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NON-NULL NORMAL CURVES IN THE SEMI-EUCLIDEAN SPACE E_2^4

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Abstract

In this study, the representation formulas of non-null curves are primarily expressed in four dimensional semi-Euclidean space E_2^4 and the non-null normal curves in E_2^4 are examined and some certain results of describing the nun-null normal curve are presented in detail in E_2^4. In addition, some mathematical conditions are expressed for a curve given in four dimensional semi-Euclidean space E_2^4 to be a nun-null normal curve as theorems.

Keywords: Semi-Euclidean space E_2^4, Frenet frame, normal curve

E_2^4 YARI ÖKLİD UZAYINDA NON-NULL NORMAL EĞRİLER

Özet

Bu çalışmada, öncelikle non-null eğrilerin temsil formülleri dört boyutlu E_2^4 yarı Öklid uzayında ifade edildi ve E_2^4 'teki nun-null normal eğriler incelenmiş ve normal eğriyi tanımlayıp bazı kesin sonuçlar ayrıntılı olarak E_2^4 'de ifade edilmiştir. Ayrıca, E_2^4 dört boyutlu yarı Öklid uzayında verilen bir eğrinin normal bir eğri olabilmesi için bazı matematiksel koşullar teorem olarak ifade edildi.

Anahtar Kelimeler: E_2^4 yarı Öklid uzayı, Frenet çatısı, normal eğri

Cite

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1. Introduction

Since ancient times, curves have been studied. The importance of curves began long before they were the subject of mathematical study. They can be considered in countless examples of their decorative use in art and in many objects from prehistoric times to the present. The mathematicians of ancient times were curious about curved lines. They first called them the "lines" but then soon they replaced the word "line" with the term "curve" and started using a phrase "straight line" for the line that is not curved or bent. A line which is not straight with no sharp edges is called a curve. It is a smoothly flowing line. In mathematics a curve is an object similar to a line which does not have to be straight. Also, the curve theory has been a fascinating subject for differential geometers.

In [1], the authors defined special curves in semi Euclidean 4-space. In [2], the authors examined the notion of the involute-evolute curves for the curves lying the surfaces in Minkowski 3-space by using the Darboux frame of the curves. In [3], the helix and slant helices were investigated using non-degenerate curves in term of Sabban frame in de Sitter 3-space or Anti de Sitter 3-space M^3 (δ_0). In [4], the author defined characterizations of semi-real quaternionic Bertrand curves in the four dimensional space E_2^4 and he studied the Serret- Frenet formulae of the curve in E_2^4 and investigated these formulas for the quaternionic Bertrand curves. In [5], the explicit parameter equations of spacelike rectifying curves in E_1^3 whose projection onto spacelike, timelike were given. In [6], they gave some conditions for non-null osculating curves in E_2^4. In [7], the author gave some characterizations of spacelike normal curves with spacelike, timelike or null principal normal in the Minkowski-space E_1^3. The curves for which the position vector always lie in their normal plane, are for simplicity called normal curves. By definition for a normal curve, the position vector β satisfies following equation

$$\beta(s) = \lambda N(s) + \mu B(s), s \in I \subset \mathbb{R},$$

for some differentiable functions λ, μ [7]. In [8], the authors defined normal curves in Minkowski space-time E_1^4 and they characterized the spacelike normal curves in E_1^4. The rectifying and the osculating curves in null cone were examined by authors, [10, 11]. By using some differential geometry concepts in references [9] and [12], the normal curves in E_2^4 are expressed.

2. Preliminaries

Let E_2^4 denote the $4-$dimensional pseudo-Euclidean space with signature (2,4), that is, the real vector space $\mathbb{R}^4$ endowed with the metric $\langle,\rangle$ which is defined by

$$\langle,\rangle = g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_2^4. A vector v of E_2^4 is said to be

i) spacelike, if $v = 0$ or $\langle v, v\rangle > 0$,

ii) timelike, if $v \neq 0$ and $\langle v, v\rangle < 0$,

iii) null (or lightlike), if $v \neq 0$ and $\langle v, v\rangle = 0$.

The norm of a vector v is givenby $\| v \| = \sqrt{|g(v,v)|}$ and two vectors v and w are said to be ortogonal if $g(v,w) = 0$. An arbitrary curve $x(s)$ in E_2^4 can locally be spacelike, timelike or null. A spacelike or timelike curve $x(s)$ has unit speed, if $g(x', x') = \pm 1$.

It is well known that pseudosphere, the pseudohyperbolic space and lightlike cone are hyperquadrics in E_2^4, respectively as follows

a) The pseudo-Riemannian sphere $S_2^3(x_0, r)$ centered at $x_0 \in E_2^4$, with radius $r > 0$ of E_2^4 is defined by

$$S_2^3(x_0, r) = \{x \in E_2^4 : \langle x - x_0, x - x_0\rangle = r^2\}.$$

b) The pseudo-hyperbolic space $H_1^3(x_0, r)$ centered at $x_0 \in E_2^4$, with radius $r > 0$ of E_2^4 is defined by

$$H_1^3(x_0, r) = \{x \in E_2^4 : \langle x - x_0, x - x_0\rangle = -r^2\}.$$

The pseudo-Riemannian sphere $S_2^3(x_0, r)$ is diffeomorfic to $\mathbb{R}^2 \times S$ and the pseudo-hyperbolic space $H_1^3(x_0, r)$ is diffeomorfic to $S^1 \times \mathbb{R}^2$.

c) The hyperbolic space $H^3(x_0, r)$ is defined by

$$H^3(x_0, r) = \{x \in E_2^4 : \langle x - x_0, x - x_0\rangle = -r^2, x_1 > 0\}.$$

where the radius is $r > 0$ and the centre of hyperquadric is x_0 , [12].

Let $\{T, N, B_1, B_2\}$ be the non-null Frenet frame moving along a unit speed non-null curve β in E_2^4, where the frame is consisted of the tangent, the principal normal, the first binormal, the second binormal vector field, respectively. Then, the Frenet equations are given as

$$T' = k_1 N; N' = -\epsilon_0\epsilon_1 k_1 T + k_2 B_1; \tag{2.1a}$$

$$B_1' = -\epsilon_1\epsilon_2 k_2 N + k_3 B_2; B_2' = -\epsilon_2\epsilon_3 k_3 B_1, \tag{2.1b}$$

where the following conditions are satisfied:

$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_2) = g(N,B_1) = g(B_1,B_2) = 0;$$

$$g(T,T) = \epsilon_0, g(N,N) = \epsilon_1, g(B_1,B_1) = \epsilon_2, g(B_2,B_2) = \epsilon_3$$

$$\epsilon_i \in \{-1,1\}, i \in I = \{0,1,2,3\},$$

[9,12].

Let β be a non-null curve in E_2^4. It is defined that β is the normal curve in E_2^4, if its position vector according to selected origin lies on the orthogonal complement $T^\perp$. The orthogonal complement $T^\perp$ is non-degenerate hyperplanes of E_2^4, spanned by $\{N, B_1, B_2\}$. From definition, the position vector of a normal curve β in E_2^4 satisfies

$$\beta(s) = \mu(s)N(s) + \gamma(s)B_1(s) + \theta(s)B_2(s) \text{ or } g(\beta, T) = 0, \tag{2.2}$$

for some differentiable functions μ, γ, θ for $s \in I \subset \mathbb{R}$.

3. Representation formulae of non-null curves in $\mathbf{E_2^4}$

Let $\beta: I \to E_2^4$ be a non-null curve in E_2^4 with arc length parameter s. Then, for $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$, one gets

$$l = -\beta_1^2 - \beta_2^2 + \beta_3^2 + \beta_4^2,$$

then, if the previous equation is written as $\beta_1^2 + (l - \beta_3^2) = \beta_4^2 - \beta_2^2$, one can write

$$\frac{\beta_1 + i\sqrt{l - \beta_3^2}}{\beta_4 + \beta_2} = -\frac{\beta_2 - \beta_4}{\beta_1 - i\sqrt{l - \beta_3^2}} = x + iy, \tag{3.1}$$

$$\frac{\beta_1 + i\sqrt{l - \beta_3^2}}{\beta_2 - \beta_4} = -\frac{\beta_2 + \beta_4}{\beta_1 - i\sqrt{l - \beta_3^2}} = -\frac{1}{x - iy} \tag{3.2}$$

and

$$\beta_2 + \beta_4 = h. \tag{3.3}$$

Then, from (3.1), (3.2), (3.3), the following equations can be written.

$$\beta_1 + i\sqrt{l - \beta_3^2} = h(x + iy); \beta_2 + \beta_4 = h \tag{3.4a}$$

$$\beta_1 - i\sqrt{l - \beta_3^2} = h(x - iy); \beta_2 - \beta_4 = -h(x^2 + y^2), \tag{3.4b}$$

and from (3.4), one obtains

$$\beta_1 = hx; \beta_2 = \frac{h}{2}(1 - x^2 - y^2);$$

$$\beta_3 = \pm\sqrt{l^2 - h^2y^2}; \ \beta_4 = \frac{h}{2}(1 + x^2 + y^2).$$

Hence, the curve β can be written as follows

$$\beta(s) = (hx, \frac{h}{2}(1 - x^2 - y^2), \pm\sqrt{l^2 - h^2y^2}, \frac{h}{2}(1 + x^2 + y^2)), \tag{3.5}$$

and since $\beta: I \to E_2^4$ is a non-null curve in E_2^4 with arc lenght parameter s, one writes

$$\beta'(s) = (hx', \frac{h}{2}(-2xx' - 2yy'), \frac{-2yy'h^2}{\sqrt{l^2 - h^2y^2}}, \frac{h}{2}(2xx' + 2yy'));$$

$$1 = -(hx')^2 - h^2(xx' + yy')^2 + \frac{(yy'h^2)^2}{l^2 - h^2y^2} + h^2(xx' + yy')^2;$$

$$l^2 - h^2y^2 = -(hx')^2(l^2 - h^2y^2) + (yy'h^2)^2;$$

$$0 = h^4(y^2y'^2 - y^2x'^2) + h^2(y^2 - l^2x'^2) - l^2.$$

Therefore, by making the necessary calculations for $h(s)$, one gets the following expressions

$$A = y^2y'^2 + y^2x'^2; B = y^2 - l^2x'^2.$$

Here, $h(s)$ satisfies $h = \pm\sqrt{\frac{-B \pm \sqrt{B^2 + 4Al}}{2A}}$ =constant. Hence, we can give the following theorem.

Theorem 1. *Let $\beta: I \to E_2^4$ be a non-null curve in E_2^4 with arc lenght parameter s. Then, the curve β can be written as*

$$\beta(s) = \begin{pmatrix} hx, \\ \frac{h}{2}(1 - x^2 - y^2), \\ \pm\sqrt{l^2 - h^2y^2}, \\ \frac{h}{2}(1 + x^2 + y^2) \end{pmatrix},$$

for some functions x, y, h, where

$$h=\pm\sqrt{\frac{-B\pm\sqrt{B^2+4Al}}{2A}}=constant.$$

4. The non-null normal curves in $\mathbf{E_2^4}$

In this section, some theorems for non-null normal curves are given in E_2^4.

Theorem 2. *Let β be a unit speed non-null normal curve in E_2^4 with $k_1,k_2,k_3\neq 0$ for each $s\in I\subset\mathbb{R}$. Then, the following statements hold:*

a) *The principal normal, the first binormal and the second binormal components of the position vector of the curve are given respectively by*

$$g(\beta,N)=\frac{-1}{\epsilon_0\epsilon_1}\left(\frac{1}{k_1}\right); \tag{4.1a}$$

$$g(\beta,B_1)=\frac{1}{\epsilon_0\epsilon_2}\left(\frac{k_1'}{k_1^2k_2}\right); \tag{4.1b}$$

$$g(\beta,B_2)=\frac{-1}{\epsilon_0\epsilon_2}\left(\int\frac{k_1'k_3}{k_1^2k_2}ds\right). \tag{4.1c}$$

In this case, the vector equation is given as

$$\beta(s)=\frac{-1}{\epsilon_0\epsilon_1}\left(\frac{1}{k_1}\right)N(s)+\frac{1}{\epsilon_0\epsilon_2}\left(\frac{k_1'}{k_1^2k_2}\right)B_1(s)-\frac{1}{\epsilon_0\epsilon_2}\left(\int\frac{k_1'k_3}{k_1^2k_2}ds\right)B_2(s).$$

b) *The distance function $l=\parallel\beta\parallel$ is constant.*

c) *The curvatures k_1,k_2,k_3 satisfy the following equality*

$$-\epsilon_2\frac{k_2}{k_1}+\epsilon_1\left(\frac{k_1'}{k_1^2k_2}\right)'+\epsilon_3\epsilon_2\epsilon_1\left(k_3\int\frac{k_1'k_3}{k_1^2k_2}ds\right)=0. \tag{4.2}$$

Proof. (a) Let β be a unit speed non-null normal curve in E_2^4, with non zero k_1,k_2,k_3. From definition, for the position vector of the curve β using the Frenet equations (2.1) and the equation (2.2), one gets

$$T=(-\epsilon_0\epsilon_1k_1\mu)T+\begin{pmatrix}\mu'\\-\gamma\epsilon_1\epsilon_2k_2\end{pmatrix}N+\begin{pmatrix}k_2\mu\\+\gamma'\\-\epsilon_3\epsilon_2k_3\theta\end{pmatrix}B_1+(k_3\gamma+\theta')B_2. \tag{4.3}$$

By using equation (4.3), one can write

$$\begin{gathered}-\epsilon_0\epsilon_1k_1\mu=1;\\(\mu'-\gamma\epsilon_1\epsilon_2k_2)\epsilon_1=0;\\(k_2\mu+\gamma'-\epsilon_3\epsilon_2k_3\theta)\epsilon_2=0;\\(k_3\gamma+\theta')\epsilon_3=0\end{gathered} \tag{4.4}$$

and therefore, one has

$$\mu=\frac{-1}{\epsilon_0\epsilon_1k_1},\gamma=\frac{k_1'}{\epsilon_0\epsilon_2k_1^2k_2},\theta=\frac{-1}{\epsilon_0\epsilon_2}\int\frac{k_1'k_3}{k_1^2k_2}ds. \tag{4.5}$$

Finally, using (2.2) and (4.5) we easily obtain (4.1). Conversely, assume that the statement (a) holds, by derivative of the equations (4.1) with respect to s and using (2.2), respectively. One obtains $g(\beta(s),T)=0$.

(b) From (4.4), one can write

$$\mu'=\gamma\epsilon_1\epsilon_2k_2;\gamma'=\epsilon_3\epsilon_2k_3\theta-k_2\mu;\theta'=-k_3\gamma \tag{4.6}$$

and multiplying the first equation with μ, the second equation with γ, the last equation with θ in (4.6), respectively, and adding, one obtains

$$\begin{aligned}\epsilon_1\mu\mu'+\epsilon_2\gamma\gamma'+\epsilon_3\theta\theta'&\\&=\gamma\epsilon_2k_2\mu-\gamma\epsilon_2k_2\mu+\gamma\epsilon_3k_3\theta\\&-\gamma\epsilon_3k_3\theta,\end{aligned}$$

where β is non-null normal curve. Hence, one get

$$\epsilon_1\mu\mu'+\epsilon_2\gamma\gamma'+\epsilon_3\theta\theta'=0 \tag{4.7}$$

and consequently, one writes

$$\epsilon_1\mu^2+\epsilon_2\gamma^2+\epsilon_3\theta^2=d^2;d\in\mathbb{R}_0^+. \tag{4.8}$$

From (4.5) and (2.2), one can write

$$l^2=\parallel\beta\parallel^2=\mu^2\epsilon_1+\gamma^2\epsilon_2+\theta^2\epsilon_3, \tag{4.9}$$

and using together with (4.8) and (4.9), one can say l =constant. Conversely, the proof is obvious.

(c) Using the third equation in (4.4) and the expressions in (4.5), one can find equation (4.2).

Conversely, assume that the statement (a) holds (4.2) and let $C\in E_2^4$ be a vector field, one writes

$$C=\beta+\left(\frac{1}{\epsilon_0\epsilon_1k_1}\right)N-\left(\frac{k_1'}{\epsilon_0\epsilon_2k_1^2k_2}\right)B_1+\left(\int\frac{k_1'k_3}{\epsilon_0\epsilon_1k_1^2k_2}ds\right)B_2$$

and by taking the previous equation and using (2.2) and (4.2), one obtains $C'=0$, which means that β is non-null normal curve.

Theorem 3. *Let $\beta:I\to E_2^4$ be a non-null normal curve in E_2^4 given by $\beta(t)=\Omega(t)\omega(t)$, where $\Omega(t)$ is an arbitrary positive function and $\omega(t)$ a unit speed curve in E_2^4. Then, the following expressions hold:*

a) *ω is a non-null normal curve.*

b) *The pair $\{\beta,\omega\}$ is an evolute-involute pair.*

c) $\Omega(t)=\pm\sqrt{\frac{D}{\xi}}$ *=constant.*

Proof. Let $\beta:I\to E_2^4$ be a non-null normal curve in E_2^4 given by

$$\beta(t)=\Omega(t)\omega(t). \tag{4.10}$$

By derivative of the equation (4.10) with respect to s, one writes

$$\beta'=\Omega'\omega+\Omega\omega'. \tag{4.11}$$

Furthermore, the unit tangent vector of β is given as follow

$$T_\beta=\frac{\Omega'}{\xi}\omega+\frac{\Omega}{\xi}\omega', \tag{4.12}$$

where ξ is the speed of β. Differentiating (4.12), one obtains

$$T_\beta'=\left(\frac{\Omega'}{\xi}\right)'\omega+\left(\left(\frac{\Omega}{\xi}\right)'+\frac{\Omega'}{\xi}\right)\omega'+\frac{\Omega}{\xi}\omega''. \tag{4.13}$$

Furthermore, let $\{T_\omega,N_\omega,B_{1_\omega},B_{2_\omega}\}$ be an orthonormal frame in E_2^4 satisfying as follows

$$\begin{aligned}&\langle\omega'',T_\omega\rangle=\langle T_\omega',T_\omega\rangle=\langle k_1^\omega N_\omega,T_\omega\rangle=0;\\&\langle\omega'',N_\omega\rangle=\langle T_\omega',N_\omega\rangle=k_1^\omega\epsilon_1;\\&\langle\omega'',B_{1_\omega}\rangle=\langle T_\omega',B_{1_\omega}\rangle=\langle k_1^\omega N_\omega,B_{1_\omega}\rangle=0;\\&\langle\omega'',B_{2_\omega}\rangle=\langle T_\omega',B_{2_\omega}\rangle=\langle k_1^\omega N_\omega,B_{2_\omega}\rangle=0.\end{aligned} \tag{4.14}$$

Hence, decomposition of ω'' with respect to $\{T_\omega,N_\omega,B_{1_\omega},B_{2_\omega}\}$, one gets

$$\omega''=\langle\omega'',T_\omega\rangle T_\omega+\langle\omega'',N_\omega\rangle N_\omega+\langle\omega'',B_{1_\omega}\rangle B_{1_\omega}+\langle\omega'',B_{2_\omega}\rangle B_{2_\omega},$$

by using (4.14) into previous equation, one gets

$$\omega''=k_1^\omega N_\omega, \tag{4.15}$$

by using (4.15) into (4.13), the following equation is found

$$k_1^\beta N_\beta=T_\beta'=\left(\frac{\Omega'}{\xi}\right)'\omega+\left(\left(\frac{\Omega}{\xi}\right)'+\frac{\Omega'}{\xi}\right)T_\omega+\frac{\Omega}{\xi}k_1^\omega N_\omega. \tag{4.16}$$

By definition, β is a non-null normal curve in E_2^4. one writes $\langle T_\beta,\beta\rangle=0$. Furthermore, taking the scalar product of (4.16) with T_ω and ω is non-null normal curve, one writes $\langle T_\omega,\omega\rangle=0$. Therefore, the following equation is written

$$k_1^{\beta}\langle N_\beta, T_z\rangle = \left(\left(\frac{\Omega}{\xi}\right)' + \frac{\Omega'}{\xi}\right)\epsilon_0. \qquad (4.17)$$

If the pair $\{\beta, \omega\}$ is an evolute-involute pair, one writes $\langle N_\beta, T_z\rangle = 0$. Hence, one gets

$$\frac{\Omega'}{\xi} + \frac{\Omega'\xi + \Omega\xi'}{\xi^2} = 0 \Longrightarrow \Omega = \pm\sqrt{\frac{D}{\xi}}; D \in \mathbb{R}^+.$$

Example 1. We consider a non-null normal curve β with arc length given by

$$\beta(s) = \Big(0, \frac{1}{\sqrt[4]{\sinh s^2(\sinh^2 s + \cosh^2 s)}}\cosh s,$$
$$\sqrt{1 - \frac{1}{\sqrt{(\sinh^2 s + \cosh^2 s)}}}\sinh s, \frac{1}{\sinh s\sqrt{(\sinh^2 s + \cosh^2 s)}}\cosh^2 s\Big),$$

where s$\in \left(-\frac{5\pi}{2}, \frac{7\pi}{2}\right)$.

Hence, the graphics of this non-null normal curve and rotational surfaces generated by using non-null normal curve are given as follows

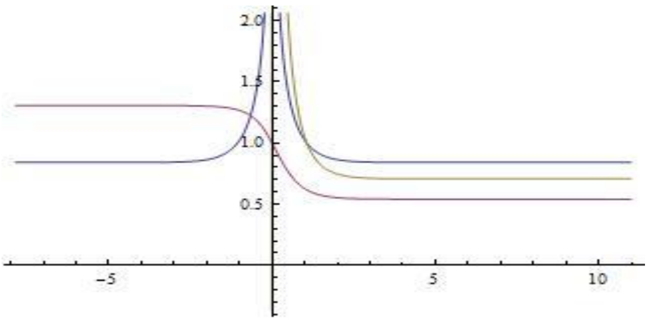

Figure 1: The non-null normal curve in E_2^4

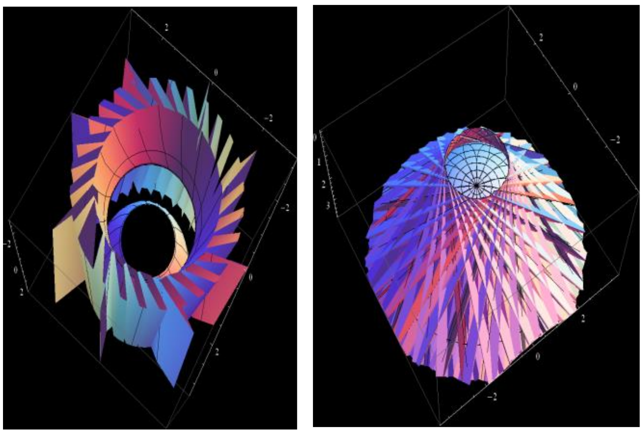

Figure 2: Rotational surfaces generated by the non-null normal curves according to parameters s$\in \left(-\frac{5\pi}{2}, \frac{7\pi}{2}\right)$ *and* s $\in (-\pi, \pi)$.

5. Conclusion

In this paper, the non-null normal curves in E_2^4 are examined and some certain results of describing the non-null normal curve are presented in detail. As a first instance, it is explored that the conditions of being nun-null normal curve in pseudo Euclidean space and some characterizations are given . Also, for the non-null normal curve and the surface of rotation formed by using this curve the graphics are given as example.

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7. Authors' contributions

FA put forward the first idea on the stated title and wrote, analyzed and commented on data. MAK analyzed the data and reinterpreted it in a well-organized form. All authors read and approved the final manuscript.

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