Sadybekov, Stability of basis property of a periodic problem with nonlocal perturbation of boundary conditions. Ufa Mathematical Journal. **3** (2011) 27 – 32.

Nurlan Imanbaev,
South Kazakhstan State Pedagogical University, A.Baitursynov st. 13, Shymkent, Kazakhstan, Phone: +7 777 9635974 ORCID ID: 0000-0002-5220-9899
Email address: imanbaevnur@mail.ru