OPTIMUM CONTROL IN THE PROBLEM OF MINIMIZATION OF HARMFUL IMPURITIES IN THE ATMOSPHERE BY PONTRYAGIN'S MAXIMUM PRINCIPLE AND SPHERICAL HARMONICS METHOD

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Protection of the environment from the industrial pollution is one of the most actual problems of modern science and engineering. This paper is devoted to the investigation of the problem, related to the disposition of industrial objects, which provides the minimal pollution of nearby economically important objects. It is supposed that all of the industrial objects in the given region throw out respective quantities of the harmful impurity in the atmosphere. The problem consist of the determining for each of the industrial object of such admissible amount of harmful impurity, which provide the minimum for the integral of their squares. At the same time, the density of the harmful particles must be as much as possible close, on the average, to the sanitary allowable norms.

1. Statement of the problem

Consider the area G of *n*–dimensional space R^n with a border Γ, which has a form of cylinder with bases Γ_0, Γ_H and lateral surface Γ_l. We assume that *r* industrial objects are located in the points $x^i = (x_1^i, x_2^i, ..., x_n^i)$, $i = 1, 2, \ldots, r$ of G, and throw out $p_i(t)$, $(i = 1, 2, ..., r)$ harmful impurities in the atmosphere. As a result, we come to the following problem setting [1].

It is given the integro-differential equation of the pollution matter diffusion of the *r* industrial objects,

$$\begin{aligned} &\frac{\partial\psi}{\partial t} + \vec{v}\,grad\,\psi + \sigma\psi(t,x,\vec{v}) - \eta\sum_{i=1}^{n-1}\frac{\partial^2\psi}{\partial x_i^2} - \xi\frac{\partial^2\psi}{\partial x_n^2} = \\ &= \sum_{i=1}^{r} p_i(t)\delta(x - x^i)\delta(\vec{v} - \vec{v}^i) + \frac{\lambda}{m(\Omega)}\int_\Omega \Theta(t,x,\vec{v},\vec{v}')\psi(t,x,\vec{v}')d\Omega'. \end{aligned} \tag{1.1}$$

Here, $\psi(t,x,\vec{v})$ is a concentration of the impurity particles located in the point $x = (x_1, x_2, ..., x_n)$ at the moment t and having a velocity $\vec{v} = (v_1, v_2, ..., v_n)$,

$grad\psi = (\frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial x_2}, ..., \frac{\partial\psi}{\partial x_n})$ is a vector-gradient, $\vec{v} = (v_1, v_2, ..., v_n) \in \Omega -$ is a velocity vector, satisfying to the continuity condition $div(\vec{v}) = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0$, and $v_n = 0$ at $x_n = 0$ and $x_n = H$, that is on Γ_0 and Γ_H (Γ_0 and Γ_H are the bases of the n – dimensional cylinder G), Ω is a sphere of unit radius in R_n described by the equation $\sum_{i=1}^{n} v_i^2 = 1$, σ, λ are the positive constants describing the medium $G \subset R^n$ where harmful impurities diffuse, η, ξ are the coefficients of a "horizontal" and "vertical" turbulent exchange. $x = (x_1, x_2, ..., x_n)$ is a spatial point of the area G, $\Theta(t, x, \bar{v}, \bar{v}')$ is the function describing dispersion of the harmful impurity particles, $\delta(x - x_i)$, $\delta(\bar{v} - \bar{v}_i)$ are the Dirak's δ-functions, $m(\Omega)$ is the area of the surface of a unit sphere Ω in R^n [2]:

$$m(\Omega) = \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}, \ \Gamma(\xi) = \int_0^\infty e^{-t} t^{\xi-1} dt .$$

The non-stationary integro-differential equation (1.1) must be supplemented with the boundary conditions

$$\psi(t, x, \vec{v})\big|_{t\le 0} = \psi_0(x, \vec{v}), \qquad (\frac{\partial\psi}{\partial x_n} - \alpha\psi)\big|_{\Gamma_0\times\Omega} = 0, \tag{1.2}$$

$$\frac{\partial\psi}{\partial x_n}\big|_{\Gamma_H\times\Omega} = 0, \quad \psi(t, x, \vec{v})\big|_{\Gamma_\delta x\Omega} = 0 \quad at \quad (\vec{v}, \vec{n}) < 0, \tag{1.3}$$

where $\vec{n}$ is a normal unit vector to the external side of surface Γ of the cylinder G.

Factor α in the condition (1.2), in the case of three-dimensional space R^3, characterizes a probability of the substances, laid-down to the ground surface, to get back into the atmosphere. Condition (1.3), in the case of $n = 3$, means that the particles which leave the domain G, do not return back into the this area.

The problem is to find such functions $p_i(t)$, $(i = 1,2,...,r)$, on which the functional

$$J[p] = \sum_{i=1}^{r} \beta_i \int_0^T p_i^{\,2}(t) dt + \int_G dG \int_\Omega [\psi(T, x, \vec{v}) - \psi_i(x, \vec{v})]^2 d\Omega \tag{1.4}$$

reach the least possible value. Here $\psi(t,x,\vec{v})$ is the solution of the problem (1.1) - (1.3), T> 0 is defined, $\psi_1(x,\vec{v})$ is the known function from $W_2^{1,0}[G\times\Omega]$, $\beta_i = const > 0, \quad (i = 1,2,...,r)\cdot$

Admissible controls are the various functions $p = (p_1, p_2, ..., p_r)$ from $L_r^2[0,T]$. The control $p = (p_1, p_2, ..., p_r)$, which gives the solution of the considered problem, will be called the optimal and denoted by $p^0 = (p_1^0, p_2^0, ..., p_r^0)$.

2. Optimality Conditions

To determine the optimality conditions, we give some admissible increment Δ p = (Δ $_{p1}$, Δ $_{p2}$, …, Δ p^r) of the control p and denote by $\Delta\psi$ the corresponding increment of the function $\psi(t,x,v)$. It is obvious that the function $\Delta\psi(t,x,v)$ is the solution of the boundary-value problem [2]

$$\begin{aligned}&\frac{\partial\Delta\psi}{\partial t}+\vec{v}grad\Delta\psi+\sigma\Delta\psi(t,x,\vec{v})-\eta\nabla^2\Delta\psi-\xi\frac{\partial^2\Delta\psi}{\partial z^2}=\\&=\sum_{i=1}^{r}\Delta p_i(t)\delta(x-x_i)\delta(\vec{v}-\vec{v}_i)+\frac{\lambda}{m(\Omega)}\int_{\Omega}\Theta(t,x,\vec{v},\vec{v}')\Delta\psi(t,x,\vec{v})d\Omega.\end{aligned} \tag{2.1}$$

$$\Delta\psi\big|_{t\le 0}=0,\quad \left(\frac{\partial\Delta\psi}{\partial x_n}-\alpha\Delta\psi\right)\Big|_{\Gamma_0\times\Omega}=0,$$

$$\Delta\psi\big|_{\Gamma\delta\times\Omega}=0\quad at\quad (\vec{v},\vec{n})<0,\quad \frac{\partial\Delta\psi}{\partial x_n}\Big|_{\Gamma_H\times\Omega}=0 \tag{2.2}$$

By the direct calculations we find that the functional $J[p]$ (see (1.4)) has the increment

$$\begin{aligned}&\Delta J[p]=\sum_{i=1}^{r}\beta_i\left[\int_0^T 2p_i(t)\Delta p_i(t)dt+\int_0^T[\Delta p_i(t)]^2dt\right]+\\&+2\int_G dG\int_\Omega[\psi(T,x,\vec{v})-\psi_i(x,\vec{v})]\Delta\psi(T,x,\vec{v})d\Omega+\int_G dG\int_\Omega[\Delta\psi(T,x,\vec{v})]^2d\Omega..\end{aligned} \tag{2.3}$$

Let's consider the arbitrary function $\Phi(t, x, v) \in W_2^{0,1,0}$. Then, obviously that the next equality takes a place,

$$\int_0^T dt \int_G dG \int_\Omega \Phi(t,x,\vec{v}) \left\{ \frac{\partial \psi}{\partial t} + \vec{v} grad \psi + \sigma \psi - \eta \sum_{i=1}^{n-1} \frac{\partial^2 \psi}{\partial x_i^2} - \xi \frac{\partial^2 \psi}{\partial x_n^2} - \right.$$
$$\left. - \sum_{i=1}^{r} p_i(t) \delta(x - x^i) \delta(\vec{v} - \vec{v}^i) - \frac{\lambda}{m(\Omega)} \int_\Omega \Theta(t,x,\vec{v},\vec{v}') \psi(t,x,\vec{v}') d\Omega' \right\} d\Omega = 0.$$

Denoting the left hand side of this equality by $A[\Phi, p]$, we obtain

$$\Delta A[\Phi, p] = \int_0^T dt \int_G dG \int_\Omega \Phi(t,x,\vec{v}) \left\{ \frac{\partial \psi}{\partial t} + \vec{v} grad \psi + \sigma \psi - \eta \sum_{i=1}^{n-1} \frac{\partial^2 \psi}{\partial x_i^2} - \xi \frac{\partial^2 \psi}{\partial x_n^2} - \right.$$
$$\left. - \sum_{i=1}^{r} p_i(t) \delta(x - x^i) \delta(\vec{v} - \vec{v}^i) - \frac{\lambda}{m(\Omega)} \int_\Omega \Theta(t,x,\vec{v},\vec{v}') \psi(t,x,\vec{v}') d\Omega' \right\} d\Omega = 0. \tag{2.4}$$

Integrating by parts, we transform the equality (2.4) to the form of

$$\int_g dG \int_\Omega \Delta\psi(T,x,\vec{v}) \Phi(T,x,\vec{v}) d\Omega + \int_0^T dt \int_G dG \int_\Omega \Delta\psi(t,x,\vec{v}) \times$$
$$\times \left\{ -\frac{\partial \Phi}{\partial t} - \vec{v} grad \Phi + \sigma \Phi - \eta \sum_{i=1}^{n-1} \frac{\partial^2 \Phi}{\partial x_i^2} - \xi \frac{\partial^2 \Phi}{\partial x_n^2} - \frac{\lambda}{m(\Omega)} \int_\Omega \Theta(t,x,\vec{v},\vec{v}') \Phi(t,x,\vec{v}') d\Omega' \right\} \times$$
$$\times d\Omega - \sum_{i=1}^{r} \int_0^T \Delta p_i(t) \Phi(t,x_i,\vec{v}_i) dt + \int_0^T dt \int_\Gamma d\Gamma \int_\Omega \left[\vec{v}_{\vec{n}} \Delta\psi \Phi - \eta \left(\Phi \frac{\partial \Delta\psi}{\partial \vec{n}} - \Delta\psi \frac{\partial \Phi}{\partial \vec{n}} \right) \right] d\Omega + \tag{2.5}$$
$$+ \int_0^T dt \int_\Omega d\Omega \left[\int_{\Gamma_0} \xi \left(\Phi \frac{\partial \Delta\psi}{\partial x_n^2} - \Delta\psi \frac{\partial \Phi}{\partial x_n^2} \right) - \int_{\Gamma_H} \xi \left(\Phi \frac{\partial \Delta\psi}{\partial x_n^2} - \Delta\psi \frac{\partial \Phi}{\partial x_n^2} \right) \right] d\Gamma = 0.$$

Here $\vec{v}_{\vec{n}}$ is a projection of the vector $\vec{v}$ to the unit vector $\vec{n}$.

Up to now $\Phi(t, x, \bar{v})$ was the arbitrary function from $W_2^{0,1,0}([0,T] \times G \times \Omega)$. Let's define it now as a generalized solution of the boundary-value problem

$$
\begin{gathered}
\frac{\partial \Phi}{\partial t} + \vec{v}\, grad \Phi - \sigma \Phi(t, x, \vec{\upsilon}) + \eta \sum_{i=1}^{n-1} \frac{\partial^2 \Phi}{\partial x_i^2} + \xi \frac{\partial^2 \Phi}{\partial x_n^2} + \\
+ \frac{\lambda}{m(\Omega)} \int_{\Omega} \Theta(t, x, \vec{\upsilon}', \vec{\upsilon}) \Phi(t, x, \vec{\upsilon}') d\Omega' = 0
\end{gathered} \tag{2.6}
$$

$$
\left(\frac{\partial \Phi}{\partial x_n} - \alpha \Phi \right)\Bigg|_{\Gamma_0 \times \Omega} = 0, \quad \frac{\partial \Phi}{\partial x_n}\Bigg|_{\Gamma_H \times \Omega} = 0, \quad \Phi\Big|_{\Gamma_0 \times \Omega} = 0 \quad at \quad (\vec{\upsilon}, \vec{n}) \ge 0
$$

$$
\Phi(T, x, \vec{\upsilon}) = -2\left[\psi(T, x, \vec{\upsilon}) - \psi_1(x, \vec{\upsilon})\right]. \tag{2.7}
$$

Taking into account (2.1), (2.2), (2.6), and (2.7), the equality (2.5) can be simplified. Namely, the second term at the left side in (2.5) vanishes due to (2.6). Because of the third equality from conditions (2.2) and the second of the conditions (2.7), and since $\cos(\vec{n}, x_1) = \cos(\vec{n}, x_2) = ... = \cos(\vec{n}, x_{n-1}) = 0$ and $\vec{v}_{\vec{n}} = 0$ on Γ_0 and Γ_H,

$$
\int_0^T dt \int_{\Gamma} d\Gamma \int_{\Omega} \vec{\upsilon}_n \Delta \psi \Phi d\Omega = 0 \cdot
$$

By virtue of the second condition in (2.2) and first of the conditions in (2.7), we have

$$
\int_0^T dt \int_{\Gamma_O} d\Gamma \int_{\Omega} \nu \left(\Phi \frac{\partial \Delta \psi}{\partial x_n} - \Delta \psi \frac{\partial \Phi}{\partial x_n} \right) d\Omega = 0 \cdot
$$

In view of the last condition from (2.2) and a penultimate condition from (2.7), the equation (2.5) takes the form of

$$
2 \int_G dG \int_{\Omega} \Delta \psi(T, x, \vec{\upsilon}) \left[\psi(T, x, \vec{\upsilon}) - \psi_1(x^i, \vec{\upsilon}^i)\right] d\Omega + \sum_{j=1}^{r} \int_0^T \Delta p_j(t) \Phi(t, x^i, \vec{\upsilon}^j) dt = 0.
$$

From this, by virtue of the last condition from (2.7), it follows that the increment $\Delta J[p]$ of the minimized functional from (2.3) is transformed to the form of

$$
\begin{gathered}
\Delta J[p] = \sum_{i=1}^{r} \int_0^T \Delta p_i(t) \left[2\beta_i p_i(t) - \Phi(t, x^i, \vec{\upsilon}^i)\right] dt + \\
+ \sum_{i=1}^{r} \beta_i \int_0^T \left[\Delta p_i(t)\right]^2 dt + \int_G dG \int_{\Omega} \left[\Delta \psi(T, x, \vec{\upsilon}\right]^2 d\Omega.
\end{gathered} \tag{2.8}
$$

Now, applying a technique of the work [3], the following theorem can be proved.

Theorem (principle of the maximum**)**. Necessary and enough condition of optimality of the admissible control $p^0 = \left(p_1^0,\dots,p_r^0\right)$ and corresponding to it solution of the boundary-value problem (1.1) - (1.3) is a satisfying by the functions

$$H_i(\Phi(t, x^i, \upsilon^i), \psi, p_i) = p_i \Phi(t, x^i, \upsilon^i) - \beta_i p_i^2, \; i=1,\dots,r \tag{2.9}$$

of the conditions

$$H_i(\Phi_i^0, \psi^0, p_i^0) = \max H_i(\Phi_i^o, \psi^0, P_i), \; i=1,\dots, r, \tag{2.10}$$

where $\Phi_i^0 = \Phi^0\left(t, x^i, v^i\right)$, $\left(i = 1,2,\dots,r\right)$ is the solution to the boundary-value problem (2.6) - (2.7) subject to $\psi = \psi_0$.

3. Construction of optimal control

For the optimal control construction, first we assume, that no restrictions are imposed on the domain of admissible control parameters. Then it follows from (2.9), (14) that optimal control $p^0 = \left(p_1^0,\dots,p_r^0\right)$ must satisfy the conditions

$$p_i(t) = \frac{1}{2\beta_i}\Phi(t, x^i, \vec{\upsilon}^i), \quad i = 1,\dots,r \tag{3.1}$$

Thus, the problem of construction of optimal control is reduced to the determining of $p^0 = \left(p_1^0,\dots,p_r^0\right)$, ψ^0 and Φ^0 from the equations (1.1) - (1.3), (2.1), (2.2) and (2.6), (2.7).

For the simplicity of reasoning henceforward, we assume that n = 3 and then $x_1 = x, x_2 = y, x_3 = z$, and the unit velocity vector in this case is $\vec{v} = (v_1, v_2, v_3)$, where $v_1 = \sin\theta\cos\varphi, v_2 = \sin\theta\sin\varphi, v_3 = \cos\theta$. We will investigate the boundary-value problem (2.6) - (2.7), where, in accordance with $[3, 4]$, we assume

$$\Theta(t, x, y, z, \zeta, \varphi) = g(\mu_0), \quad \mu_0 = \zeta'\zeta + \sqrt{1-\zeta^2}\sqrt{1-\zeta'^2}\cos(\varphi - \varphi') \tag{3.2}$$

Then the equations (2.6) and (1.1) take the form of

$$\frac{\partial \Phi}{\partial t} + \sin\theta\cos\varphi\frac{\partial \Phi}{\partial x} + \sin\theta\sin\varphi\frac{\partial \Phi}{\partial y} + \cos\theta\frac{\partial \Phi}{\partial z} - \eta\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) + \xi\frac{\partial^2 \Phi}{\partial z^2} + \\ + \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} g(\mu_0)\Phi(t, x, y, z, \zeta', \varphi')d\zeta' = 0 \tag{3.3}$$

$$\frac{\partial\psi}{\partial t}+\sin\theta\cos\varphi\frac{\partial\psi}{\partial x}+\sin\theta\sin\varphi\frac{\partial\psi}{\partial y}+\cos\theta\frac{\partial\psi}{\partial z}+\sigma\psi(t,x,y,z,\varphi,\theta)-$$
$$-\eta\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)-\xi\frac{\partial^2\psi}{\partial z^2}=\frac{\lambda}{4\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{1}g(\mu_0)\psi(t,x,y,z,\zeta',\varphi')d\zeta'+ \tag{3.4}$$
$$+\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\Phi(t,x^i,y^i,z^i,\varphi,\theta)\delta(x-x^i,y-y^i,z-z^i)\delta(\varphi-\varphi^i,\theta-\theta^i)$$

We apply the spherical harmonics method to equation (3.3). For that, we consider the system of spherical functions [4] :

$$C_k^0=P_k^0(\cos\theta),\quad C_k^m=P_k^m(\cos\theta)\cos m\varphi,\quad S_k^m=P_k^m(\cos\theta)\sin m\varphi \tag{3.5}$$

k = 0, 1, 2...; m = 0, 1, 2, …, k.
Here

$$P_k^0(\mu)=P_k(\mu)=\frac{1}{2^k k!}\frac{d^k}{d\mu^k}\left[(\mu^2-1)^k\right],\quad k=0,1,2,... \tag{3.6}$$

are Legendre polynomials [5],

$$P_k^m(\mu)=(1-\mu^2)^{m/2}\frac{d^m P_k(\mu)}{d\mu^m}=\frac{(1-\mu^2)^{m/2}}{2^k k!}\frac{d^{k+m}}{d\mu^{k+m}}\left[(\mu^2-1)^k\right],\ k=0,1,2,...;\ m=0,1,2,...,k \tag{3.7}$$

are the attached Legendre polynomials [4, 5]. It is known, that functions (3.6) and (3.7) satisfy the orthogonally conditions of on the interval $[-1,1]$,

$$\int_{-1}^{1}P_k^m(\mu)P_j^m(\mu)d\mu=\frac{2}{2k+1}\frac{(k+m)!}{(k-m)1}\delta_k^j,\quad where\quad \delta_k^j=\begin{cases}1, & j=k\\ 0, & j\neq k\end{cases} \tag{3.8}$$

Function $g(\mu_0)$ can be presented as (see (3.2))

$$g(\mu_0)=\frac{1}{2}\sum_{k=0}^{\infty}(2k+1)g_k P_k(\mu_0)\quad where\quad g_k=\int_{-1}^{1}P_k(\mu_0)g(\mu_0)d\mu_0 \tag{3.9}$$

Here

$$P_k(\mu_0)=P_k(\zeta)P_k(\zeta')+2\sum_{j=1}^{k}\frac{(k-j)!}{(k+j)!}P_k^j(\zeta)P_k^j(\zeta')\cos(\varphi-\varphi') \tag{3.10}$$

Solution of the equation (3.3) will be found in the form of

$$\Phi = \frac{1}{2\pi}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{2k+1}{1+\delta_m^0}\frac{(k-m)!}{(k+m)!}C_k^m A_k^m + \sum_{k=1}^{\infty}\sum_{m=1}^{k}(2k+1)\frac{(k-m)!}{(k-m)!}S_k^m B_k^m\right\} \quad (3.11)$$

where C_k^m , S_k^m are determined by formulas (3.5) - (3.7), and A_k^m , B_k^m are unknown functions of arguments *t, x, y, z.*

The system of spherical functions (3.5) forms the orthogonal functions on the unit sphere and complete function set in the Hilbert space. Therefore any continuous function $\Phi(t,x,y,z,\varphi,\theta)$ can be decomposed on the spherical functions to any accuracy. In the decomposition (3.11), coefficients are defined by means of the integrals

$$A_k^0 = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^0(\mu)\Phi d\mu , \qquad A_k^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\cos m\varphi \Phi d\mu ,$$
$$B_k^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\sin m\varphi \Phi d\mu \quad (3.12)$$

For convenience, we present function (3.11) as

$$\Phi = \frac{1}{4\pi}\left\{\sum_{k=0}^{\infty}(2k+1)P_k^0(\zeta)A_k^0 + 2\sum_{k=1}^{\infty}(2k+1)\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}P_k^m(\zeta)(A_k^m \cos m\varphi + B_k^m \sin m\varphi)\right\}$$

Using this function and equalities (3.8) - (3.10), integral term in the equation (3.3) can be transformed to

$$J = \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi' \int_{-1}^{1} g(\mu_0)\Phi(t,x,y,z,\varphi',\zeta')d\zeta' = \frac{\lambda}{8\pi}\left\{\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)A_i^0 + \right.$$
$$\left. + 2\sum_{i=1}^{\infty}(2i+1)g_i \sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)(A_i^j \cos j\varphi + B_i^j \sin j\varphi)\right\} \quad (3.13)$$

Now, the equation (3.3) can be presented as

$$\frac{\partial\Phi}{\partial t} + \sqrt{1-\mu^2}\cos\varphi\frac{\partial\Phi}{\partial x} + \sqrt{1-\mu^2}\sin\varphi\frac{\partial\Phi}{\partial y} + \mu\theta\frac{\partial\Phi}{\partial z} - \sigma\Phi + \eta\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \xi\frac{\partial^2\Phi}{\partial z^2} +$$
$$+ \frac{\lambda}{8\pi}\left\{\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)A_i^0 + 2\sum_{i=1}^{\infty}(2i+1)g_i \sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)(A_i^j \cos j\varphi + B_i^j \sin j\varphi)\right\} = 0 \quad (3.14)$$

Equation (3.14) can be reduced to the system of differential equations with respect to A_k^m, B_k^m, $(k = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, k)$. For that, we multiply the equation (3.14) in turn by $P_k^0(\mu)$, $(k = 0, 1, 2, \ldots)$, $C_k^m = P_k^m(\mu)\cos m\varphi$, and $S_k^m = P_k^m(\mu)\sin m\varphi$ $(k = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, k)$, and integrate with respect to angular variables φ and μ in the limits from 0 up to 2π and from -1 up to 1, respectively. The following recurrence relation from [4, 5] are used:

$$
\begin{aligned}
&(2k+1)\mu P_k^m(\mu) = (k-m+1)P_{k+1}^m(\mu) + (k+m)P_{k-1}^m(\mu),\\
&\sqrt{1-\mu^2}\,P_k^m(\mu) = \frac{1}{2k+1}\left[P_{k+1}^{m+1}(\mu) - P_{k-1}^{m+1}(\mu)\right],\\
&\sqrt{1-\mu^2}\,P_k^m(\mu) = \frac{1}{2k+1}\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu) - (k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right],\\
&0 \le m \le k-1
\end{aligned}
\tag{3.15}
$$

So, we multiply (3.14) by $P_k(\mu) = P_k^0(\mu)$ and integrate a result with respect to variables φ and μ. Then, by virtue of the first of the formulas (3.12), we get

$$
\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\partial \Phi}{\partial t} P_k^0(\mu)d\mu = \frac{\partial}{\partial t}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi P_k^0(\mu)d\mu = \frac{\partial A_k^0}{\partial t} \tag{3.16}
$$

Thus, we have found a transformation of the first term in the equation (3.14).
The second term of this equation will be equal to

$$
\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\,\cos\varphi \frac{\partial \Phi}{\partial x} P_k^0(\mu)d\mu = \frac{1}{2k+1}\frac{\partial}{\partial x}\left(A_{k+1}^1 + A_{k-1}^1\right) \tag{3.17}
$$

Here, we used the first two formulae in (3.12), and the second identity from (3.15):

$$
\begin{aligned}
&\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\,P_k^0(\mu)\cos\varphi \frac{\partial \Phi}{\partial x} d\mu = \frac{\partial}{\partial x}\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{1}{2k+1}\left[P_{k+1}^1(\mu) - P_{k-1}^1(\mu)\right]\cos\varphi\Phi d\mu = \\
&= \frac{1}{2k+1}\frac{\partial}{\partial x}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k+1}^1(\mu)\cos\varphi\Phi d\mu - \int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k-1}^1(\mu)\cos\varphi\Phi d\mu\right] = \frac{1}{2k+1}\frac{\partial}{\partial x}\left(A_{k+1}^1 + A_{k-1}^1\right)
\end{aligned}
$$

The third term is treated by the similar way,

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\, P_k^0(\mu) \sin\varphi \frac{\partial \Phi}{\partial y} d\mu = \frac{1}{2k+1} \frac{\partial}{\partial y} \int_0^{2\pi} d\varphi \int_{-1}^{1} \left[P_{k+1}^1(\mu) - P_{k-1}^1(\mu)\right] \sin\varphi \Phi d\mu =$$

$$= \frac{1}{2k+1} \frac{\partial}{\partial y} \left[\int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k+1}^1(\mu) \sin\varphi \Phi d\mu - \int_0^{2\pi} d\varphi \int_{-1}^{1} P_{k-1}^1(\mu) \sin\varphi \Phi d\mu \right]$$

As a result, we have

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\, P_k^0(\mu) \sin\varphi \frac{\partial \Phi}{\partial y} d\mu = \frac{1}{2k+1} \frac{\partial}{\partial y} \left(B_{k+1}^1 - B_{k-1}^1\right)$$

Now, we consider the fourth term of that equation. We use the first identity from (3.14) for m=0:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu P_k^0(\mu) \frac{\partial \Phi}{\partial z} d\mu = \frac{1}{2k+1} \int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\partial \Phi}{\partial z} \left[(k+1) P_{k+1}^0(\mu) + k P_{k-1}^0(\mu)\right] d\mu =$$

$$= \frac{1}{2k+1} \frac{\partial}{\partial z} \left\{ (k+1) \int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi P_{k+1}^0(\mu) d\mu + k \int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi P_{k-1}^0(\mu) d\mu \right\} = \tag{3.18}$$

$$= \frac{1}{2k+1} \frac{\partial}{\partial z} \left\{ (k+1) A_{k+1}^0 + k A_{k-1}^0 \right\}$$

The fifth, sixth, seventh and eighth terms contain constant coefficients, therefore they are transformed to the following expression,

$$-\sigma A_л^0 + \eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A_k^0 + \xi \frac{\partial^2 A_k^0}{\partial z^2} \tag{3.19}$$

Now we consider the next term,

$$\frac{\lambda}{8\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^0(\mu) \sum_{i=0}^{\infty} (2i+1) g_i P_i^0(\mu) A_i^0 d\mu = \frac{\lambda}{8\pi} 2\pi \sum_{i=0}^{\infty} (2i+1) g_i A_i^0 \int_{-1}^{1} P_k^0(\mu) P_i^0(\mu) d\mu =$$

$$= \frac{\lambda}{4} \sum_{i=0}^{\infty} (2i+1) g_i A_i^0 \frac{2}{2k+1} \delta_k^i = \frac{\lambda}{4} (2k+1) g_k A_k^0 \frac{2}{2k+1} = \frac{\lambda}{2} g_k A_k^0 \qquad (k = 0,1,2,...)$$

Combining the formulae (3.15) - (3.19) and $\frac{\lambda}{2} g_k A_k^0$, we obtain the system with respect to A_k^0 , A_k^1 , B_k^1,

$$\frac{\partial A_k^0}{\partial t}+\frac{1}{2k+1}\left\{\frac{\partial}{\partial x}\left(A_{k+1}^1-A_{k-1}^1\right)+\frac{\partial}{\partial y}\left(B_{k+1}^1-B_{k-1}^1\right)+\frac{\partial}{\partial x}\left((k+1)A_{k+1}^0+kA_{k-1}^0\right)\right\}+$$
$$+\left(\frac{\lambda}{2}g_k-\sigma\right)A_k^0+\eta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)A_k^0+\xi\frac{\partial^2}{\partial z^2}A_k^0=0, \qquad k=0,1,2,... \tag{3.20}$$

Let's multiply now equation (3.14) by $C_k^m=P_k^m(\mu)\cos m\varphi$ for $(k=0,1,2,\text{K}\,,\quad m=0,1,\text{K}\,,k)$ and integrate the result with respect to φ and μ from 0 to 2π and from -1 to 1, respectively. As a result, the first term takes a form of $\frac{\partial}{\partial t}A_k^m$. To find expression for the second term, we use the second and third of identities (3.14),

$$\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}\cos\varphi\frac{\partial\Phi}{\partial x}P_k^m(\mu)\cos m\varphi d\mu d\varphi=$$
$$=\frac{\partial}{\partial x}\int_0^{2\pi}\int_{-1}^{1}\Phi\sqrt{1-\mu^2}P_k^m(\mu)\frac{1}{2}\left[\cos(m+1)\varphi+\cos(m-1)\varphi\right]d\mu d\varphi=$$
$$=\frac{1}{2}\frac{\partial}{\partial x}\int_0^{2\pi}\int_{-1}^{1}\Phi\frac{1}{2k+1}\left[P_{k+1}^{m+1}(\mu)-P_{k-1}^{m+1}(\mu)\right]\cos(m+1)\varphi d\mu d\varphi+$$
$$+\frac{1}{2}\frac{\partial}{\partial x}\int_0^{2\pi}\int_{-1}^{1}\Phi\frac{1}{2k+1}\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu)-\right.$$
$$\left.-(k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right]\cos(m-1)\varphi d\mu d\varphi=$$
$$=\frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left(A_{k+1}^{m+1}-A_{k-1}^{m+1}\right)+$$
$$+\frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left[(k+m)(k+m-1)A_{k-1}^{m-1}-(k-m+1)(k-m+2)A_{k+1}^{m-1}\right]$$

Let's transform the third term in the equation (3.14),

$$\int_0^{2\pi}d\varphi\int_{-1}^{1}\sqrt{1-\mu^2}\sin\varphi\frac{\partial\Phi}{\partial y}P_k^m(\mu)\cos m\varphi d\mu=$$
$$=\frac{\partial}{\partial y}\int_0^{2\pi}d\varphi\int_{-1}^{1}\sqrt{1-\mu^2}P_k^m(\mu)\Phi\sin\varphi\cos m\varphi d\mu=$$

$$= \frac{\partial}{\partial y}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\, P_k^m(\mu)\Phi\, \frac{1}{2}\left[\sin(m+1)\varphi - \sin(m-1)\varphi\right] d\mu\right] =$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\frac{1}{2k+1}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[P_{k+1}^{m+1}(\mu) - P_{k-1}^{m+1}(\mu)\right]\sin(m+1)\varphi d\mu -$$

$$- \frac{1}{2}\frac{\partial}{\partial y}\frac{1}{2k+1}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu) -\right.$$

$$\left.- (k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right]\sin(m-1)\varphi d\mu =$$

$$= \frac{1}{2}\frac{\partial}{\partial y}\frac{1}{2k+1}\left[B_{k+1}^{m+1} - B_{k-1}^{m+1}\right] -$$

$$- \frac{1}{2}\frac{\partial}{\partial y}\frac{1}{2k+1}\left[(k+m)(k+m-1)B_{k-1}^{m-1} - (k-m+1)(k-m+2)B_{k+1}^{m-1}\right]$$

Now we transform the fourth item in the equation (3.14), using the first of identities (3.15):

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu \frac{\partial \Phi}{\partial z} P_k^m(\mu)\cos m\varphi d\mu =$$

$$= \frac{1}{2k+1}\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\partial \Phi}{\partial z}\left[(k-m+1)P_{k+1}^m(\mu) + (k+m)P_{k-1}^m(\mu)\right]\cos m\varphi d\mu =$$

$$= \frac{1}{2k+1}\frac{\partial}{\partial z}\int_0^{2\pi} d\varphi \int_{-1}^{1} \left[(k-m+1)P_{k+1}^m(\mu) + (k+m)P_{k-1}^m(\mu)\right]\Phi \cos m\varphi d\mu =$$

$$= \frac{1}{2k+1}\frac{\partial}{\partial z}\left\{(k-m+1)A_{k+1}^m + (k+m)A_{k-1}^m\right\}$$

The fifth-eight terms in the equation (3.13) take the form of

$$\left[-\sigma + \eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \xi \frac{\partial^2}{\partial z^2}\right] A_k^m$$

Now we find the last term at the left hand side of the equation (3.14):

$$\frac{\lambda}{8\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\cos m\varphi \left[\sum_{i=0}^{\infty}(2i+1)g_i P_i^0(\mu)\cos 0\varphi A_i^0 + \right.$$
$$\left. + 2\sum_{i=0}^{\infty}(2i+1)g_i \sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)\left(A_i^j \cos j\varphi + B_i^j \sin j\varphi\right)\right] d\mu =$$
$$= \sum_{i=0}^{\infty}\frac{\lambda}{8\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^m(\mu)\cos m\varphi P_i^0(\mu)\cos 0\varphi d\mu (2i+1) g_i A_i^0 +$$
$$+ \frac{\lambda}{8\pi} 2\sum_{i=0}^{\infty}(2i+1)g_i \sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu P_k^m(\mu)\times$$
$$\times \cos m\varphi P_i^j(\mu)\left(A_i^j \cos j\varphi + B_i^j \sin j\varphi\right) =$$
$$= \frac{\lambda}{4\pi}\sum_{i=0}^{\infty}(2i+1)g_i \sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}\int_{-1}^{1} P_k^m(\mu) P_i^m(\mu) d\mu \delta_m^j =$$
$$= \frac{\lambda}{4\pi}\sum_{i=0}^{\infty}(2i+1)g_i \frac{(i-m)!}{(i+m)!}\frac{2}{2k+1}\delta_k^i A_i^m = \frac{\lambda}{2} g_k \frac{(k-m)!}{(k+m)!} A_k^m$$

Thus, we have a system for determining of A_k^m and B_k^m,

$$\frac{\partial}{\partial t}A_k^m + \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left(A_{k+1}^{m+1} - A_{k-1}^{m+1}\right) +$$
$$+ \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left[(k+m)(k+m-1)A_{k-1}^{m-1} - (k-m+1)(k-m+2)A_{k+1}^{m-1}\right] +$$
$$+ \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left(B_{k+1}^{m+1} - B_{k-1}^{m+1}\right) -$$
$$- \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left[(k+m)(k+m-1)B_{k-1}^{m-1} - (k-m+1)(k-m+2)B_{k+1}^{m-1}\right] +$$
$$+ \frac{1}{(2k+1)}\frac{\partial}{\partial z}\left[(k-m+1)A_{k+1}^m + (k+m)A_{k-1}^m\right] + \left[\eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \xi\frac{\partial^2}{\partial z^2} - \sigma\right]A_k^m +$$
$$+ \frac{\lambda}{2} g_k \frac{(k-m)!}{(k+m)!} A_k^m = 0, \quad k = 1,2,..., \quad m = 1,2,...,k; \tag{3.21}$$

Let's multiply the equation (3.14) by $S_k^m = P_k^m(\mu)\sin m\varphi, \quad (k = 1,2,...; \quad m = 1,2,...,k)$ and integrate with respect to φ and μ from 0 to 2π and from -1 to 1, accordingly. Clearly that $\frac{\partial \Phi}{\partial t}$ is transformed to $\frac{\partial}{\partial t}B_k^m$, $(k = 1,2,...; \quad m = 1,2,...,k)$. Then,

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\cos\varphi \frac{\partial \Phi}{\partial x} P_k^m(\mu)\sin m\varphi d\mu =$$
$$= \frac{\partial}{\partial x}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} P_k^m(\mu)\Phi \frac{1}{2}\left[\sin(m+1)\varphi + \sin(m-1)\varphi\right] d\mu\right] =$$
$$= \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[P_{k+1}^{m+1}(\mu) - P_{k-1}^{m+1}(\mu)\right]\sin(m+1)\varphi d\mu + \qquad (3.22)$$
$$+ \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu) - \right.$$
$$\left. -(k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right]\sin(m-1)\varphi d\mu =$$
$$= \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left[B_{k+1}^{m+1} - B_{k-1}^{m+1}\right] +$$
$$+ \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left[(k+m)(k+m-1)B_{k-1}^{m-1} - (k-m+1)(k-m+2)B_{k+1}^{m-1}\right]$$

Further we have:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2}\sin\varphi \frac{\partial \Phi}{\partial y} P_k^m(\mu)\sin m\varphi d\mu =$$
$$= \frac{\partial}{\partial y}\left[\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} P_k^m(\mu)\Phi \frac{1}{2}\left[\cos(m-1)\varphi - \cos(m+1)\varphi\right] d\mu\right] = \qquad (3.23)$$
$$= \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[(k+m)(k+m-1)P_{k-1}^{m-1}(\mu) - \right.$$
$$\left. -(k-m+1)(k-m+2)P_{k+1}^{m-1}(\mu)\right]\cos(m-1)\varphi d\mu -$$
$$- \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi\left[P_{k+1}^{m+1}(\mu) - P_{k-1}^{m+1}(\mu)\right]\cos(m+1)\varphi d\mu =$$
$$= \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left[(k+m)(k+m-1)A_{k-1}^{m-1} - (k-m+1)(k-m+2)A_{k+1}^{m-1}\right] -$$
$$- \frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left[A_{k+1}^{m+1} - A_{k-1}^{m+1}\right]$$

Then we have:

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu \frac{\partial \Phi}{\partial z} P_k^m(\mu) \sin m\varphi d\mu =$$

$$= \frac{\partial}{\partial z} \int_0^{2\pi} d\varphi \int_{-1}^{1} \Phi \frac{1}{2k+1} \left[(k-m+1) P_{k+1}^m(\mu) + (k+m) P_{k-1}^m(\mu) \right] \sin m\varphi d\mu =$$

$$= \frac{1}{2k+1} \frac{\partial}{\partial z} \left\{ (k-m+1) B_{k+1}^m + (k+m) B_{k-1}^m \right\} \quad (3.24)$$

The fifth-eighth terms from the equation (3.13) are transformed to the next form,

$$\left[-\sigma + \eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \xi \frac{\partial^2}{\partial z^2} \right] B_k^m . \quad (3.25)$$

Finally, the last term in the equation (3.14) is transformed to

$$\frac{\lambda}{8\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} d\mu \left[\sum_{i=0}^{\infty} (2i+1) g_i P_i^0(\mu) A_i^0 + \right.$$

$$\left. + 2 \sum_{i=0}^{\infty} (2i+1) g_i \sum_{j=1}^{i} \frac{(i-j)!}{(i+j)!} P_i^j(\mu) \left(A_i^j \cos j\varphi + B_i^j \sin j\varphi \right) \right] P_k^m(\mu) \sin m\varphi =$$

$$= \frac{\lambda}{4\pi} \sum_{i=0}^{\infty} (2i+1) g_i \sum_{j=1}^{i} \frac{(i-j)!}{(i+j)!} \int_0^{2\pi} \int_{-1}^{1} P_i^j(\mu) B_i^j P_k^m(\mu) \sin j\varphi \sin m\varphi d\varphi d\mu = \quad (3.26)$$

$$= \frac{\lambda}{4\pi} \pi \sum_{i=0}^{\infty} (2i+1) g_i \sum_{j=1}^{i} \frac{(i-j)!}{(i+j)!} B_i^j \int_{-1}^{1} P_i^j(\mu) P_k^m(\mu) d\mu \delta_m^j =$$

$$= \frac{\lambda}{4} \sum_{i=0}^{\infty} (2i+1) g_i \frac{(i-m)!}{(i+m)!} B_i^m \int_{-1}^{1} P_i^m(\mu) P_k^m(\mu) d\mu = \frac{\lambda}{2} g_k \frac{(k-m)!}{(k+m)!} B_k^m$$

Combining all of these expressions, we obtain the another one system with respect to A_k^m , B_k^m ,

$$\frac{\partial}{\partial t} B_k^m + \frac{1}{2(2k+1)} \frac{\partial}{\partial x} \left(B_{k+1}^{m+1} - B_{k-1}^{m+1} \right) +$$

$$+ \frac{1}{2(2k+1)} \frac{\partial}{\partial x} \left[(k+m)(k+m-1) B_{k-1}^{m-1} - (k-m+1)(k-m+2) B_{k+1}^{m-1} \right] +$$

$$+ \frac{1}{2(2k+1)} \frac{\partial}{\partial y} \left[(k+m)(k+m-1) A_{k-1}^{m-1} - (k-m+1)(k-m+2) A_{k+1}^{m-1} \right] -$$

$$-\frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left(A_{k+1}^{m+1}-A_{k-1}^{m+1}\right)+\frac{1}{(2k+1)}\frac{\partial}{\partial z}\left[(k-m+1)B_{k+1}^{m}+(k+m)B_{k-1}^{m}\right]+$$

$$+\left[\eta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\xi\frac{\partial^2}{\partial z^2}-\sigma\right]B_k^m+\frac{\lambda}{2}g_k\frac{(k-m)!}{(k+m)!}B_k^m=0\,, \qquad (3.27)$$

$$k=1,2,..., \quad m=1,2,...,k;$$

4. Derivation of the spherical harmonics method equations for the initial state

Let us rewrite equation (3.4) in the form of

$$\frac{\partial\psi}{\partial t}+\sqrt{1-\mu^2}\cos\varphi\frac{\partial\psi}{\partial x}+\sqrt{1-\mu^2}\sin\varphi\frac{\partial\psi}{\partial y}+\mu\frac{\partial\psi}{\partial z}+$$

$$[\ \sigma-\eta(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})-\xi\frac{\partial^2}{\partial z^2}\ ]\psi(t,x,y,z,\varphi,\theta)=$$

$$=\frac{\lambda}{4\pi}\int_0^{2\pi}d\varphi'\int_{-1}^{1}g(\mu_0)\psi(t,x,y,z,\zeta',\varphi')d\zeta'+$$

$$+\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\Phi\left(t,x^i,y^i,z^i,\varphi,\theta\right)\delta\left(x-x^i,y-y^i,z-z^i\right)\delta\left(\varphi-\varphi^i,\theta-\theta^i\right).\ (4.1)$$

In order to apply a spherical harmonics method to this equation, we multiply (4.1) by spherical functions (3.5) and integrate the result with respect to φ and μ by turns, within the limits from 0 to 2π and from -1 to 1, respectively. We use again the recurrence relations (3.15).

First, we decompose function $\psi(t,x,y,z,\varphi,\theta)$ on the spherical functions (3.5),

$$\psi=\frac{1}{4\pi}\left\{\sum_{k=0}^{\infty}(2k+1)P_k^0(\zeta)a_k^0+2\sum_{k=1}^{\infty}(2k+1)\sum_{m=1}^{k}\frac{(k-m)!}{(k+m)!}P_k^m(\zeta)(a_k^m\cos m\varphi+b_k^m\sin m\varphi)\right\},\ (4.2)$$

where decomposition coefficients are determined according with the formulae

$$a_k^0=\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^0(\mu)\psi d\mu\,,\qquad a_k^m=\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^m(\mu)\psi\cos m\varphi d\mu\,,$$

$$b_k^m=\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^m(\mu)\psi\sin m\varphi d\mu \qquad (4.3)$$

Multiplying (4.1) by $P_k(\mu) = P_k^0(\mu)$ and integrating with respect to the variables φ and μ, analogically with (3.16), we get

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \frac{\partial \psi}{\partial t} P_k^0(\mu) d\mu = \frac{\partial}{\partial t} \int_0^{2\pi} d\varphi \int_{-1}^{1} \psi P_k^0(\mu) d\mu = \frac{\partial a_k^0}{\partial t} \qquad (4.4)$$

Analogically with (3.17), we have

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} \frac{\partial \psi}{\partial x} P_k^0(\mu) cos\varphi d\mu = \frac{1}{2k+1} \frac{\partial}{\partial x} \left(a_{k+1}^1 - a_{k-1}^1 \right) \qquad (4.5)$$

The third term of the equation (4.1) takes a form of

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \sqrt{1-\mu^2} \frac{\partial \psi}{\partial y} P_k^0(\mu) \sin\varphi d\mu = \frac{1}{2k+1} \frac{\partial}{\partial y} \left(b_{k+1}^1 - b_{k-1}^1 \right). \qquad (4.6)$$

We rewrite the next term by analogy with (3.18),

$$\int_0^{2\pi} d\varphi \int_{-1}^{1} \mu P_k^0(\mu) \frac{\partial \psi}{\partial z} d\mu == \frac{1}{2k+1} \frac{\partial}{\partial z} \left\{ (k+1) a_{k+1}^0 + k a_{k-1}^0 \right\}. \qquad (4.7)$$

By analogy with (3.18), we obtain

$$\left[\sigma - \eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \xi \frac{\partial^2}{\partial z^2} \right] a_k^0 . \qquad (4.8)$$

The first term from the right hand side in the equation (4.1) is transformed to the form of

$$\begin{aligned} &\frac{\lambda}{8\pi} \int_0^{2\pi} d\varphi \int_{-1}^{1} P_k^0(\mu) \sum_{i=0}^{\infty} (2i+1) g_i P_i^0(\mu) a_i^0 d\mu = \frac{\lambda}{8\pi} 2\pi \sum_{i=0}^{\infty} (2i+1) g_i a_i^0 \int_{-1}^{1} P_k^0(\mu) P_i^0(\mu) d\mu = \\ &= \frac{\lambda}{4} \sum_{i=0}^{\infty} (2i+1) g_i a_i^0 \frac{2}{2k+1} \delta_k^i = \frac{\lambda}{4} (2k+1) g_k a_k^0 \frac{2}{2k+1} = \frac{\lambda}{2} g_k a_k^0 \end{aligned} \qquad (4.9)$$

Now, we turn to the transformation of the second from the right term of the equation (4.1).

$$\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\int_0^{2\pi}d\varphi\int_{-1}^{1} P_k^0(\mu)\Phi(t,x^i,y^i,z^i,\varphi,\theta)\delta(\varphi-\varphi^i,\theta-\theta^i)d\mu\delta(x-x^i,y-y^i,z-z^i)= \qquad (4.10)$$
$$=\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\delta(x-x^i,y-y^i,z-z^i)P_k^0(\mu^i)\Phi(t,x^i,y^i,z^i,\varphi^i,\theta^i),$$

where

$$P_k^0(\mu^i)\Phi(t,x^i,y^i,z^i,\varphi^i,\theta^i)=\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^0(\mu)\delta(\varphi-\varphi^i,\theta-\theta^i)d\mu, \qquad (4.11)$$

From (4.4) - (4.10), we get the system of equations for determining of $a_k^0, a_k^1, b_k^0, b_k^1$,

$$\frac{\partial a_k^0}{\partial t}+\frac{1}{2k+1}\left\{\frac{\partial}{\partial x}\left(a_{k+1}^1-a_{k-1}^1\right)+\frac{\partial}{\partial y}\left(b_{k+1}^1-b_{k-1}^1\right)+\frac{\partial}{\partial x}\left((k+1)a_{k+1}^0+ka_{k-1}^0\right)\right\}+$$
$$+\left[\frac{\lambda}{2}g_k-\sigma+\eta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\xi\frac{\partial^2}{\partial z^2}\right]a_k^0=$$
$$=\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\delta(x-x^i,y-y^i,z-z^i)P_k^0(\mu^i)\Phi(t,x^i,y^i,z^i,\varphi^i,\theta^i),\ k=0,1,2,3,\ldots \qquad (4.12)$$

Now, we multiply equation (4.1) by $C_k^m = P_k^m(\mu)\cos m\varphi$ for (k = 0,1,2,…; m =1,2,…,k) and integrate it with respect to φ and μ over the limits from 0 to 2π and from -1 to 1, respectively. As a result, we obtain the first term in the form of $\frac{\partial}{\partial t}a_k^m$. To find the expression for the second term, we use the previous reasoning for the A_k^m, B_k^m,

$$\int_0^{2\pi}\int_{-1}^{1}\sqrt{1-\mu^2}\cos\varphi\frac{\partial\psi}{\partial x}P_k^m(\mu)\cos m\varphi d\mu d\varphi=$$
$$=\frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left\{\left(a_{k+1}^{m+1}-a_{k-1}^{m+1}\right)+\left[(k+m)(k+m-1)a_{k-1}^{m-1}-(k-m+1)(k-m+2)a_{k+1}^{m-1}\right]\right\} \qquad (4.13)$$

$$\int_0^{2\pi}d\varphi\int_{-1}^{1}\sqrt{1-\mu^2}\sin\varphi\frac{\partial\psi}{\partial y}P_k^m(\mu)\cos m\varphi d\mu= \qquad (4.14)$$
$$=\frac{1}{2}\frac{1}{2k+1}\frac{\partial}{\partial y}\left\{\left[b_{k+1}^{m+1}-b_{k-1}^{m+1}\right]-\left[(k+m)(k+m-1)b_{k-1}^{m-1}-(k-m+1)(k-m+2)b_{k+1}^{m-1}\right]\right\}$$

The firth term in the equation (4.1) turns to

$$\left[\ \sigma-\eta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\xi\frac{\partial^2}{\partial z^2}\ \right]a_k^m \tag{4.15}$$

By analogy with (3.14), the last from the right term in the equation (4.1) takes a form of

$$\begin{aligned}&\frac{\lambda}{8\pi}\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^m(\mu)\cos m\varphi\left[\sum_{i=0}^{\infty}(2i+1)g_iP_i^0(\mu)\cos 0\varphi a_i^0+\right.\\&\left.+2\sum_{i=0}^{\infty}(2i+1)g_i\sum_{j=1}^{i}\frac{(i-j)!}{(i+j)!}P_i^j(\mu)\left(a_i^j\cos j\varphi+a_i^j\sin j\varphi\right)\right]d\mu=\\&=\frac{\lambda}{4\pi}\sum_{i=0}^{\infty}(2i+1)g_i\frac{(i-m)!}{(i+m)!}\frac{2}{2k+1}\delta_k^ia_i^m=\frac{\lambda}{2}g_k\frac{(k-m)!}{(k+m)!}a_k^m\end{aligned} \tag{4.16}$$

By analogy with (4.10), we transform the first term of (4.1) ,

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\int_0^{2\pi}d\varphi\int_{-1}^{1}P_k^m(\mu)\cos m\varphi\Phi\left(t,x^i,y^i,z^i,\varphi,\theta\right)\delta\left(\varphi-\varphi^i,\theta-\theta^i\right)d\mu\delta\left(x-x^i,y-y^i,z-z^i\right)=\\&=\frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\delta(x-x^i,y-y^i,z-z^i)P_k^m(\mu^i)\cos m\varphi^i\Phi(t,x^i,y^i,z^i,\varphi^i,\theta^i)\end{aligned}$$

Taking this into account and combining (4.12) – (4.16) , we have

$$\begin{aligned}&\frac{\partial}{\partial t}a_k^m+\frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left(a_{k+1}^{m+1}-a_{k-1}^{m+1}\right)+\\&+\frac{1}{2(2k+1)}\frac{\partial}{\partial x}\left[(k+m)(k+m-1)a_{k-1}^{m-1}-(k-m+1)(k-m+2)a_{k+1}^{m-1}\right]+\\&+\frac{1}{2(2k+1)}\frac{\partial}{\partial y}\left\{\ \left(b_{k+1}^{m+1}-b_{k-1}^{m+1}\right)-\left[(k+m)(k+m-1)b_{k-1}^{m-1}-(k-m+1)(k-m+2)b_{k+1}^{m-1}\right]\ \right\}+\\&+\frac{1}{(2k+1)}\frac{\partial}{\partial z}\left[(k-m+1)a_{k+1}^m+(k+m)a_{k-1}^m\right]+\left[\sigma-\eta\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-\xi\frac{\partial^2}{\partial z^2}-\frac{\lambda}{2}g_k\frac{(k-m)!}{(k+m)!}\right]a_k^m=\\&=\frac{1}{2}\sum_{i01}^{m}\frac{1}{\beta_i}\delta(x-x^i,y-y^i,z-z^i)P_k^m(\mu^i)\cos m\varphi^i\Phi(t,x^i,y^i,z^i,\varphi^i,\theta^i)\ ,\end{aligned}$$

k = 1,2,3,…; m = 1,2,3,…,m. (4.17)

Now, we multiply equation (4.1) by $S_k^m = P_k^m(\mu)\sin m\varphi$ for (k = 0,1,2,…; m =1,2,…,k) and integrate the result over the limits from 0 to 2π and from -1 to 1 with respect to φ and μ, respectively. Repeating last reasoning, we obtain a system with respect to a_k^m , b_k^m :

$$\frac{\partial}{\partial t}b_k^m + \frac{1}{2(2k+1)}\frac{\partial}{\partial x}\{\left(b_{k+1}^{m+1} - b_{k-1}^{m+1}\right) + \left[(k+m)(k+m-1)b_{k-1}^{m-1} - (k-m+1)(k-m+2)b_{k+1}^{m-1}\right]\} +$$

$$+\frac{1}{2(2k+1)}\frac{\partial}{\partial y}\{\left[(k+m)(k+m-1)a_{k-1}^{m-1} - (k-m+1)(k-m+2)a_{k+1}^{m-1}\right] - \left(a_{k+1}^{m+1} - a_{k-1}^{m+1}\right)\} +$$

$$+\frac{1}{(2k+1)}\frac{\partial}{\partial z}\left[(k-m+1)b_{k+1}^m + (k+m)b_{k-1}^m\right] + \left[\sigma - \eta\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - \xi\frac{\partial^2}{\partial z^2} - \frac{\lambda}{2}g_k\frac{(k-m)!}{(k+m)!}\right]b_k^m =$$

$$= \frac{1}{2}\sum_{i=1}^{r}\frac{1}{\beta_i}\delta(x-x^i, y-y^i, z-z^i)P_k^m(\mu^i)\sin m\varphi^i\,\Phi(t, x^i, y^i, z^i, \varphi^i, \theta^i), k = 1,2,3,\ldots; m = 1,2,3,\ldots,k$$

(4.18)

Equations derived from the boundary conditions for (1.1) and (2.6), namely the equations (3.3) и (3.4), written not for general *n*-dimensional case, but for the *n*=3, should be added to the equations (3.2), (3.21), (3.27), (4.12), (4.17) and (4.18). Since the domain G in this particular case of three-dimensional space R^3 represents the region with the boundary Γ of the form of a cylinder with the bases Γ_0 and Γ_H (z = 0 and z = H), and lateral surface Γ_l ($x^2 + y^2 = \rho^2$ with ρ being the radius of the cylinder Γ), we have

$$\left(\frac{\partial\psi}{\partial z} - \alpha\psi\right)\Big|_{z=0} = 0\ ,\ \frac{\partial\psi}{\partial я}\Big|_{z=H} = 0,\quad \psi(t,x,y,z,\varphi,\xi)\big|_{\Gamma_l} = 0 \quad for \quad -1 < \mu < 0 \quad (4.19)$$

$$\left(\frac{\partial\Phi}{\partial z} - \alpha\Phi\right)\Big|_{z=0} = 0,\quad \frac{\partial\Phi}{\partial z}\Big|_{z=H} = 0,\quad \Phi(t,x,y,z,\varphi,\xi)\big|_{\Gamma_l} = 0 \quad for \quad 1 > \mu > 0 \quad (4.20)$$

Here, μ is determined by the second of the equations (3.2). Conditions on the ends of the control time interval $[0, T]$ take the form of

$$\psi(t,x,y,z,\varphi,\xi)\big|_{t\le 0} = \psi^0(x,y,z,\varphi,\xi) \qquad (4.21)$$

$\Phi(T,x,y,z,\varphi,\xi) = -2[\psi(T,x,y,z,\varphi,\xi) - \psi_1(x,y,z,\varphi,\xi)]$ (4.22) Boundary conditions (4.19) – (4.22) for coefficients A_k^m , B_k^m and a_k^m , b_k^m (k=0, 1, 2,…; m = 0, 1, 2,…, k) in decompositions (3.11) и (4.2) of functions $\Phi(t,x,y,z,\varphi,\xi)$ and $\psi(t,x,y,z,\varphi,\xi)$ on spherical functions (3.5) have the form of

$$\left(\frac{\partial A_k^m}{\partial z} - \alpha A_k^m\right)\bigg|_{z=0} = \left(\frac{\partial B_k^m}{\partial z} - \alpha B_k^m\right)\bigg|_{z=0} = 0\ , \ \frac{\partial A_k^m}{\partial z}\bigg|_{z=H} = \frac{\partial B_k^m}{\partial z}\bigg|_{z=H} = 0\ ,$$

$$A_k^m(t,x,y,z,\varphi,\xi)\Big|_{\Gamma_\delta} = B_k^m(t,x,y,z,\varphi,\xi)\Big|_{\Gamma_\delta} = 0 \quad for \quad 1 > \mu > 0 \quad (4.23)$$

To derive the conditions, similar to (1.2) и (4.20) for decomposition coefficients (3.2) и (4.2), on the ends of the control time interval $[0,T]$, the functions $\psi_0(x,y,z,\varphi,\xi)$ and $\psi_1(x,y,z,\varphi,\xi)$ from (1.2) and (4.20) must be decomposed on spherical functions (3.5),

$$\psi_i(x,y,z,\varphi,\xi) = \frac{1}{2\pi}\left\{\sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{2k+1}{1+\delta_m^0}\frac{(k-m)!}{(k+m)!}C_k^m a_{ik}^m + \sum_{k=1}^{\infty}\sum_{m=1}^{k}(2k+1)\frac{(k-m)!}{(k-m)!}S_k^m b_{ik}^m\right\}$$

Here a_{ik}^m , b_{ik}^m are Fourier coefficients of functions $\psi_0(x,y,z,\varphi,\xi)$ and $\psi_1(x,y,z,\varphi,\xi)$:

$$a_{ik}^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} \psi_i(x,y,z,\varphi,\xi) P_k^m(\mu) \cos m\varphi d\mu\ , \quad b_{ik}^m = \int_0^{2\pi} d\varphi \int_{-1}^{1} \psi_i(x,y,z,\varphi,\xi) P_k^m(\mu) \sin m\varphi d\mu,$$

where $i = 0,1$. Now, from (3.2), (4.2), (4.19) - (4.23), it follows that

$$a_k^m(t,x,y,z)\big|_{t\le 0} = a_{0k}^m(x,y,z)\ , \quad b_k^m(t,x,y,z)\big|_{t\le 0} = b_{0k}^m(x,y,z)\ , \quad (4.24)$$

$$\begin{aligned} A_k^m(T,x,y,z) &= -2\left[a_k^m(T,x,y,z) - a_{1k}^m(x,y,z)\right], \\ B_k^m(T,x,y,z) &= -2\left[b_k^m(T,x,y,z) - b_{1k}^m(x,y,z)\right] \ , \end{aligned} \quad (4.25)$$

$k = 0,1,2,\ldots;\ m = 0,1,2,\ldots,k.$

Thus, optimal control functions $p_i(t),\ (i = 1,2,\ldots,r)$ in the problem of minimization of pollution of the environment by detrimental impurities particles, where the performance criterion has a form of integral quadratic functional (1.4), are determined according with formulae (3.1). In particular, in the case of R^3 , function from these formulae, are transformed into function $\Phi(t, x, y, z, \varphi, \xi)$, which is determined as a solution of the equations (3.3) and (3.4) subject to the boundary conditions (4.19) – (4.22). Method of spherical harmonics [3, 4, 6, 9] of this problem is reduced to the infinite system of partial differential equations (3.2), (3.21) , (3.27) and (4.12), (4.17), (4.18), subject to the special boundary conditions (4.23), (4.24) and (4.25).

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