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Improved bounds for the number of spanning trees of graphs

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ABSTRACT. For a given a simple connected graph, we present some new bounds via a new approach for the number of spanning trees. Usage this approach presents an advantage not only to derive old and new bounds on this topic but also gives an idea how some previous results in similar area can be developed.

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1. INTRODUCTION

Let G be a simple connected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \dots, e_m\}$ be the edge set of G . If any two vertices v_i and v_j of G are adjacent, that is, $v_i v_j \in E(G)$, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i , denoted by d_i , is the number of the vertices adjacent to v_i .

Let $R_\alpha = R_\alpha(G) = \sum_{v_i \sim v_j} (d_i d_j)^\alpha$ be the *general Randić index* ([3]) of the graph G , where $\alpha \neq 0$ is a fixed real number.

Note that the *Randić index* $R_{-1} = R_{-1}(G) = \sum_{v_i \sim v_j} \frac{1}{d_i d_j}$ is also well studied in the literature. For more details on R_{-1} , see ([6, 22]).

The *Laplacian matrix* of the graph G is the matrix $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the $(0, 1)$ -adjacency matrix and the diagonal matrix of vertex degrees of G , respectively. The *normalized Laplacian matrix* of G is defined as $l(G) = D(G)^{-\frac{1}{2}} L(G) D(G)^{\frac{1}{2}}$, where $D(G)^{-\frac{1}{2}}$ is the matrix which is obtained by taking $(-\frac{1}{2})$ power of each entry of $D(G)$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \mu_n = 0$ be the *Laplacian eigenvalues* and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = 0$ be the *normalized Laplacian eigenvalues*. For more details on Laplacian and normalized Laplacian eigenvalues, see [8, 10, 18, 19].

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Among various indices in mathematical chemistry, the *Kirchhoff index* $K_f(G)$ and a relative of it, the *degree Kirchhoff index* $K'_f(G)$, have received a great deal of attention, recently. For a connected undirected graph G , the *Kirchhoff index* was defined by Klein and Randić ([16]) as

$$K_f = K_f(G) = \sum_{i < j} r_{ij},$$

where r_{ij} is the effective resistance of the edge $v_i v_j$. We refer the reader to the references ([1, 16]), and their bibliographies, to get a taste of the variety of approaches used to study this descriptor. In [26], Zhou et al. studied the extremal graphs with given matching number, connectivity and the minimal Kirchhoff index. Also in [21] and [23], the authors determined independently the extremality on the unicyclic graphs with respect to the Kirchhoff index. Moreover, in [25], Zhou et al. presented some lower bounds for the Kirchhoff index of a connected (molecular) graph via the number of vertices (atoms), the number of edges (bands), valency (maximum vertex degree), connectivity and chromatic number.

The *degree Kirchhoff index* was proposed by Chen and Zhang in [7], is defined as

$$K'_f(G) = \sum_{i \sim j} d_i d_j r_{ij}.$$

The *degree Kirchhoff index* has been taken attention as much as the *Kirchhoff index*. It can be seen the reference [7] for some bounds on the degree Kirchhoff index and for some relations on the degree Kirchhoff and Kirchhoff indices. We finally give the reference [20] for further studies over degree Kirchhoff index.

We just want to remind the expression of Kirchhoff index in terms of the Laplacian eigenvalues (see [15]) as in the equality

$$K_f = K_f(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}. \quad (1.1)$$

Moreover, in [7], by considering normalized Laplacian eigenvalues, the degree Kirchhoff index is defined as

$$K'_f = K'_f(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\lambda_i}. \quad (1.2)$$

Hence, by taking into account (1.1) and (1.2), we can easily conclude that the degree Kirchhoff index is the normalized Laplacian analogue of the ordinary Kirchhoff index.

The *number of spanning trees*, $t(G)$, of a graph G is equal to the total number of distinct spanning subgraphs of G that are trees. This quantity is also known as the complexity of G , and is given by the following formula in terms of the Laplacian eigenvalues ([9]):

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i.$$

It is well known that the number of spanning trees of G can be expressed by the normalized Laplacian eigenvalues as ([8, 9])

$$t(G) = \left(\frac{\Delta'}{2m} \right) \prod_{i=1}^{n-1} \lambda_i$$

where $\Delta' = \prod_{i=1}^n d_i$.

In the literature ([11–13, 17, 24]), it was obtained so many bounds on $t(G)$.

We organize this paper in the following way. In Section 2, we give some previously known results which will be needed later. In Section 3, we obtain some bounds on the number of spanning trees of connected graphs in terms of the number of vertices, the number of edges, degree Kirchhoff index and Randić index (R_{-1}). We also improve some bounds which was obtained previously for the number of spanning trees of graphs.

2. PRELIMINARIES

In this section, we give some useful lemmas which will be used later.

Let a_1, a_2, \dots, a_r be positive real numbers. For a positive number k among the values $1 \leq k \leq r$, let us suppose that each P_k is defined as in the following:

$$\begin{aligned} P_1 &= \frac{a_1 + a_2 + \cdots + a_r}{r}, \\ P_2 &= \frac{a_1 a_2 + a_1 a_3 + \cdots + a_1 a_r + a_2 a_3 + \cdots + a_{r-1} a_r}{\frac{1}{2}r(r-1)}, \\ &\vdots \\ P_{r-1} &= \frac{a_1 a_2 \cdots a_{r-1} + a_1 a_2 \cdots a_{r-2} a_r + \cdots + a_2 a_3 \cdots a_{r-1} a_r}{r}, \\ P_r &= a_1 a_2 \cdots a_r. \end{aligned}$$

Hence the *arithmetic mean* is simply P_1 while the *geometric mean* is $P_r^{1/r}$. In fact the following famous lemma (see [2]) gives a relationship among them.

Lemma 2.1 (Maclaurin's Symmetric Mean Inequality [2]). *For $a_1, a_2, \dots, a_r \in \mathbb{R}^+$, it is true that*

$$P_1 \geq P_2^{1/2} \geq P_3^{1/3} \geq \cdots \geq P_r^{1/r}.$$

Equality among them holds if and only if $a_1 = a_2 = \cdots = a_r$.

Let K_n and $K_{p,q}$ ($p+q=n$) denote the complete graph and the complete bipartite graph, respectively.

Lemma 2.2 ([10]). *Let G be a graph with n vertices and without isolated vertices. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ if and only if G is a complete graph K_n .*

Lemma 2.3 ([10]). *Let G be a connected graph with $n \geq 2$ vertices. Then $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

3. MAIN RESULTS

Now we present our main results on $t = t(G)$.

Theorem 3.1. *Let G be a connected graph with n vertices, m edges and degree Kirchhoff index K'_f . Then*

$$(2m)^{n-2} \Delta' \left(\frac{n-1}{K'_f} \right)^{n-1} \leq t(G) \leq \frac{\Delta'(n-1)}{K'_f} \left(\frac{n}{n-1} \right)^{n-2} \quad (3.1)$$

with equality holding if and only if $G \cong K_n$.

Proof. Upper bound: Setting $r = n-1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n-1$, by Lemma 2.1, we get

$$P_1 \geq P_{n-2}^{1/(n-2)},$$

where

$$P_1 = \frac{\sum_{i=1}^{n-1} \lambda_i}{n-1}$$

and

$$\begin{aligned} P_{n-2} &= \frac{\sum_{i=1}^{n-1} \prod_{j=1, j \neq n-i+1}^{n-1} \lambda_j}{n-1} \\ &= \frac{\prod_{j=1}^{n-1} \lambda_j}{n-1} \times \sum_{i=1}^{n-1} \frac{1}{\lambda_i} = \left[\frac{tK'_f}{\Delta'(n-1)} \right]. \end{aligned}$$

From the above,

$$\frac{n}{n-1} \geq \left[\frac{tK'_f}{\Delta'(n-1)} \right]^{\frac{1}{n-2}},$$

that is,

$$t \leq \frac{\Delta'(n-1)}{K'_f} \left(\frac{n}{n-1} \right)^{n-2}$$

which gives the upper bound in (3.1). First part of the proof is over.

Lower Bound: Now setting $r = n - 1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n - 1$, in Lemma 2.1, we have

$$P_{n-2}^{1/n-2} \geq P_{n-1}^{1/n-1}$$

where

$$P_{n-2} = \frac{\prod_{j=1}^{n-1} \lambda_j}{n-1} \times \sum_{i=1}^{n-1} \frac{1}{\lambda_i} = \frac{tK'_f}{\Delta'(n-1)}$$

and

$$P_{n-1} = \prod_{i=1}^{n-1} \lambda_i = \frac{2mt}{\Delta'}.$$

From this, we write

$$\left(\frac{2mt}{\Delta'} \right)^{\frac{1}{n-1}} \leq \left(\frac{tK'_f}{\Delta'(n-1)} \right)^{\frac{1}{n-2}}$$

and hence we obtain the required result.

Now suppose that both the equality holds in (3.1). Then all the inequalities in above must be equalities.

The inequality for both lower and upper bounds, $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$, by Lemma 2.1. Consequently, from Lemma 2.2, we have $G \cong K_n$.

Conversely, one can see easily that the equality holds in (3.1) for complete graph K_n . \square

In [4], for a non-zero real number α , it was obtained the sum of the α -th power of the non-zero normalized Laplacian eigenvalues, $s_\alpha(G)$, of a graph G without isolated vertices as

$$s_\alpha(G) = \sum_{i=1}^h \lambda_i^\alpha$$

where h is the number of non-zero normalized Laplacian eigenvalues of G .

Now we present the following result which contains s_{-2} particularly for $\alpha = -2$.

Theorem 3.2. *Let G be a connected graph with n vertices, m edges and degree Kirchhoff index K'_f . Then*

$$\frac{\Delta'}{2m} \left[\frac{(n-1)(n-2)}{\left(\frac{K'_f}{2m} \right)^2 - s_{-2}} \right]^{\frac{n-1}{2}} \leq t(G) \leq \frac{\left(\frac{n}{n-1} \right)^{n-3} \Delta' (n-1)(n-2)}{2m \left[\left(\frac{K'_f}{2m} \right)^2 - s_{-2} \right]} \quad (3.2)$$

with equality holding if and only if $G \cong K_n$.

Proof. Upper Bound : Setting $r = n - 1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n - 1$, by Lemma 2.1, we get

$$P_1 \geq P_{n-3}^{1/n-3},$$

where

$$P_1 = \frac{\sum_{i=1}^{n-1} \lambda_i}{n-1}$$

and

$$P_{n-3} = \frac{\prod_{i=1}^{n-1} \lambda_i}{(n-1)(n-2)} \left[\left(\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \right)^2 - \sum_{i=1}^{n-1} \frac{1}{\lambda_i^2} \right].$$

From the above

$$\frac{n}{n-1} \geq \left\{ \frac{2mt}{\Delta' (n-1)(n-2)} \left[\left(\frac{K'_f}{2m} \right)^2 - s_{-2} \right] \right\}^{\frac{1}{n-3}},$$

that is,

$$t(G) \leq \frac{\left(\frac{n}{n-1} \right)^{n-3} (n-1)(n-2) \Delta'}{\left(\frac{K'_f}{2m} \right)^2 - s_{-2}} \frac{1}{2m}$$

which gives the upper bound in (3.2). First part of the proof is over.

Lower Bound: Now we take $r = n - 1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n - 1$, in Lemma 2.1, we have

$$P_{n-3}^{1/n-3} \geq P_{n-1}^{1/n-1},$$

where

$$P_{n-1} = \prod_{i=1}^{n-1} \lambda_i$$

and

$$P_{n-3} = \frac{\prod_{i=1}^{n-1} \lambda_i}{(n-1)(n-2)} \left[\left(\sum_{i=1}^{n-1} \frac{1}{\lambda_i} \right)^2 - \sum_{i=1}^{n-1} \frac{1}{\lambda_i^2} \right].$$

From this, we write

$$\left(\frac{2mt}{\Delta'} \right)^{\frac{1}{n-1}} \leq \left\{ \frac{2mt}{\Delta' (n-1)(n-2)} \left[\left(\frac{K'_f}{2m} \right)^2 - s_{-2} \right] \right\}^{\frac{1}{n-3}}$$

where $s_{-2} = \sum_{i=1}^{n-1} \frac{1}{\lambda_i^2}$. Hence we obtain the required result.

Suppose that both the equality holds in (3.2). Then all the inequalities in above must be equalities.

The inequality for both lower and upper bounds, $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$, by Lemma 2.1. Consequently, from Lemma 2.2, we have $G \cong K_n$.

Conversely, one can see easily that the equality holds in (3.2) for complete graph K_n . \square

Now we give an another upper bound for $t(G)$ that we think as good result.

Theorem 3.3. Let G be a connected graph with n vertices, m edges, degree Kirchhoff index K'_f and Randić index R_{-1} . Then

$$t(G) \leq \frac{\Delta' (n-1)}{K'_f} \left[\frac{n^2 - 2 - 2R_{-1}}{(n-1)(n-2)} \right]^{\frac{n-2}{2}} \quad (3.3)$$

with equality holding if and only if $G \cong K_n$.

Proof. Taking $r = n - 1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n - 1$ and using $P_2^{1/2} \geq P_{n-2}^{1/n-2}$, by Lemma 2.1, where

$$\begin{aligned} P_2 &= \frac{1}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=1, j \neq i}^{n-1} \lambda_i \lambda_j \\ &= \frac{1}{(n-1)(n-2)} \left[\left(\sum_{i=1}^{n-1} \lambda_i \right)^2 - \sum_{i=1}^{n-1} \lambda_i^2 \right] \\ &= \frac{1}{(n-1)(n-2)} [n^2 - n - 2R_{-1}] \end{aligned}$$

as $\sum_{i=1}^{n-1} \lambda_i = n$ and $\sum_{i=1}^{n-1} \lambda_i^2 = n + 2R_{-1}$

and

$$\begin{aligned} P_{n-2} &= \frac{\sum_{i=1}^{n-1} \prod_{j=1, j \neq n-i+1}^{n-1} \lambda_j}{n-1} \\ &= \frac{\prod_{j=1}^{n-1} \lambda_j}{n-1} \times \sum_{i=1}^{n-1} \lambda_i = \frac{t K'_f}{\Delta' (n-1)}, \end{aligned}$$

we obtain the inequality (3.3).

The equality holds in (3.3) if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$ from Lemma 2.1. Also, by Lemma 2.2, $G \cong K_n$. Conversely the equality follows easily. \square

Remark 3.4. We take $r = n - 1$ and $a_i = \lambda_i$, $i = 1, 2, \dots, n - 1$, in Lemma 2.1. If we consider the inequality $P_1 \geq P_{n-1}^{1/n-1}$, we have the following result for $t(G)$ which was obtained in [14]

$$t(G) \leq \frac{\Delta'}{2m} \left(\frac{n}{n-1} \right)^{n-1}. \quad (3.4)$$

Similarly, if we consider the inequality $P_2^{1/2} \geq P_{n-2}^{1/n-2}$, we get the following upper bound

$$\frac{\Delta'}{2m} \left[\frac{n^2 - (n-2)(n+2R_{-1})}{(n-1)} \right]^{\frac{n-1}{2}} \leq t(G) \leq \frac{\Delta'}{2m} \left[\frac{n^2 - n - 2R_{-1}}{(n-1)(n-2)} \right]^{\frac{n-1}{2}} \quad (3.5)$$

which was obtained in [5].

We now consider connected bipartite graphs.

Theorem 3.5. Let G be a connected bipartite graphs with $n > 2$ vertices, m edges and degree Kirchhoff index K'_f . Then

$$\frac{\Delta'}{m} \left[\frac{n-2}{\frac{K'_f}{2m} - \frac{1}{2}} \right]^{n-2} \leq t(G) \leq \frac{\Delta'}{m} \left[\frac{n-2}{\frac{K'_f}{2m} - \frac{1}{2}} \right] \quad (3.6)$$

with equality holding if and only if $G \cong K_{p,q}$.

Proof. Lower Bound : Setting $r = n - 2$ and $a_i = \lambda_i$, $i = 2, 3, \dots, n - 1$, by Lemma 2.1, we have

$$P_{n-3}^{1/n-3} \geq P_{n-2}^{1/n-2} \quad (3.7)$$

where

$$\begin{aligned} P_{n-3} &= \frac{\sum_{i=2}^{n-1} \prod_{j=2, j \neq n-i+1}^{n-1} \lambda_j}{n-2} \\ &= \frac{\prod_{j=2}^{n-1} \lambda_j}{n-2} \times \sum_{i=2}^{n-1} \frac{1}{\lambda_i} = \frac{2mt}{\Delta' \lambda_1 (n-2)} \left[\frac{K'_f}{2m} - \frac{1}{\lambda_1} \right] \end{aligned}$$

and

$$P_{n-2} = \prod_{i=2}^{n-1} \lambda_i = \frac{2mt}{\Delta' \lambda_1}.$$

Since G is bipartite, we also have $\lambda_1 = 2$ [8]. Then, we combine this fact and the inequality (3.7), we arrive the result.

Upper Bound: Now using $P_1 \geq P_{n-3}^{1/n-3}$ with $r = n - 2$ and $a_i = \lambda_i$, $i = 2, 3, \dots, n - 1$ from Lemma 2.1, for bipartite graph G , where

$$P_1 = \frac{\prod_{i=2}^{n-1} \lambda_i}{n-2} = \frac{n - \lambda_1}{n-2}$$

and P_{n-3} is as in the proof of lower bound, we have the result.

All equalities hold if and only if $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1}$.

Now we suppose that all equalities hold. Then, by Lemma 2.3, we conclude that $G \cong K_{p,q}$.

Conversely, we can easily see that the equalities hold for the complete bipartite graph $K_{p,q}$. \square

Example 3.6. Let G be a graph with $V_G = \{1, 2, 3, 4\}$ and $E_G = \left\{ \{1, 2\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \right\}$ and H be a graph with $V_H = \{1, 2, 3, 4\}$ and $E_H = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$. For these graphs $t(G) = 3$ and $t(H) = 1$. The lower and upper bounds for $t(G)$ are as follows:

<i>Upper bounds</i>	(3.1)	(3.2)	(3.3)	(3.4)	(3.5)
G	3.14	3	3	3.55	3.308
H	1.07	1	1	1.185	1.075
<i>Lower bounds</i>	(3.1)	(3.2)	(3.5)		
G	2.76	2.76	2.6		
H	0.864	0.918	0.769		

The above table shows that in some cases, the upper bounds (3.2) and (3.3) are the best among the above mentioned bounds for $t(G)$. Moreover we see that the upper bounds (3.2) and (3.3), in many graph examples, are equal to $t(G)$. We actually see that our results are better than the results which were obtained previously.

Now we give some results for K'_f which are obtained from the bounds (3.1), (3.2) and (3.3), respectively.

Corollary 3.7. Let G be a connected graph with n vertices, m edges and t spanning trees. Then

$$(n-1) \left[\frac{(2m)^{n-2} \Delta'}{t} \right]^{1/n-1} \leq K'_f \leq \frac{\Delta'(n-1)}{t} \left(\frac{n}{n-1} \right)^{n-2}, \quad (3.8)$$

$$2m \sqrt{(n-1)(n-2) \left(\frac{\Delta'}{2mt} \right)^{\frac{2}{n-1}} + s_{-2}} \leq K'_f \quad (3.9)$$

$$K'_f \leq 2m \sqrt{\frac{\left(\frac{n}{n-1} \right)^{n-3} \Delta' (n-1)(n-2) + s_{-2}}{2mt}} \quad (3.10)$$

and

$$K'_f \leq \frac{\Delta'(n-1)}{t} \left(\frac{n^2 - n - 2R_{-1}}{(n-1)(n-2)} \right)^{\frac{n-2}{2}}. \quad (3.11)$$

with equality holding if and only if $G \cong K_n$.

For trees and unicyclic graphs, we obtain the following results over K'_f

Corollary 3.8. Let T be a tree of order n with m edges. Then

$$(n-1) \left[(2m)^{n-2} \Delta' \right]^{1/n-1} \leq K'_f(T) \leq \Delta'(n-1) \left(\frac{n}{n-1} \right)^{n-2},$$

$$2m \sqrt{(n-1)(n-2) \left(\frac{\Delta'}{2m} \right)^{\frac{2}{n-1}} + s_{-2}} \leq K'_f(T)$$

$$K'_f(T) \leq 2m \sqrt{\frac{\left(\frac{n}{n-1} \right)^{n-3} \Delta' (n-1)(n-2)}{2m} + s_{-2}}$$

and

$$K'_f(T) \leq \Delta'(n-1) \left(\frac{n^2 - n - 2R_{-1}}{(n-1)(n-2)} \right)^{\frac{n-2}{2}}.$$

Proof. Since T is a tree, $t = 1$. From (3.8), (3.9), (3.10) and (3.11) we get the required result. \square

Corollary 3.9. *Let U be a unicyclic graph of order n with m edges. Then*

$$(n-1) \left[\frac{(2m)^{n-2} \Delta'}{n} \right]^{1/n-1} \leq K'_f(U) \leq \frac{\Delta'(n-1)}{3} \left(\frac{n}{n-1} \right)^{n-2},$$

$$2m \sqrt{(n-1)(n-2) \left(\frac{\Delta'}{2mn} \right)^{\frac{2}{n-1}} + s_{-2}} \leq K'_f(U)$$

$$K'_f(U) \leq 2m \sqrt{\frac{\left(\frac{n}{n-1} \right)^{n-3} \Delta' (n-1)(n-2)}{6m} + s_{-2}}$$

and

$$K'_f(U) \leq \frac{\Delta'(n-1)}{3} \left(\frac{n^2 - n - 2R_{-1}}{(n-1)(n-2)} \right)^{\frac{n-2}{2}}.$$

Proof. For unicyclic graph U , $3 \leq t \leq n$. From (3.8), (3.9), (3.10) and (3.11), we get the required result. \square

In [4], it was obtained the following results for K'_f :

$$K'_f \geq \frac{2m}{P} + 2(n-2)m \left(\frac{\Delta' P}{2mt} \right)^{\frac{1}{n-2}} \quad (3.12)$$

and

$$K'_f \geq \frac{2m}{P} + \frac{2(n-2)^2 m}{P} \quad (3.13)$$

where $P = 1 + \sqrt{\frac{2}{n(n-1)} R_{-1}}$.

For graphs in Example 3.6, we consider the following table where $K'_f(G) = 20.3$ and $K'_f(H) = 15$.

<i>Lower</i>				
<i>Bounds</i>	(3.8)	(3.9)	(3.12)	(3.13)
$K'_f(G)$	19.05	20.26	19.09	18.01
$K'_f(H)$	14.28	14.73	14.39	13.52
<i>Upper</i>				
<i>bounds</i>	(3.8)	(3.10)	(3.11)	
$K'_f(G)$	21.3	20.33	20.33	
$K'_f(H)$	16	15	15	

Finally, from (3.6), considering connected bipartite graphs, we give the following result for K'_f .

Corollary 3.10. *Let G be a connected bipartite graph with n vertices, m edges and t spanning trees. Then*

$$m + 2(n-2)m \left(\frac{\Delta'}{mt} \right)^{\frac{1}{n-2}} \leq K'_f \leq m + 2(n-2) \frac{\Delta'}{t}$$

with equality holding if and only if $G \cong K_{p,q}$.

Remark 3.11. We note that the lower bound which is obtained in Corollary 3.10 is same with the lower bound in [4, Corollary 3.6].

Remark 3.12. Note that if G is a k -regular graph, then $\lambda_i = \frac{\mu_i}{k}$ for $i = 1, 2, \dots, n$ (see [8]). Hence we have $K_f = \frac{n}{2m} \frac{1}{k^{n-1}} K'_f$ for any k -regular graph. Therefore, in the case of G is regular, results obtained for K'_f can be immediately re-stated for K_f .

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

REFERENCES

- [1] Banchev, D., Balaban, A.T., Liu, X., Klein, D.J., *Molecular cyclicity and centrality of polycyclic graphs. I. Cyclicity based on resistance distances or reciprocal distances*, Int. J. Quantum Chem., **50**(1994), 2978–2981. [1](#)
- [2] Biler, P., Witkowski, A., *Problems in Mathematical Analysis*, New York, 1990. [2](#), [2.1](#)
- [3] Bollobas, B., Erdős, P., *Graphs of extremal weights*, Ars Combinatoria, **50**(1998), 225–233. [1](#)
- [4] Bozkurt, Ş.B., Bozkurt, D., *On the sum of powers of normalized Laplacian eigenvalues of graphs*, MATCH Commun. Math. Comput. Chem., **68**(2012), 917–930. [3](#), [3](#), [3.11](#)
- [5] Bozkurt, Ş. B., Bozkurt, D., *On the number of spanning trees of graphs*, The Scientific World Journal, **2014**(2014), Article ID 294038, 5 pages. [3.4](#)
- [6] Covers, M., Fallat, S., Kirkland, S., *On the normalized Laplacian energy and the general Randic index of graphs*, Linear Algebra and its Appl., **433**(2010), 172–190. [1](#)
- [7] Chen, H., Zhang, F., *Resistance distance and the normalized Laplacian spectrum*, Discrete App. Math., **155**(2007), 654–661. [1](#), [1](#)
- [8] Chung, F.R.K., *Spectral Graph Theory*, CBMS Lecture Notes, Providence, 1997. [1](#), [1](#), [3](#), [3.12](#)
- [9] Cvetković, D., Doob, M., Sachs, H., *Spectra of Graphs*, Academic Press, New York, 1980. [1](#)
- [10] Das, K.Ch., Maden, A.D., Bozkurt, Ş. B., *On the normalized Laplacian eigenvalues of graphs*, Ars Combinatoria, in press. [1](#), [2.2](#), [2.3](#)
- [11] Das, K.Ch., *A sharp upper bound for the number of spanning trees of a graph*, Graphs and Combinatorics, **23**(2007), 625–632. [1](#)
- [12] Das, K.Ch., Çevik, A.S., Cangül, I. N., *The number of spanning trees of a graph*, Journal of Inequalities and Applications 2013, **395**(2013), 13 pages. [1](#)
- [13] Feng, L., Yu, G., Jiang, Z., Ren, L., *Sharp upper bounds for the number of spanning trees of a graph*, Applicable Analysis and Discrete Mathematics, **2**(2008), 255–259. [1](#)
- [14] Grone R., Merris, R., *A bound for the complexity for a simple graph*, Discrete Mathematics, **69**(1988), 97–99. [3.4](#)
- [15] Gutman, I., Mohar, B., *The quasi-Wiener and the Kirchhoff indices Coincide*, J. Chem. Inf. Comput. Sci., **36**(1996), 982–985. [1](#)
- [16] Klein, D.J., Randić, M., *Resistance distance. Applied graph theory and discrete mathematics in chemistry (Saskatoon, SK, 1991)*, J. Math. Chem., **12**(1993), 81–95. [1](#)
- [17] Li, J., Shiu, W. C., Chang, A., *The number of spanning trees of a graph*, Applied Mathematics Letters, **23**(2010), 286–290. [1](#)
- [18] Merris, R., *Laplacian matrices of graphs. A Survey*, Linear Algebra and its Appl., **197**(1994), 143–176. [1](#)
- [19] Merris, R., *A survey of graph Laplacians*, Linear and Multilinear Algebra, **39**(1995), 19–31. [1](#)
- [20] Palacios, J., Renom, J.M., *Another look at the degree Kirchhoff index*, Int. J. Quantum Chem., **111**(2011), 3453–3455. [1](#)
- [21] Yang, Y.J., Jiang, X.Y., *Unicyclic graphs with extremal Kirchhoff index*, MATCH Commun. Math. Comput. Chem., **60**(2008), 107–120. [1](#)
- [22] Yu, G., Feng, L., *Randić index and eigenvalues of graphs*, Rocky Mount. J. Math., **40**(2010), 713–721. [1](#)
- [23] Zhang, W., Deng, H., *The second maximal and minimal Kirchhoff indices of unicyclic graphs*, MATCH Commun. Math. Comput. Chem., **61**(2009), 683–695. [1](#)
- [24] Zhang, X.D., *A new bound for the complexity of a graph*, Utilitas Mathematica, **67**(2005), 201–203. [1](#)
- [25] Zhou, B., *On sum of powers of the Laplacian eigenvalues of graphs*, Linear Algebra and its Appl., **429**(2008), 2239–2246. [1](#)
- [26] Zhou, B., Trinajstić, N., *The Kirchhoff index and the matching number*, Int. J. Quantum Chem., **109**(2009), 2978–2981. [1](#)