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TITLE: Solutions Of The Rational Difference Equations $X(n+1)=x(n-3)/(1+x(n)x(n-1)x(n-2))$

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Solutions of The Rational Difference Equations

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}$$

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Abstract: In this paper the solutions of the following difference equation is examined,

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\dots \quad (1)$$

where the initial conditions are positive real numbers.

Keywords: Difference Equation, Period Four Solution

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}$$

Rasyonel Fark Denkleminin Çözümleri

Öz: Bu çalışmada aşağıdaki fark denkleminin çözümleri incelenmiştir,

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\dots \quad (1)$$

Burada başlangıç şartları pozitif reel sayılardır.

Anahtar Kelimeler: Fark Denklemi, Dört Periyotlu Çözüm

INTRODUCTION

Recently there has been a lot of interest in studying the periodic nature of non-linear difference equations. For some recent results concerning among other problems, the periodic nature of scalar nonlinear difference equations see, [1-25].

Cinar, studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}$$

$$x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}$$

for $n=0,1,\dots$, in [2,3,4], respectively.

In [18] Stevic assumed that $\beta=1$ and solved the following problem

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n} \text{ for } n = 0,1,2, \dots$$

where $x_{-1}, x_0 \in (0, \infty)$. Also, this results was generalized to the equation of the following form:

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)} \text{ for } n = 0,1,2, \dots$$

where $x_{-1}, x_0 \in (0, \infty)$.

Simsek et. al., studied the following problems with positive initial values

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}}$$

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}}$$

for $n=0,1,\dots$, in [19,20,21] respectively.

In this paper we investigated the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_n x_{n-1} x_{n-2}}, \quad n=0,1,2,\dots \quad (1)$$

where $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

MAIN RESULT

Let $\bar{x}$ be the unique positive equilibrium of Eq. (1), then clearly

$$\bar{x} = \frac{\bar{x}}{1 + \bar{x}\bar{x}} \Rightarrow \bar{x} + \bar{x}^{-4} = \bar{x} \Rightarrow \bar{x}^{-4} = 0 \Rightarrow \bar{x} = 0$$

We can obtain $\bar{x} = 0$.

Theorem 1. Consider the difference equation (1). Then the following statements are true.

- a) The sequences (x_{4n-3}) , (x_{4n-2}) , (x_{4n-1}) , and (x_{4n}) are decreasing and there exist $p, q, r, s \geq 0$ such that

$$\lim_{n \rightarrow \infty} x_{4n-3} = p, \lim_{n \rightarrow \infty} x_{4n-2} = q, \lim_{n \rightarrow \infty} x_{4n-1} = r \text{ and } \lim_{n \rightarrow \infty} x_{4n} = s.$$

- b) $(p, q, r, s, p, q, r, s, \dots)$ is a solution of equation (1) of period four.

- c) $p, q, r, s = 0$.

- d) If there exist $n_0 \in \mathbb{N}$ such that $x_n x_{n-1} x_{n-2} \geq x_{n+1} x_n x_{n-1}$ for all $n \geq n_0$, then

$$\lim_{n \rightarrow \infty} x_n = 0.$$

- e) The following formulas hold:

$$\begin{aligned} x_{4n+1} &= x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ x_{4n+2} &= x_{-2} \left(1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ x_{4n+3} &= x_{-1} \left(1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ x_{4n+4} &= x_0 \left(1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right). \end{aligned}$$

- f) If $x_{4n+1} \rightarrow p \neq 0$, $x_{4n+2} \rightarrow q \neq 0$ and $x_{4n+3} \rightarrow r \neq 0$ then $x_{4n+1} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. a) Firstly, we consider the equation (1). From this equation we obtain

$$x_{n+1}(1 + x_n x_{n-1} x_{n-2}) = x_{n-3}.$$

If $x_n, x_{n-1}, x_{n-2} \in (0, +\infty)$, then $(1 + x_n x_{n-1} x_{n-2}) \in (1, +\infty)$. Since $x_{n+1} < x_{n-3}$, $n \in \mathbb{N}$, we obtain that

$$\lim_{n \rightarrow \infty} x_{4n-3} = p, \lim_{n \rightarrow \infty} x_{4n-2} = q, \lim_{n \rightarrow \infty} x_{4n-1} = r \text{ and } \lim_{n \rightarrow \infty} x_{4n} = s.$$

- b) $(p, q, r, s, p, q, r, s, \dots)$ is a solution of equation (1) of period four.

- c) In view of the equation (1), we obtain

$$x_{4n+1} = \frac{x_{4n-3}}{1 + x_{4n} x_{4n-1} x_{4n-2}}.$$

Taking limit as $n \rightarrow \infty$ on both sides of the above equality, we get

$$\lim_{n \rightarrow \infty} x_{4n+1} = \lim_{n \rightarrow \infty} \frac{x_{4n-3}}{1 + x_{4n} x_{4n-1} x_{4n-2}}.$$

Then

$$p = \frac{p}{1 + s, r, q} \Rightarrow p + p, q, r, s = p \Rightarrow p, q, r, s = 0.$$

d) If there exist $n_0 \in \mathbb{N}$ such that $x_n x_{n-1} x_{n-2} \geq x_{n+1} x_n x_{n-1}$ for all $n \geq n_0$, then $p \leq q \leq r \leq s \leq p$. Since $p.q.r.s = 0$ we obtain the result.

e) Subtracting x_{n-3} from the left and right-hand sides of equation (1) we obtain

$$x_{n+1} - x_{n-3} = \frac{1}{1 + x_n x_{n-1} x_{n-2}} (x_n - x_{n-4})$$

and the following formula

$$n \geq 1 \text{ for } \left\{ x_n - x_{n-4} = (x_1 - x_{-3}) \prod_{i=1}^{n-1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right. \quad (2)$$

holds. Replacing n by $4j$ in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+1} - x_{-3} = (x_1 - x_{-3}) \sum_{j=0}^n \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \dots). \quad (3)$$

Also, replacing n by $4j+1$ in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+2} - x_{-2} = (x_1 - x_{-3}) \sum_{j=0}^n \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \dots). \quad (4)$$

Also, replacing n by $4j+2$ in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+3} - x_{-1} = (x_1 - x_{-3}) \sum_{j=0}^n \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \dots). \quad (5)$$

Also, replacing n by $4j+3$ in (2) and summing from $j = 0$ to $j = n$ we obtain

$$x_{4n+4} - x_0 = (x_1 - x_{-3}) \sum_{j=0}^n \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (n = 0, 1, 2, \dots). \quad (6)$$

From the formulas above, we obtain

$$x_{4n+1} = x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (7)$$

$$x_{4n+2} = x_{-2} \left(1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (8)$$

$$x_{4n+3} = x_{-1} \left(1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \quad (9)$$

$$x_{4n+4} = x_0 \left(1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right). \quad (10)$$

f) Suppose that $p = q = r = s = 0$. By **e)** we have

$$\begin{aligned} \lim_{n \rightarrow \infty} x_{4n+1} &= \lim_{n \rightarrow \infty} x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^n \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ p &= x_{-3} \left(1 - \frac{x_0 x_{-1} x_{-2}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ p = 0 &\Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \end{aligned} \quad (11)$$

Similarly,

$$\begin{aligned} q &= x_{-2} \left(1 - \frac{x_0 x_{-1} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ q = 0 &\Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \end{aligned} \quad (12)$$

Similarly,

$$\begin{aligned} r &= x_{-1} \left(1 - \frac{x_0 x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ r = 0 &\Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \end{aligned} \quad (13)$$

Similarly,

$$\begin{aligned} s &= x_0 \left(1 - \frac{x_{-1} x_{-2} x_{-3}}{1 + x_0 x_{-1} x_{-2}} \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \right) \\ s = 0 &\Rightarrow \frac{1 + x_0 x_{-1} x_{-2}}{x_{-1} x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i}. \end{aligned} \quad (14)$$

From the equations (11) and (12),

$$\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-2}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j} \frac{1}{1 + x_{i-2} x_{i-1} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (15)$$

thus, $x_{-3} > x_{-2}$.

From the equations (12) and (13),

$$\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-1} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+1} \frac{1}{1 + x_{i-2} x_{i-1} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (16)$$

thus, $x_{-2} > x_{-1}$.

From the equations (13) and (14),

$$\frac{1 + x_0 x_{-1} x_{-2}}{x_0 x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+2} \frac{1}{1 + x_{i-2} x_{i-1} x_i} > \frac{1 + x_0 x_{-1} x_{-2}}{x_{-1} x_{-2} x_{-3}} = \sum_{j=0}^{\infty} \prod_{i=1}^{4j+3} \frac{1}{1 + x_{i-2} x_{i-1} x_i} \quad (17)$$

thus, $x_{-1} > x_0$.

From here we obtain $x_{-3} > x_{-2} > x_{-1} > x_0$. We arrive at a contradiction which completes the proof of theorem.

EXAMPLES

Example 1: If the initial conditions are selected as follows:

$$x[-3]=2; x[-2]=3; x[-1]=4; x[0]=5;$$

The following solutions are obtained:

$$x(n)=\{ 0.0327869, 1.81188, 3.08397, 4.22581, 0.0013321, 1.78096, 3.05336, 4.19542, 0.000055937, 1.77969, 3.05208, 4.19414, 2.35212 \times 10^{-6}, 1.77963, 3.05203, 4.19409, 9.89108 \times 10^{-8}, 1.77963, 3.05203, 4.19409, 4.15939 \times 10^{-9}, 1.77963, 3.05203, 4.19409, 1.7491 \times 10^{-10}, 1.77963, 3.05203, 4.19409, 7.35532 \times 10^{-12}, 1.77963, 3.05203, 4.19409, 3.09306 \times 10^{-13}, 1.77963, 3.05203, 4.19409, \dots \}$$

The graph of the solutions is given below.

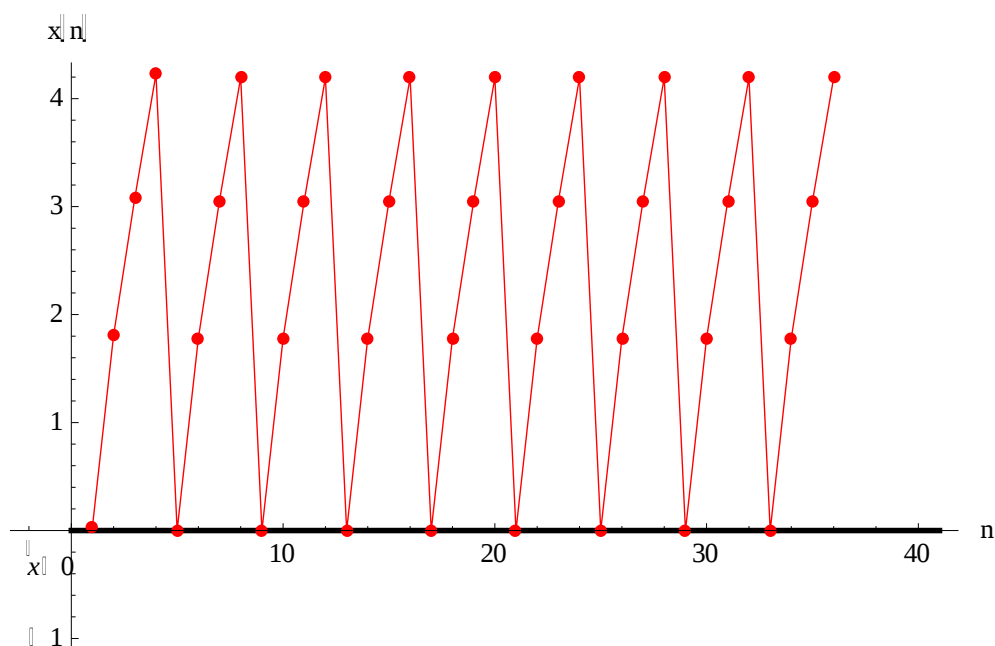


Figure 3.1. $x(n)$ graph of the solutions

Example 2: If the initial conditions are selected as follows:

$$x[-3]=5; x[-2]=4; x[-1]=3; x[0]=2;$$

The following solutions are obtained:

$x(n)=\{0.2, 1.81818, 1.73684, 1.22581, 0.0410596, 1.67202, 1.60202, 1.10435, 0.0103735, 1.64189, 1.57245, 1.07554, 0.00274663, 1.63429, 1.56489, 1.06804, 0.000736065, 1.63229, 1.56289, 1.06604, 0.000197891, 1.63175, 1.56235, 1.0655, 0.0000532489, 1.6316, 1.5622, 1.06536, 0.0000143316, 1.63156, 1.56217, 1.06532, 3.85751 \times 10^{-6}, 1.63155, 1.56216, 1.06531, \dots\}$

The graph of the solutions is given below.

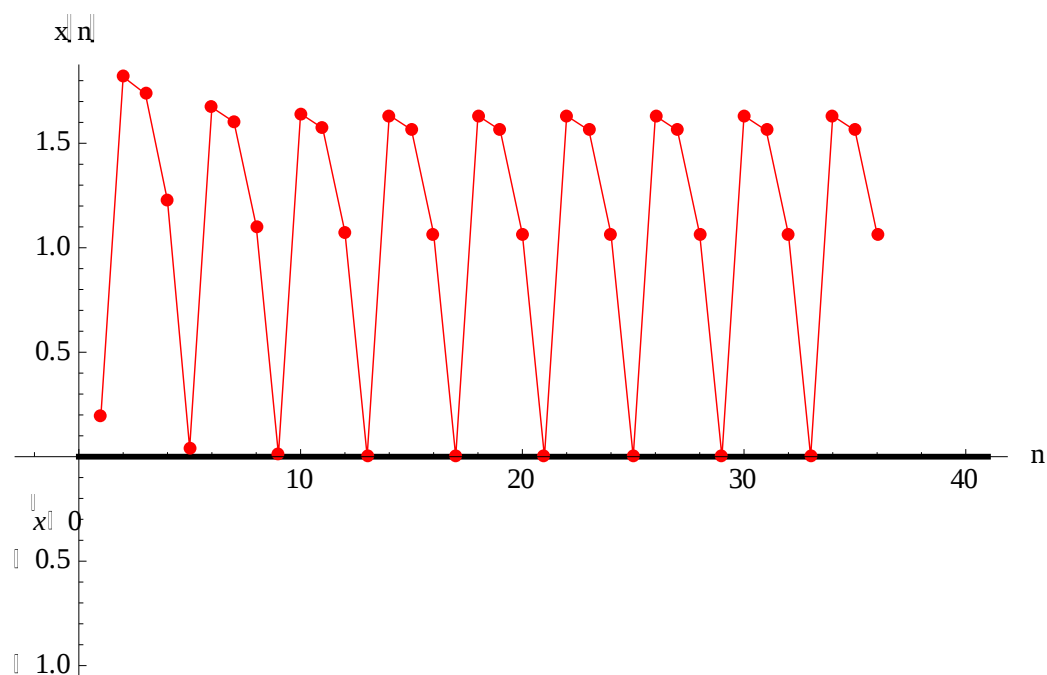


Figure 3.2. $x(n)$ graph of the solutions

Example 3: If the initial conditions are selected as follows:

$$x[-3]=2; x[-2]=0.1; x[-1]=0.01; x[0]=0.001;$$

The following solutions are obtained:

$x(n)=\{2, 0.099998, 0.009998, 0.000998004, 2, 0.099996, 0.00999601, 0.000996013, 1.99999, 0.099994, 0.00999401, 0.000994027, 1.99999, 0.099992, 0.00999203, 0.000992044, 1.99999, 0.09999, 0.00999005, 0.000990066, 1.99999, 0.0999881, 0.00998807, 0.000988093, 1.99999, 0.0999861, 0.0099861, 0.000986123, 1.99998, 0.0999841, 0.00998413, 0.000984159, 1.99998, 0.0999822, 0.00998216, 0.000982198, \dots\}$

The graph of the solutions is given below.

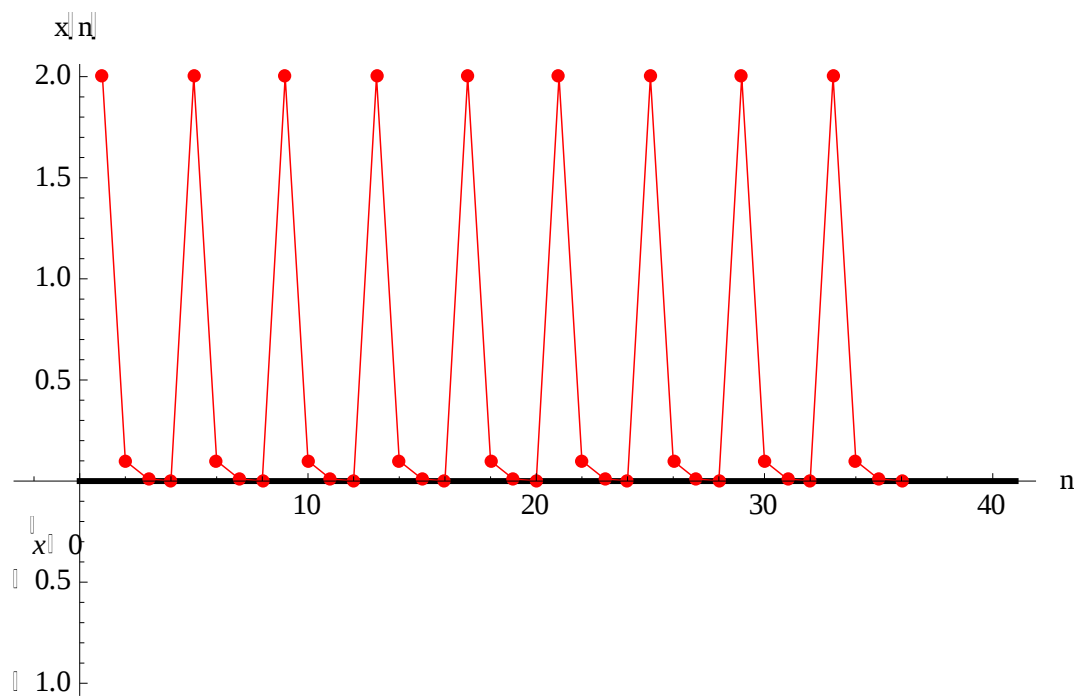


Figure 3.3. $x(n)$ graph of the solutions

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