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A Note on The Equivalence of Some Metric and Non-Newtonian Metric Results

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Abstract. In this short note is on the equivalence between non-Newtonian metric (particularly multiplicative metric) and metric. We present a different proof the fact that the notion of a non-Newtonian metric space is not more general than that of a metric space. Also, we emphasize that a lot of fixed point results in the non-Newtonian metric setting can be directly obtained from their metric counterparts.

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1. Introduction and Preliminaries

Arithmetic is any system that satisfies the whole of the ordered field axioms whose domain is a subset of $\mathbb{R}$. There are infinitely many types of arithmetic, all of which are isomorphic, that is, structurally equivalent.

In non-Newtonian calculus, a *generator* α is a one-to-one function whose domain is all real numbers and whose range is a subset of real numbers. Each generator generates exactly one arithmetic, and conversely each arithmetic is generated by exactly one generator. By α-*arithmetic*, we mean the arithmetic whose operations and whose order are defined as

$$\begin{array}{llcl}
\alpha\text{-}\mathit{addition} & x \dot{+} y & = & \alpha\{\alpha^{-1}(x) + \alpha^{-1}(y)\} \\
\alpha\text{-}\mathit{subtraction} & x \dot{-} y & = & \alpha\{\alpha^{-1}(x) - \alpha^{-1}(y)\} \\
\alpha\text{-}\mathit{multiplication} & x \dot{\times} y & = & \alpha\{\alpha^{-1}(x) \times \alpha^{-1}(y)\} \\
\alpha\text{-}\mathit{division} & x \dot{/} y & = & \alpha\{\alpha^{-1}(x) \div \alpha^{-1}(y)\} \quad (\alpha^{-1}(y) \neq 0) \\
\alpha\text{-}\mathit{order} & x \dot{<} y & \Leftrightarrow & \alpha^{-1}(x) < \alpha^{-1}(y)
\end{array}$$

for all x and y in the range $\mathbb{R}_\alpha$ of α. In the special cases the identity function I and the exponential function exp generate the classical and geometric arithmetics, respectively.

α	α-*addition*	α-*subtraction*	α-*multiplication*	α-*division*	α-*order*
I	$x+y$	$x-y$	xy	x/y	$x<y$
exp	xy	x/y	$x^{\ln y}\left(y^{\ln x}\right)$	$x^{1/\ln y}$	$\ln x < \ln y$

For further information about α-arithmetics, we refer to [6].

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Now, we give the definitions of non-Newtonian metric [4] and multiplicative metric [12] with new notations.

Definition 1.1. Let X be a non-empty set and let $\mathbb{R}_\alpha$ be an ordered field generated by a generator α on $\mathbb{R}$. The map $d^\alpha : X \times X \to \mathbb{R}_\alpha$ is said to be a *non-Newtonian metric* if it satisfies the following properties:
(αm1) $\dot{0} = \alpha(0) \dot{\leq} d^\alpha(x, y)$ and $d^\alpha(x, y) = \dot{0} \Leftrightarrow x = y$,
(αm2) $d^\alpha(x, y) = d^\alpha(y, x)$
(αm3) $d^\alpha(x, y) \dot{\leq} d^\alpha(x, z) \dot{+} d^\alpha(z, y)$
for all $x, y, z \in X$. Also the pair (X, d^α) is said to be a *non-Newtonian metric space*.

When $\alpha = \exp$, the non-Newtonian metric $d^{\exp}$ is called multiplicative metric. Then, $\mathbb{R}_{\exp} = \mathbb{R}_+$ and $\dot{0} = 1$.

Definition 1.2. Let X be a non-empty set. The map $d^{\exp} : X \times X \to \mathbb{R}_+$ is said to be a *multiplicative metric* if it satisfies the following properties:
(mm1) $1 \leq d^{\exp}(x, y)$ and $d^{\exp}(x, y) = 1 \Leftrightarrow x = y$,
(mm2) $d^{\exp}(x, y) = d^{\exp}(y, x)$
(mm3) $d^{\exp}(x, y) \leq d^{\exp}(x, z).d^{\exp}(z, y)$
for all $x, y, z \in X$. Also the pair $(X, d^{\exp})$ is said to be a *multiplicative metric space*.

In the present work we show that some topological results of non-Newtonian metric can be obtained in an easier way. Therefore, a lot of fixed point and common fixed point results from the metric setting can be proved in the non-Newtonian metric (particularly the multiplicative metric) setting.

2. Main Results

Let α be a generator on $\mathbb{R}$ and $\mathbb{R}_\alpha = \{\alpha(u) : u \in \mathbb{R}\}$. By the injectivity of α we have

$$\begin{array}{rclcrcl}
\alpha(u+v) & = & \alpha(u) \dot{+} \alpha(v) & & \alpha^{-1}(x \dot{+} y) & = & \alpha^{-1}(x) + \alpha^{-1}(y) \\
\alpha(u-v) & = & \alpha(u) \dot{-} \alpha(v) & & \alpha^{-1}(x \dot{-} y) & = & \alpha^{-1}(x) - \alpha^{-1}(y) \\
\alpha(u \times v) & = & \alpha(u) \dot{\times} \alpha(v) & \text{and} & \alpha^{-1}(x \dot{\times} y) & = & \alpha^{-1}(x) \times \alpha^{-1}(y) \\
\alpha(u / v) & = & \alpha(u) \dot{/} \alpha(v) \quad (v \neq 0) & & \alpha^{-1}(x \dot{/} y) & = & \alpha^{-1}(x) / \alpha^{-1}(y) \\
u \leq v & \Leftrightarrow & \alpha(u) \dot{\leq} \alpha(v) & & x \dot{\leq} y & \Leftrightarrow & \alpha^{-1}(x) \leq \alpha^{-1}(y)
\end{array}$$

for all $x, y \in \mathbb{R}_\alpha$ with $u, v \in \mathbb{R}$, $x = \alpha(u)$, $y = \alpha(v)$. Therefore, α and α^{-1} preserve basic operations and order.

Remark 2.1. Since the generator α and α^{-1} are order preserving, for any two elements x and y in $\mathbb{R}_\alpha$, $x \dot{\leq} y$ if and only if $x \leq y$.

Let (X, d^α) be a non-Newtonian metric space. For any $\varepsilon \dot{>} \dot{0}$ and any $x \in X$ the set

$$B_\alpha(x, \varepsilon) = \{y \in X : d^\alpha(x, y) \dot{<} \varepsilon\}$$

is called an *α-open ball of center x and radius ε*. A topology on X is obtained easily by defining open sets as in the classical metric spaces.

Now, let us emphasize that former topology is obtained by real-valued metric and vice versa.

Theorem 2.2. *For any generator α on $\mathbb{R}$ and for any non-empty set X*
(1) If d^α is a non-Newtonian metric on X, then $d = \alpha^{-1} \circ d^\alpha$ is a metric on X,
(2) If d is a metric on X, then $d^\alpha = \alpha \circ d$ is a non-Newtonian metric on X.

Proof. It is obvious that α and α^{-1} are one-to-one and order preserving. □

Corollary 2.3. *For any generator α on $\mathbb{R}$ and, let d^α and d be a non-Newtonian metric and a metric on a non-empty set X, respectively, as in Theorem 2.2. If τ_α and τ are metric topologies induced by d^α and d, respectively, then $\tau_\alpha = \tau$.*

Proof. Since $\delta_\varepsilon = \alpha^{-1}(\varepsilon) > 0$ and $\varepsilon_\delta = \alpha(\delta) \dot{>} \dot{0}$ for all $\varepsilon \dot{>} \dot{0}, \delta > 0$, we have

$$\begin{aligned}
B_\alpha(x, \varepsilon_\delta) &= \{y \in X : d^\alpha(x, y) \dot{<} \varepsilon_\delta\} = \{y \in X : \alpha\left(d(x, y)\right) \dot{<} \alpha(\delta)\} \\
&= \{y \in X : d(x, y) < \delta_\varepsilon\} = B(x, \delta_\varepsilon)
\end{aligned}$$

for all $x \in X, \varepsilon \dot{>} \dot{0}, \delta > 0$. Therefore, $\tau_\alpha = \tau$. □

Corollary 2.4. *Under the hypothesis of Corollary 2.3, the topological properties of (X, d) and (X, d^α) are equivalent. In particular, for a sequence (x_n) in X and for an element $x \in X$*
(1) $x_n \overset{d^\alpha}{\to} x$ if and only $x_n \overset{d}{\to} x$,
(2) (x_n) is d^α-Cauchy if and only if (x_n) is d-Cauchy, and
(3) (X, d^α) is complete if and only if (X, d) is complete.

3. Conclusion

The topological results obtained by non-Newtonian metrics (particularly multiplicative metrics) as in [1–5, 7–13] are equivalent the ones obtained by metrics. In [1, 2, 5, 7–9, 11–13] some results of the multiplicative metric and in [3] some results of the non-Newtonian metric have been obtained for the fixed point theory. Those results are direct consequences of Theorem 2.2 and Corollary 2.4 since any type of contraction mapping for the non-Newtonian metric space is also a contraction mapping for the metric space and vice versa. For example, the non-Newtonian contraction $T : X \to X$ as in [3] is the classical Banach contraction since

$$d^\alpha(T(x), T(y)) \dot{\leq} k \dot{\times} d^\alpha(x, y) \;\Leftrightarrow\; d(T(x), T(y)) \leq \lambda.d(x, y) \tag{3.1}$$

for all $x, y \in X$ where $k \in [\alpha(0), \alpha(1))$ is constant, $d = \alpha^{-1} \circ d^\alpha$ and $\lambda = \alpha^{-1}(k)$. In particular, by Remark 2.1 and by (3.1), the multiplicative contraction $T : X \to X$ as in [4] is the classical Banach contraction since

$$\begin{aligned} d^{\exp}(T(x), T(y)) \leq d^{\exp}(x, y)^\lambda \;&\Leftrightarrow\; d^{\exp}(T(x), T(y)) \dot{\leq} d^{\exp}(x, y)^\lambda = k \dot{\times} d^{\exp}(x, y) \\ &\Leftrightarrow\; d(T(x), T(y)) \leq \lambda.d(x, y) \end{aligned}$$

for all $x, y \in X$ where $\lambda \in [0, 1)$ is constant, $d = \ln \circ d^{\exp}$ and $\lambda = \ln k$. In this way we can obtain most of the non-Newtonian metric results and most of the multiplicative metric results applying corresponding properties from the metric setting.

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