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AUTHORS: Dagistan SIMSEK,Burak Ogul

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Solutions of the Maximum of Difference Equations

$$x_{n+1} = \max \left\{ \frac{1}{x_{n-1}}, \frac{y_n}{x_{n-3}} \right\}; y_{n+1} = \max \left\{ \frac{1}{y_{n-1}}, \frac{x_n}{y_{n-3}} \right\}$$

Dağıstan Şimşek

Kyrgyz Turkish Manas University, Bishkek, Kyrgyzstan; Selcuk University, Konya, Turkey
dagistan.simsek@manas.edu.kg ; dsimsek@selcuk.edu.tr

Burak Oğul

Kyrgyz Turkish Manas University, Bishkek, Kyrgyzstan
burak.ogul@manas.edu.kg

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Abstract: The behaviour of the solutions of the following system of difference equations is examined.

$$x_{n+1} = \max \left\{ \frac{1}{x_{n-1}}, \frac{y_n}{x_{n-3}} \right\}; y_{n+1} = \max \left\{ \frac{1}{y_{n-1}}, \frac{x_n}{y_{n-3}} \right\}, \quad (1)$$

where the initial conditions are positive real numbers.

Keywords: Difference Equation, Maximum Operations, Semicycle

$$x_{n+1} = \max \left\{ \frac{1}{x_{n-1}}, \frac{y_n}{x_{n-3}} \right\}; y_{n+1} = \max \left\{ \frac{1}{y_{n-1}}, \frac{x_n}{y_{n-3}} \right\}$$

Maksimumlu Fark Denkleminin Çözümleri

Özet: Aşağıdaki Maksimumlu fark denklemin sisteminin çözümlerinin davranışları incelendi.

$$x_{n+1} = \max \left\{ \frac{1}{x_{n-1}}, \frac{y_n}{x_{n-3}} \right\}; y_{n+1} = \max \left\{ \frac{1}{y_{n-1}}, \frac{x_n}{y_{n-3}} \right\}, \quad (1)$$

burada başlangıç şartları reel sayılardır.

Anahtar Kelimeler: Fark Denklemleri, Maksimum Operatörü, Yarı Dönmeler

INTRODUCTION

Investigations on the recent studies show that researches on the periodic nature of nonlinear difference equations have been an object of great interest. Although difference equations are relatively simple in form, it is, unfortunately, extremely difficult to understand thoroughly the periodic behavior of their solutions. The periodic nature of nonlinear difference equations of the max type has been investigated by many authors (see 1-30).

Definition 1.1. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there is $n_0 \in \{-k, \dots, -1, 0, 1\}$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$, then we say that the sequence $\{x_n\}_{n=-k}^{\infty}$ is periodic with p .

Definition 1.2. Let I be an interval of real numbers and let $f : I^{s+1} \rightarrow I$ be a continuously differentiable function, where s is a non-negative integer. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-s}) \text{ for } n = 0, 1, 2, \dots \quad (2)$$

with the initial values $x_{-s}, \dots, x_0 \in I$. A point $\bar{x}$ called an equilibrium point of Eq.(2) if $\bar{x} = f(\bar{x}, \dots, \bar{x})$.

Definition 1.3. A positive semicycle of a solutions $\{x_n\}_{n=-s}^{\infty}$ of Eq.(2) consists of a string of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all greater than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such as that either $l = -s$ or $l > -s$ ve $x_{l-1} < \bar{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \bar{x}$.

Definition 1.4. A negative semicycle of a solutions $\{x_n\}_{n=-s}^{\infty}$ of Eq.(2) consists of a string of terms $\{x_l, x_{l+1}, \dots, x_m\}$, all less than or equal to equilibrium $\bar{x}$ with $l \geq -s$ and $m \leq \infty$ such that either $l = -s$ or $l > -s$ and $x_{l-1} \geq \bar{x}$ and either $m = \infty$ or $m \leq \infty$ and $x_{m+1} \geq \bar{x}$.

MAIN RESULTS

Let $\bar{x}$ and $\bar{y}$ be the unique positive equilibrium of Eq.(1), then clearly

$$\bar{x} = \max \left\{ \frac{1}{x}, \frac{\bar{y}}{x} \right\}; \bar{y} = \min \left\{ \frac{1}{y}, \frac{\bar{x}}{y} \right\}$$

$$\bar{x} = \frac{1}{x} \Rightarrow \bar{x}^2 = 1 \Rightarrow \bar{x} = 1 \text{ and } \bar{x} = \frac{\bar{y}}{x} \Rightarrow \bar{x}^2 = \bar{y},$$

$$\bar{y} = \frac{1}{y} \Rightarrow \bar{y}^2 = 1 \Rightarrow \bar{y} = 1 \text{ and } \bar{y} = \frac{\bar{x}}{y} \Rightarrow \bar{y}^2 = \bar{x},$$

From which we can obtain $\bar{x} = 1$ and $\bar{y} = 1$.

Lemma 2.1. Assume that, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_{-3} < y_0$,

$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_0 < y_{-1}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_0 < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_{-3} < y_0$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-1} < y_{-2}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1}$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-3} < y_{-2}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_{-1} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-2} < y_{-1}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_{-1} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-3} < y_{-1}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_{-2} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2}$,
 $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_{-2} < y_0$, $1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_0 < y_{-1}$

Then the following statements are true for the solutions of Eq.(1) :

- a) (x_n, y_n) is the solution, solution x_n , for $n \geq 1$ and solution y_n , for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;
- b) (x_n, y_n) is the solution, solution x_n , for $n \geq 1$ and solution y_n , for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two;
- c) (x_n, y_n) is the solution, solution x_n , for $n \geq 1$ and solution y_n , for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. a)

$$x_1 = \max \left\{ \frac{1}{x_{-1}}, \frac{y_0}{x_{-3}} \right\} = \frac{y_0}{x_{-3}} > \bar{x}$$

$$y_1 = \max \left\{ \frac{1}{y_{-3}}, \frac{x_0}{y_{-3}} \right\} = \frac{x_0}{y_{-3}} < \bar{y}$$

$$x_2 = \max \left\{ \frac{1}{x_0}, \frac{y_1}{x_{-2}} \right\} = \max \left\{ \frac{1}{x_0}, \frac{x_0}{y_{-3}x_{-2}} \right\} = \frac{x_0}{y_{-3}x_{-2}} < \bar{x}$$

$$y_2 = \max \left\{ \frac{1}{y_0}, \frac{x_1}{y_{-2}} \right\} = \max \left\{ \frac{1}{y_0}, \frac{y_0}{x_{-3}y_{-2}} \right\} = \frac{y_0}{x_{-3}y_{-2}} < \bar{y}$$

$$\begin{aligned}
 x_3 &= \max \left\{ \frac{1}{x_1}, \frac{y_2}{x_{-1}} \right\} = \max \left\{ \frac{x_{-3}}{y_0}, \frac{y_0}{x_{-3}x_{-1}y_{-2}} \right\} = \frac{x_{-3}}{y_0} > \bar{x} \\
 y_3 &= \max \left\{ \frac{1}{y_1}, \frac{x_2}{y_{-1}} \right\} = \max \left\{ \frac{y_{-3}}{x_0}, \frac{x_0}{y_{-3}y_{-1}x_{-2}} \right\} = \frac{y_{-3}}{x_0} > \bar{y} \\
 x_4 &= \max \left\{ \frac{1}{x_2}, \frac{y_3}{x_0} \right\} = \max \left\{ \frac{y_{-3}x_{-2}}{x_0}, \frac{y_{-3}}{x_0^2} \right\} = \frac{y_{-3}x_{-2}}{x_0} > \bar{x} \\
 y_4 &= \max \left\{ \frac{1}{y_2}, \frac{x_3}{y_0} \right\} = \max \left\{ \frac{x_{-3}y_{-2}}{y_0}, \frac{x_{-3}}{y_0^2} \right\} = \frac{x_{-3}y_{-2}}{y_0} > \bar{y} \\
 x_5 &= \max \left\{ \frac{1}{x_3}, \frac{y_4}{x_1} \right\} = \max \left\{ \frac{y_0}{x_{-3}}, \frac{x_{-3}^2 y_{-2}}{y_0^2} \right\} = \frac{y_0}{x_{-3}} > \bar{x} \\
 y_5 &= \max \left\{ \frac{1}{y_3}, \frac{x_4}{y_1} \right\} = \max \left\{ \frac{x_0}{y_{-3}}, \frac{y_{-3}^2 x_{-2}}{x_0^2} \right\} = \frac{y_{-3}^2 x_{-2}}{x_0^2} > \bar{y} \\
 x_6 &= \max \left\{ \frac{1}{x_4}, \frac{y_5}{x_2} \right\} = \max \left\{ \frac{x_0}{y_{-3}x_{-2}}, \frac{y_{-3}^3 x_{-2}^2}{x_0^3} \right\} = \frac{y_{-3}^3 x_{-2}^2}{x_0^3} > \bar{x} \\
 y_6 &= \max \left\{ \frac{1}{y_4}, \frac{x_5}{y_2} \right\} = \max \left\{ \frac{y_0}{x_{-3}y_{-2}}, y_{-2} \right\} = y_{-2} > \bar{y} \\
 x_7 &= \max \left\{ \frac{1}{x_5}, \frac{y_6}{x_3} \right\} = \max \left\{ \frac{x_{-3}}{y_0}, \frac{y_0 y_{-2}}{x_{-3}} \right\} = \frac{y_0 y_{-2}}{x_{-3}} > \bar{x} \\
 y_7 &= \max \left\{ \frac{1}{y_5}, \frac{x_6}{y_3} \right\} = \max \left\{ \frac{x_0^2}{y_{-3}^2 x_{-2}}, \frac{y_{-3}^2 x_{-2}^2}{x_0^2} \right\} = \frac{y_{-3}^2 x_{-2}^2}{x_0^2} > \bar{y} \\
 x_8 &= \max \left\{ \frac{1}{x_6}, \frac{y_7}{x_4} \right\} = \max \left\{ \frac{x_0^3}{y_{-3}^3 x_{-2}^2}, \frac{y_{-3} x_{-2}}{x_0} \right\} = y_{-1} > \bar{x} \\
 y_8 &= \max \left\{ \frac{1}{y_6}, \frac{x_7}{y_4} \right\} = \max \left\{ \frac{1}{y_{-2}}, \frac{y_0^2}{x_{-3}^2} \right\} = \frac{y_0^2}{x_{-3}^2} > \bar{y} \\
 x_9 &= \max \left\{ \frac{1}{x_7}, \frac{y_8}{x_5} \right\} = \max \left\{ \frac{x_{-3}}{y_0 y_{-2}}, \frac{y_0}{x_{-3}} \right\} = \frac{y_0}{x_{-3}} > \bar{x} \\
 y_9 &= \max \left\{ \frac{1}{y_7}, \frac{x_8}{y_5} \right\} = \max \left\{ \frac{x_0^2}{y_{-3}^2 x_{-2}^2}, \frac{x_0}{y_{-3}} \right\} = \frac{x_0}{y_{-3}} < \bar{y}
 \end{aligned}$$

Consequently, we have obtained,

$$\begin{aligned}
 x_1 &> \bar{x}, \quad x_2 < \bar{x}, \quad x_3 < \bar{x}, \quad x_4 > \bar{x}, \quad x_5 > \bar{x}, \quad x_6 > \bar{x}, \quad x_7 > \bar{x}, \quad x_8 > \bar{x}, \dots \\
 y_1 &< \bar{y}, \quad y_2 < \bar{y}, \quad y_3 > \bar{y}, \quad y_4 > \bar{y}, \quad y_5 > \bar{y}, \quad y_6 > \bar{y}, \quad y_7 > \bar{y}, \quad y_8 > \bar{y}, \dots
 \end{aligned}$$

Hence, the solution x_n for $n \geq 1$ and the solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms. The proof is:

- a) The solution is x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;
- b) Using the proof a);

Therefore, the solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two.

- c) Using the proof a).

Therefore, the solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Lemma 2.2. Assume that,

$$1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < y_{-3}, 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3}, \\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-2} < y_{-3}, 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_0 < y_{-3}, \\ 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3}, 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-1} < y_{-3}$$

Then the following statements are true for the solutions of Eq.(1) :

- a) (x_n, y_n) is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;
- b) (x_n, y_n) is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every positive semicycle of length six is followed by a negative semicycle of length two;
- c) (x_n, y_n) is the solution, solution x_n for $n \geq 1$ and solution y_n for $n \geq 0$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.2 as in the proof of Lemma 2.1.

Lemma 2.3. Assume that,

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-2} < x_{-3}, 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-1} < x_{-3}, \\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-1} < x_{-3}, 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_0 < x_{-3}, \\ 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-2} < x_{-3}, 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_0 < x_{-3}$$

Then the following statements are true for the solutions of Eq.(1) :

- d) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every positive semicycle consists of six terms, every negative semicycle consists of two terms;

- e) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every positive semicycle of length six is followed by a negative semicycle of length two;
- f) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.3 as in the proof of Lemma 2.1.

Lemma 2.4. Assume that,

$$\begin{aligned}
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_{-1} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_{-2} < x_0, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_0 < x_{-2}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-1} < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-2} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-3} < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_{-3} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_0 < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_0 < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_0 < x_{-1}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-2} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-3} < x_{-1}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-3} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-1} < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-3} < x_{-2}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_{-1} < x_0, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_{-3} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_{-2} < x_0
 \end{aligned}$$

Then, the following statements are true for the solutions of Eq.(1) :

- a) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every positive semicycle consists of six terms and every negative semicycle consists of two terms;
- b) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every positive semicycle of length six is followed by a negative semicycle of length two;
- c) (x_n, y_n) is the solution, solution x_n for $n \geq 0$ and solution y_n for $n \geq 1$; every negative semicycle of length two is followed by a positive semicycle of length six.

Proof. Similarly, we can obtain the proof of Lemma 2.4 as in the proof of Lemma 2.1.

Theorem 2.1. Let (x_n, y_n) be the solution of Eq.(1) for

$$\begin{aligned}
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_{-3} < y_0, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_0 < y_{-2}, \\
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_0 < y_{-1}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_0 < y_{-2}, \\
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_{-3} < y_0, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-1} < y_{-2}, \\
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-3} < y_{-1}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-2} < y_{-1}, \\
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-3} < y_{-2}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_{-1} < y_0,
 \end{aligned}$$

$$\begin{aligned}
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_0 < y_{-2} < y_{-1}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-3} < y_{-1} < y_0, \\
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-3} < y_{-1}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-3} < y_{-2} < y_0, \\
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-3} < y_{-1} < y_{-2}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-1} < y_{-2} < y_0, \\
 &1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-3} < y_{-2} < y_0 < y_{-1}
 \end{aligned}$$

Then every (x_n, y_n) is periodic with period eight.

$$\begin{aligned}
 x_n &= \left(\frac{y_0}{x_{-3}}, \frac{x_0}{y_{-3}x_{-2}}, \frac{x_{-3}}{y_0}, \frac{y_{-3}x_{-2}}{x_0}, \frac{y_0}{x_{-3}}, \frac{y_{-3}^3x_{-2}^2}{x_0^3}, \frac{y_0y_{-2}}{x_{-3}}, \frac{y_{-3}x_{-2}}{x_0}, \dots \right) \\
 y_n &= \left(\frac{x_0}{y_{-3}}, \frac{y_0}{x_{-3}y_{-2}}, \frac{y_{-3}}{x_0}, \frac{x_{-3}y_{-2}}{y_0}, \frac{y_{-3}^2x_{-2}}{x_0^2}, y_{-2}, \frac{y_{-3}^2x_{-2}^2}{x_0^2}, \frac{y_0^2}{x_{-3}^2}, \dots \right)
 \end{aligned}$$

Proof.

$$\begin{aligned}
 x_n &= \max \left\{ \frac{1}{x_{n-1}}, \frac{y_n}{x_{n-3}} \right\}; \quad y_n = \max \left\{ \frac{1}{y_{n-1}}, \frac{x_n}{y_{n-3}} \right\} \\
 x_1 &= \max \left\{ \frac{1}{x_{-1}}, \frac{y_0}{x_{-3}} \right\} = \frac{y_0}{x_{-3}} \\
 y_1 &= \max \left\{ \frac{1}{y_{-3}}, \frac{x_0}{y_{-3}} \right\} = \frac{x_0}{y_{-3}} \\
 x_2 &= \max \left\{ \frac{1}{x_0}, \frac{y_1}{x_{-2}} \right\} = \max \left\{ \frac{1}{x_0}, \frac{x_0}{y_{-3}x_{-2}} \right\} = \frac{x_0}{y_{-3}x_{-2}} \\
 y_2 &= \max \left\{ \frac{1}{y_0}, \frac{x_1}{y_{-2}} \right\} = \max \left\{ \frac{1}{y_0}, \frac{y_0}{x_{-3}y_{-2}} \right\} = \frac{y_0}{x_{-3}y_{-2}} \\
 x_3 &= \max \left\{ \frac{1}{x_1}, \frac{y_2}{x_{-1}} \right\} = \max \left\{ \frac{x_{-3}}{y_0}, \frac{y_0}{x_{-3}x_{-1}y_{-2}} \right\} = \frac{x_{-3}}{y_0} \\
 y_3 &= \max \left\{ \frac{1}{y_1}, \frac{x_2}{y_{-1}} \right\} = \max \left\{ \frac{y_{-3}}{x_0}, \frac{x_0}{y_{-3}y_{-1}x_{-2}} \right\} = \frac{y_{-3}}{x_0} \\
 x_4 &= \max \left\{ \frac{1}{x_2}, \frac{y_3}{x_0} \right\} = \max \left\{ \frac{y_{-3}x_{-2}}{x_0}, \frac{y_{-3}}{x_0^2} \right\} = \frac{y_{-3}x_{-2}}{x_0} \\
 y_4 &= \max \left\{ \frac{1}{y_2}, \frac{x_3}{y_0} \right\} = \max \left\{ \frac{x_{-3}y_{-2}}{y_0}, \frac{x_{-3}}{y_0^2} \right\} = \frac{x_{-3}y_{-2}}{y_0} \\
 x_5 &= \max \left\{ \frac{1}{x_3}, \frac{y_4}{x_1} \right\} = \max \left\{ \frac{y_0}{x_{-3}}, \frac{x_{-3}^2y_{-2}}{y_0^2} \right\} = \frac{y_0}{x_{-3}}
 \end{aligned}$$

$$\begin{aligned}
 y_5 &= \max \left\{ \frac{1}{y_3}, \frac{x_4}{y_1} \right\} = \max \left\{ \frac{x_0}{y_{-3}}, \frac{y_{-3}^2 x_{-2}}{x_0^2} \right\} = \frac{y_{-3}^2 x_{-2}}{x_0^2} \\
 x_6 &= \max \left\{ \frac{1}{x_4}, \frac{y_5}{x_2} \right\} = \max \left\{ \frac{x_0}{y_{-3} x_{-2}}, \frac{y_{-3}^3 x_{-2}^2}{x_0^3} \right\} = \frac{y_{-3}^3 x_{-2}^2}{x_0^3} \\
 y_6 &= \max \left\{ \frac{1}{y_4}, \frac{x_5}{y_2} \right\} = \max \left\{ \frac{y_0}{x_{-3} y_{-2}}, y_{-2} \right\} = y_{-2} \\
 x_7 &= \max \left\{ \frac{1}{x_5}, \frac{y_6}{x_2} \right\} = \max \left\{ \frac{x_{-3}}{y_0}, \frac{y_0 y_{-2}}{x_{-3}} \right\} = \frac{y_0 y_{-2}}{x_{-3}} \\
 y_7 &= \max \left\{ \frac{1}{y_5}, \frac{x_6}{y_3} \right\} = \max \left\{ \frac{x_0^2}{y_{-3}^2 x_{-2}}, \frac{y_{-3}^2 x_{-2}^2}{x_0^2} \right\} = \frac{y_{-3}^2 x_{-2}^2}{x_0^2} \\
 x_8 &= \max \left\{ \frac{1}{x_6}, \frac{y_7}{x_4} \right\} = \max \left\{ \frac{x_0^3}{y_{-3}^3 x_{-2}^2}, \frac{y_{-3} x_{-2}}{x_0} \right\} = y_{-1} \\
 y_8 &= \max \left\{ \frac{1}{y_6}, \frac{x_7}{y_4} \right\} = \max \left\{ \frac{1}{y_{-2}}, \frac{y_0^2}{x_{-3}^2} \right\} = \frac{y_0^2}{x_{-3}^2} \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

If $n \geq 0$, then

$$\begin{aligned}
 x_n &= \left(\frac{y_0}{x_{-3}}, \frac{x_0}{y_{-3} x_{-2}}, \frac{x_{-3}}{y_0}, \frac{y_{-3} x_{-2}}{x_0}, \frac{y_0}{x_{-3}}, \frac{y_{-3}^3 x_{-2}^2}{x_0^3}, \frac{y_0 y_{-2}}{x_{-3}}, \frac{y_{-3} x_{-2}}{x_0}, \dots \right) \\
 y_n &= \left(\frac{x_0}{y_{-3}}, \frac{y_0}{x_{-3} y_{-2}}, \frac{y_{-3}}{x_0}, \frac{x_{-3} y_{-2}}{y_0}, \frac{y_{-3}^2 x_{-2}}{x_0^2}, y_{-2}, \frac{y_{-3}^2 x_{-2}^2}{x_0^2}, \frac{y_0^2}{x_{-3}^2}, \dots \right).
 \end{aligned}$$

Theorem 2.2.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$$\begin{aligned}
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_{-1} < y_0 < y_{-3}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-2} < y_{-1} < y_{-3}, \\
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_0 < y_{-2} < y_{-3}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-1} < y_{-2} < y_0 < y_{-3}, \\
 1 &< x_{-3} < x_{-2} < x_{-1} < x_0 < y_0 < y_{-1} < y_{-2} < y_{-3}, \quad 1 < x_{-3} < x_{-2} < x_{-1} < x_0 < y_{-2} < y_0 < y_{-1} < y_{-3}
 \end{aligned}$$

Then every (x_n, y_n) is periodic with period eight.

$$x_n = \left(\frac{y_0}{x_{-1}}, \frac{1}{x_0}, \frac{x_{-3}}{y_0}, x_0, \frac{y_0}{x_{-3}}, y_{-3}x_0, \frac{y_0y_{-2}}{x_{-3}}, x_0, \dots \right)$$

$$y_n = \left(\frac{x_0}{y_{-3}}, \frac{y_0}{x_{-3}y_{-2}}, \frac{y_{-3}}{x_0}, \frac{x_{-3}y_{-2}}{y_0}, y_{-3}, y_{-2}, x_0^2, \frac{y_0^2}{x_{-3}^2}, \dots \right)$$

Proof. Similarly, we can obtain the proof of Theorem 2.2 as in the proof of Theorem 2.1. which completes the proofs of the given theorems.

Theorem 2.3.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-2} < x_{-3}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-1} < x_{-3},$$

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-1} < x_{-3}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_0 < x_{-3},$$

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-2} < x_{-3}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_0 < x_{-3}$$

Then every (x_n, y_n) is periodic with period eight.

$$x_n = \left(\frac{y_0}{x_{-3}}, \frac{x_0}{x_{-2}y_{-3}}, \frac{x_{-3}}{y_0}, \frac{y_{-3}x_{-2}}{x_0}, x_{-3}, x_{-2}, y_0^2, \frac{x_0^2}{y_{-3}^2}, \dots \right)$$

$$y_n = \left(\frac{x_0}{y_{-3}}, \frac{1}{y_0}, \frac{y_{-3}}{x_0}, y_0, \frac{x_0}{y_{-3}}, x_{-3}y_0, \frac{x_0x_{-2}}{y_{-3}}, y_0, \dots \right)$$

Proof. Similarly, we can obtain the proof of Theorem 2.3 as the in the proof of Theorem 2.1. which completes the proofs of the given theorems.

Theorem 2.4.

Let (x_n, y_n) be the solution of the system of difference equations (1) for

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_{-1} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_{-2} < x_0,$$

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-1} < x_0 < x_{-2}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-1} < x_{-2},$$

$$1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-3} < x_{-2} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_0 < x_{-3} < x_{-2},$$

$$\begin{aligned}
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-2} < x_{-3} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_0 < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_0 < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_{-2} < x_0 < x_{-1}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-2} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_0 < x_{-3} < x_{-1}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-2} < x_{-3} < x_{-1}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-3} < x_0 < x_{-1} < x_{-2}, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_0 < x_{-1} < x_{-3} < x_{-2}, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-3} < x_{-1} < x_0, \\
 &1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-2} < x_{-1} < x_{-3} < x_0, \quad 1 < y_{-3} < y_{-2} < y_{-1} < y_0 < x_{-1} < x_{-3} < x_{-2} < x_0
 \end{aligned}$$

Then every (x_n, y_n) is periodic with period eight.

$$\begin{aligned}
 x_n &= \left(\frac{y_0}{x_{-3}}, \frac{x_0}{x_{-2}y_{-3}}, \frac{x_{-3}}{y_0}, \frac{x_{-2}y_{-3}}{x_0}, \frac{y_{-2}x_{-3}^2}{y_0^2}, x_{-2}, \frac{x_{-3}^2 y_{-2}^2}{y_0^2}, \frac{x_0^2}{y_{-3}^2}, \dots \right) \\
 y_n &= \left(\frac{x_0}{y_{-3}}, \frac{y_0}{y_{-2}x_{-3}}, \frac{y_{-3}}{x_0}, \frac{y_{-2}x_{-3}}{y_0}, \frac{x_0}{y_{-3}}, \frac{y_{-2}^2 x_{-3}^3}{y_0^3}, \frac{x_0 x_{-2}}{y_{-3}}, \frac{x_{-3} y_{-2}}{y_0}, \dots \right)
 \end{aligned}$$

Proof. Similarly, we can obtain the proof of Theorem 2.4 as in the proof of Theorem 2.1. which completes the proofs of the shown theorems.

EXAMPLES

EXAMPLE 3.1: If the initial conditions are selected as follows:

$$x[-3]=2; x[-2]=3; x[-1]=4; x[0]=5; y[-3]=8; y[-2]=7; y[-1]=6; y[0]=9;$$

then the following solutions are obtained:

$$x(n) = \{ 4.5, 0.208333, 0.222222, 4.8, 4.5, 36.864, 31.5, 4.8, 4.5, 0.208333, 0.222222, 4.8, 4.5, 36.864, 31.5, 4.8, \dots \}$$

$$y(n) = \{ 0.625, 0.642857, 1.6, 1.55556, 7.68, 7., 23.04, 20.25, 0.625, 0.642857, 1.6, 1.55556, 7.68, 7., 23.04, 20.25, \dots \}$$

The graph of the solutions is given below.

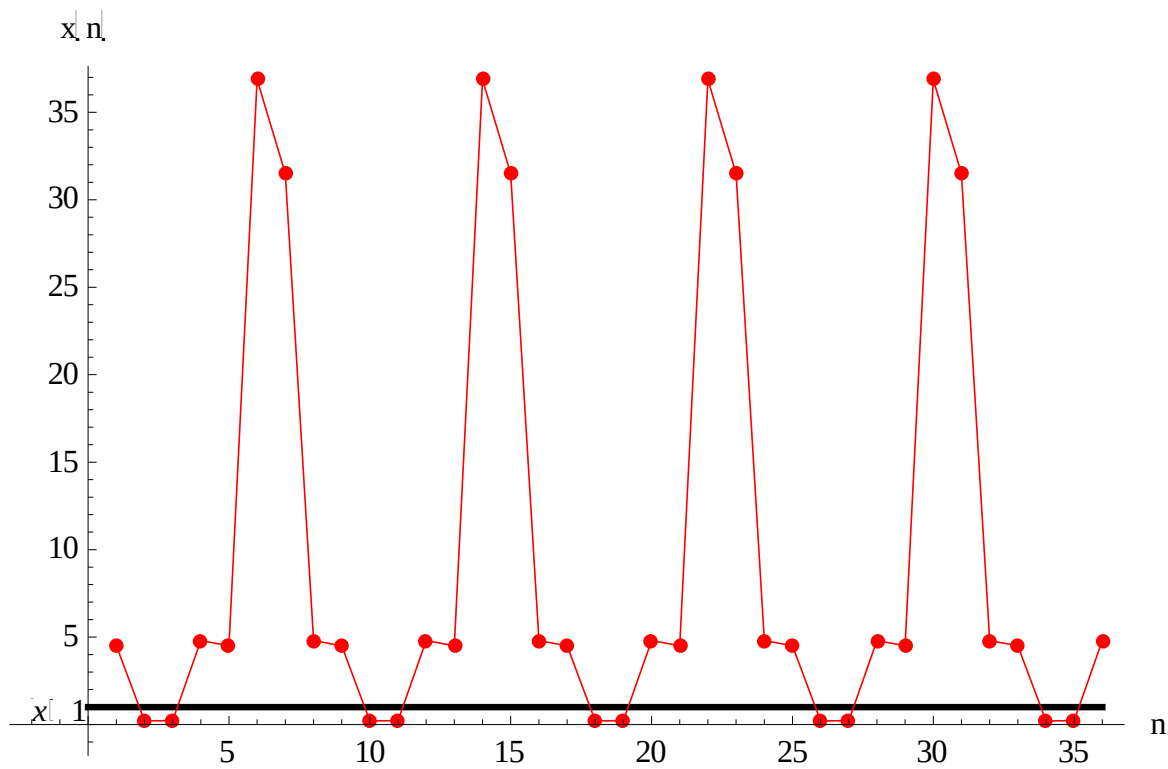


Figure 3.1. $x(n)$ graph of the solutions

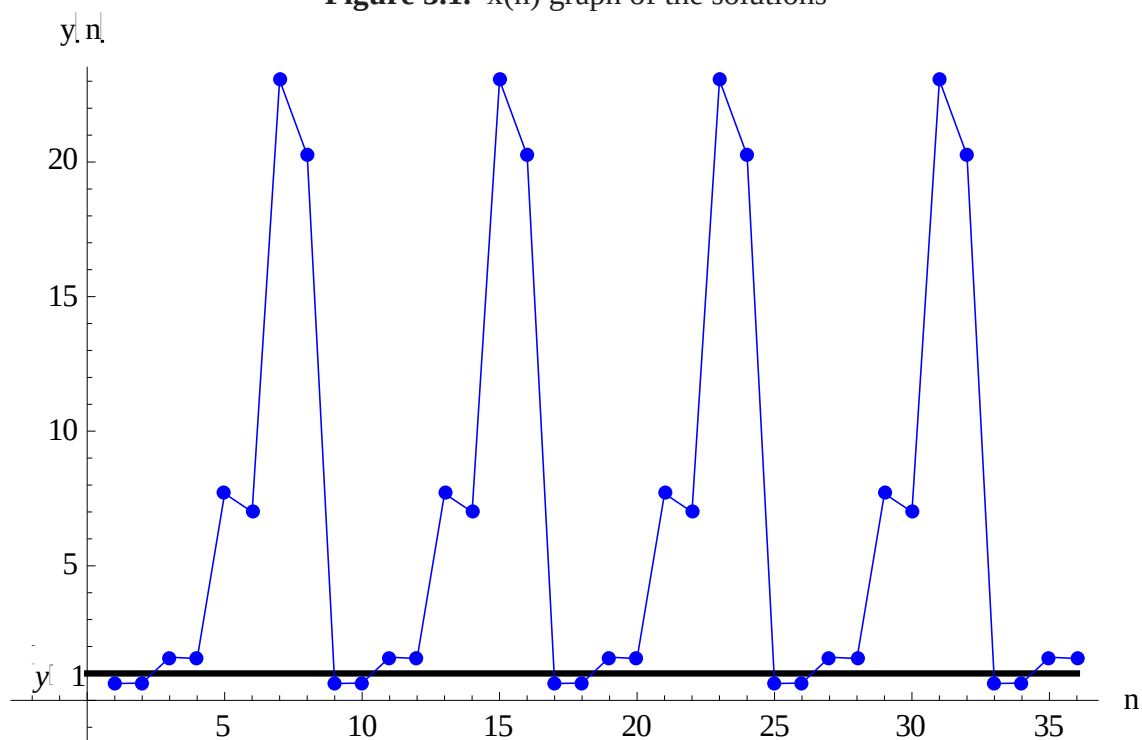


Figure 3.2. $y(n)$ graph of the solutions

EXAMPLE 3.2: If the initial conditions are selected as follows:

$$x[-3]=9; x[-2]=8; x[-1]=7; x[0]=6; y[-3]=2; y[-2]=3; y[-1]=4; y[0]=5;$$

then the following solutions are obtained:

$$x(n) = \{ 0.555556, 0.375, 1.8, 2.66667, 9, 8, 25, 9, 0.555556, 0.375, 1.8, 2.66667, 9, 8, 25, 9, \dots \}$$

$$y(n) = \{ 3, 0.2, 0.333333, 5, 3, 45, 24, 5, 3, 0.2, 0.333333, 5, 3, 45, 24, 5, \dots \}$$

The graph of the solutions is given below.

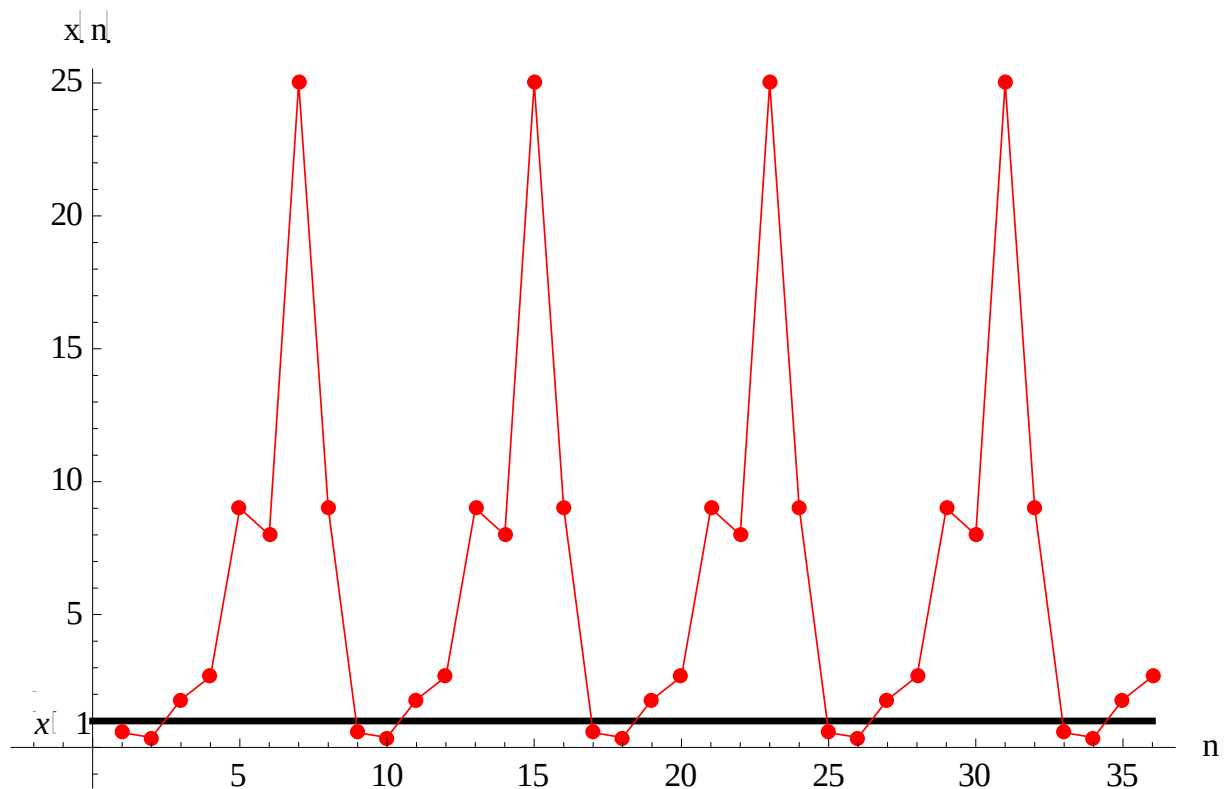


Figure 3.3. $x(n)$ graph of the solutions

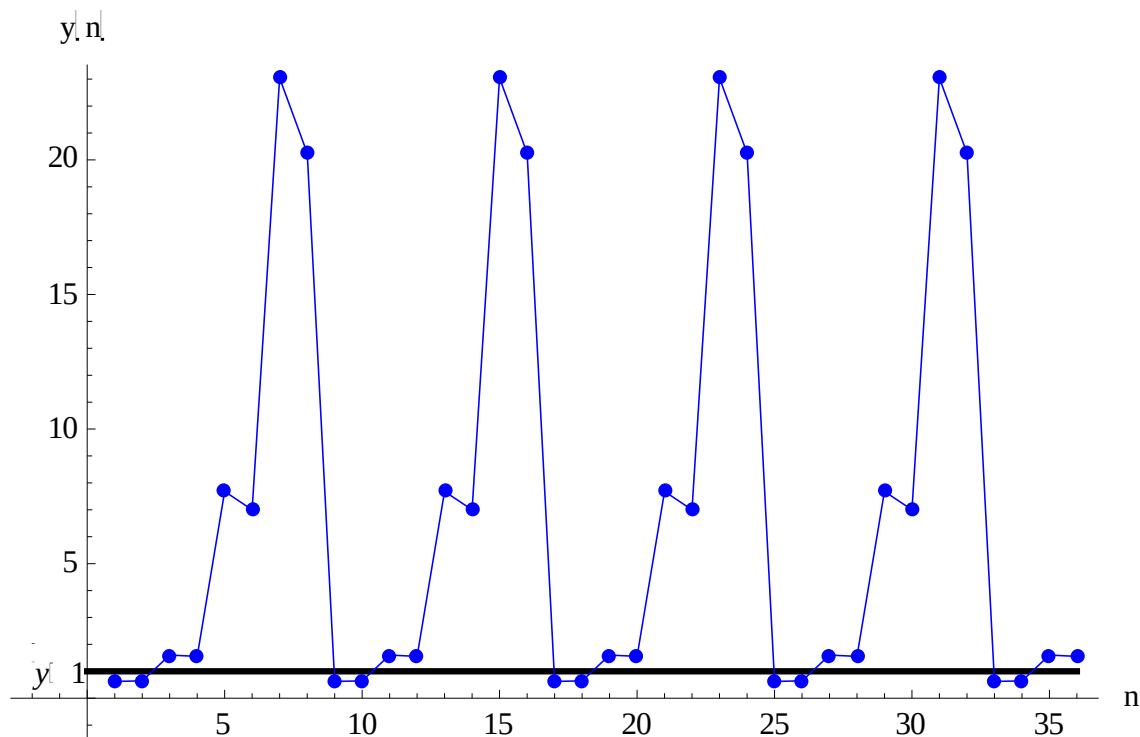


Figure 3.4. $y(n)$ graph of the solutions

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