TITLE: Tubular Surfaces According to a Focal Curve in E^3

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# Tubular Surfaces According to a Focal Curve in E³

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Abstract. A spine curve moves through the middle of a canal or a tubular surface. It might be asked whether it is possible to carry a spine curve over a tubular surface. For a tubular surface, we have seen that it can be done. In this study, we have given the general equations of a canal surface and a tubular surface according to a focal curve. In this case, we found the fundamental curvatures of a tubular surface. We gave theorems and proofs about the focal curve being a special curve.

## 1. Introduction

A canal surface appears as the envelope of a collection of 1-parameter spheres. The center of these spheres is on a curve, which is named the center curve or the spine curve. If the radii of these spheres are constant, they are named a tubular surfaces. Canal surfaces occupy an important place in the surface theory of differential geometry. These surfaces provide a great convenience in engineering applications. Since canal surfaces were first described by the French mathematician Gaspard Monge in 1850, much work has been done algebraic and geometrical in Euclidean and non-Euclidean spaces, [1–3, 5, 8, 11, 12]. For these purposes, we carry out this study.

In studies done so far, the spine curve moves through the middle of a canal or a tubular surface. In this study, we have seen that if we take the focal curve instead of the spine curve of a tubular surface , it can be expressed according to a curve traveling on a tubular surface. We reinforced this with results, theorems and examples.

## 2. Preliminaries

In this section, some basic notions of regular curves, which have all velocity vectors different from zero, are given.

**Definition 2.1.** Let us give a function $f : I \longmapsto \mathbf{R}$. The derivative of the function $f(\delta)$ of a variable $\delta$ is

$$f'(\delta) = \frac{\mathrm{d}f}{\mathrm{d}\delta} = \lim_{\triangle\delta \to 0} \frac{f(\delta + \triangle\delta) - f(\delta)}{\triangle\delta},$$

provided that the limit exists and the function $f(\delta)$ is said to differentiable at $\delta_0$. If it is differenetiable at every point in domain I, then it is said to be differntiable on I.

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For $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in \mathbf{R}^3$, the scalar product of $X$, and $Y$ is defined by

$$g(X, Y) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The cross product of $X$, and $Y$ is defined as

$$X \times Y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

Let a curve $\alpha = \alpha(\delta) : I \longrightarrow \mathbf{E}^3$ be given by arclength $\delta$. In other words, let the curve $\alpha$ be a unit speed curve. Its velocity vector is $T(\delta) = \alpha'(\delta) = \dfrac{d\alpha(\delta)}{d\delta}$. The principal normal $N$ is defined as $\dfrac{T'}{\|T'\|}$ and the binormal vectors $B$ is defined as $N \times T$. The family $\{T, N, B\}$ is an orthonormal triad and is named the Frenet frame.

For a curve $\alpha$, the rate of change of the Frenet-Serret vector equations may be expressed as

$$\begin{aligned}
T' &= \kappa N, \\
N' &= -\kappa T + \tau B, \\
B' &= -\tau N,
\end{aligned}$$

where the coefficients $\kappa$, and $\tau$ are the first, and the second curvatures of the curve $\alpha$, respectively [8].

In $\mathbf{R}^3$ curvatures of an arbitrary curve $X$ is derived as

$$\kappa = \frac{\|X' \times X''\|}{\|X'\|^3}, \qquad \tau = \frac{g(X' \times X'', X''')}{\|X' \times X''\|^2}, \tag{2.1}$$

where $\times$ is the cross product in $\mathbf{R}^3$ [5, 9, 10]. According to (2.1), the first, and the second curvatures of the curve $\alpha$

$$\kappa_\alpha = \|\alpha''\| \text{ and } \tau_\alpha = \frac{g(\alpha' \times \alpha'', \alpha''')}{\|\alpha''\|^2},$$

respectively

If $\alpha'$, and $\alpha''$ are linearly independent in I, then the curve $\alpha$ is named well defined [11].

From now on, we will assume that the given curves are good.

## 3. Focal Curves in $\mathbf{E}^3$

Let $\alpha = \alpha(\delta) : I \longrightarrow \mathbf{E}^3$ be given by arclength $\delta$. The points of $C_\alpha$ are the centers of the osculating spheres of $\alpha$, and it is named the focal curve of the curve $\alpha$. The sphere $\|C_\alpha - \alpha\|^2 = r^2$ with the center $C_\alpha$ is maned the osculating sphere [12].

Let us assume

$$f(\delta) = \frac{1}{2} \left( \|C_\alpha(\delta) - \alpha(\delta)\|^2 - r^2 \right).$$

If there are infinitely close joint 4-points between the curve $\alpha$ with its osculating sphere at $\delta = \delta_0$, then we have

$$f(\delta_0) = f'(\delta_0) = f''(\delta_0) = f'''(\delta_0) = 0.$$

The plane spanned by both the tangent vector, and the principal normal vector of a curve is named the osculating plane. A point of a smooth curve in $\mathbf{E}^3$ for which the derivative of the curve of order 3 belongs to the osculating plane is named a flattening.

If there are infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $\delta = \delta_0$ of the curve $\alpha$, it is named a vertex of the curve. Conversely, unless there are infinitely close 5-points in the neighbourhood of a point with the osculator sphere at $\delta = \delta_0$ of the curve $\alpha$, it is named non-vertex of the curve.

From now on, we assume that all points of the given curves are non-vertex.

**Lemma 3.1.** [1] *Let $\alpha$ be given by arclength in $\mathbf{E}^3$, and its Frenet frame be $\{T, N, B\}$. Then, the focal curve $C_\alpha$ of $\alpha$ is*

$$C_\alpha = \alpha + c_1 N + c_2 B, \tag{3.1}$$

*and the focal coefficients of $C_\alpha$ are given by*

$$c_1 = \frac{1}{\kappa_\alpha}, \qquad c_2 = c_1' \frac{1}{\tau_\alpha}, \tag{3.2}$$

*where $\kappa_\alpha \neq 0$, and $\tau_\alpha \neq 0$ are the curvature, and the torsion of the curve $\alpha$.*

**Lemma 3.2** ( [1]). *Let $\alpha = \alpha(\delta) : I \longrightarrow \mathbf{E}^3$ be given by arclength. If a non-flattening point of $\alpha$ is a vertex, then*

$$c_2' + c_1 \tau_\alpha = 0.$$

*The opposite is also possible.*

The forthcoming theorem, lemmas, and corollaries state the relations between a curve $\alpha$, and its focal curve $C_\alpha$.

**Theorem 3.3.** *Let $\alpha : I \longrightarrow \mathbf{E}^3$ be given by arclength. Let $\{T, N, B\}$ (resp. $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$) be the Frenet frame of the curve $\alpha$ (resp. $C_\alpha$). Let $\kappa_\alpha$, and $\tau_\alpha$ be the curvature, and the torsion of the curve $\alpha$, respectively. Then, we have the connections as follows;*

$$\mathbf{t} = \varepsilon_\mathbf{t} B,$$
$$\mathbf{n} = \varepsilon_\mathbf{t} \varepsilon_\mathbf{n} N,$$
$$\mathbf{b} = -\varepsilon_\mathbf{n} T,$$

*where $\varepsilon_\mathbf{t} = \dfrac{c_2' + c_1 \tau_\alpha}{\left| c_2' + c_1 \tau_\alpha \right|}$, and $\varepsilon_\mathbf{n} = \dfrac{\tau_\alpha}{\left| \tau_\alpha \right|}$.*

*Proof.* Let $\sigma$ be the arc length parameter of the focal curve $C_\alpha$. If we take the derivative of both sides of (3.1) concerning the arclength parameter $\delta$, we reach

$$\frac{dC_\alpha}{d\delta} = \frac{dC_\alpha}{d\sigma} \frac{d\sigma}{d\delta} = \left[ c_2' + c_1 \tau_\alpha \right] B, \tag{3.3}$$

and if we take the norm of both sides of (3.3), we reach

$$\frac{d\delta}{d\sigma} = \frac{1}{\left| c_2' + c_1 \tau_\alpha \right|},$$

and

$$\mathbf{t} = \varepsilon_\mathbf{t} B = \frac{\left( c_2' + c_1 \tau_\alpha \right)}{\left| c_2' + c_1 \tau_\alpha \right|} B = \frac{dC_\alpha}{d\sigma}. \tag{3.4}$$

Now, differentiating both sides of (3.4) concerning the arclength parameter $\delta$ we obtain

$$\mathbf{n} = \varepsilon_\mathbf{t} \varepsilon_\mathbf{n} N,$$

and

$$\kappa_{C_\alpha} = \frac{\left| \tau_\alpha \right|}{\left| c_2' + c_1 \tau_\alpha \right|}. \tag{3.5}$$

On the other hand, we may write

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = (\varepsilon_\mathbf{t} B) \times (\varepsilon_\mathbf{t} \varepsilon_\mathbf{n} N) = -\varepsilon_\mathbf{n} T.$$

$\square$

Let $\kappa_\alpha$, and $\tau_\alpha$ (resp. $\kappa_{C_\alpha}$, and $\tau_{C_\alpha}$) be the curvature, and the torsion of the curve $\alpha$ (resp. the first and the second curvatures of the focal curve $C_\alpha$ ). Accordingly to this, we can express the following corollaries.

**Corollary 3.4.** *Taking the derivative of (3) concerning the arclength parameter $\delta$, we obtain*

$$\kappa_\alpha = \left| \tau_{C_\alpha} \right| \left| c_2' + c_1 \tau_\alpha \right|. \tag{3.6}$$

**Corollary 3.5.** *Let $\alpha = \alpha(\delta) : I \longrightarrow \mathbf{E}^3$ be given by arclength. If the curve $\alpha$ is spherical, then*

$$\begin{aligned} r^2 &= \left\| C_\alpha - \alpha \right\|^2, \\ &= \left\| c_1 N + c_2 B \right\|^2, \\ &= c_1^2 + c_2^2, \end{aligned} \tag{3.7}$$

*where $r$ is the radius of spherical, and differentiating (3.7) with respect to the arc length parameter $\delta$, we obtain*

$$\left( r^2 \right)' = 2c_2 \left( c_2' + c_1 \tau_\alpha \right). \tag{3.8}$$

*The opposite is also true. According to (3.8), if $r$ is a constant, then*

$$c_2 = 0.$$

*Since the curve $\alpha$ is a non-vertex curve, $c_2' + c_1\tau_\alpha \neq 0$.*

**Corollary 3.6.** *If we consider (3.2) and (3.8), the focal coefficients of $c_1$, $c_2$ of the curve $\alpha$ satisfy the following matrix-vector equation*

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' - \dfrac{(r^2)'}{2c_2} \end{bmatrix} = \begin{bmatrix} 0 & -\kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix}.$$

**Corollary 3.7.** *If we consider (3.5) and (3.6), then we obtain that*

$$\frac{\kappa_{C_\alpha}}{|\tau_\alpha|} = \frac{\left|\tau_{C_\alpha}\right|}{\kappa_\alpha} = \frac{1}{\left|c_2' + c_1\tau_\alpha\right|} = \frac{2\left|c_2\right|}{\left|(r^2)'\right|}.$$

**Lemma 3.8.** *Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be given by arclength. If its osculating sphere radius $r$ is constant, then its curvature $\kappa$ is constant, and*

$$r = |c_1| = \frac{1}{\kappa_\alpha},$$

*where $c_1$ is the first focal coefficient of the focal curve $C_\alpha$, so*

$$C_\alpha' = c_1\tau_\alpha B.$$

*Proof.* Since the curve $\alpha$ is a non-vertex curve, $c_2' + c_1\tau_\alpha \neq 0$. Using (3.2), and (3.8) we find that $c_1 = \dfrac{1}{\kappa_\alpha}$ is a constant. $\qquad\square$

**Lemma 3.9.** *According to the parameter $\delta$, derivative changes of Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \nu\kappa_{C_\alpha} & 0 \\ -\nu\kappa_{C_\alpha} & 0 & \nu\tau_{C_\alpha} \\ 0 & -\nu\tau_{C_\alpha} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix},$$

*where $\nu = \dfrac{\mathrm{d}\sigma}{\mathrm{d}\delta} = \left|c_2' + c_1\tau_\alpha\right|$. If the radius of the osculating sphere $r$ is a constant, then*

$$\nu = \frac{\mathrm{d}\sigma}{\mathrm{d}\delta} = r\left|\tau_\alpha\right|,$$

*where $\delta$ and $\sigma$ are the arc length parameters of the curve $\alpha$, and the focal curve $C_\alpha$, respectively.*

## 4. Canal Surfaces in $\mathbf{E}^3$

Let us state equations for canal or tubular surface around any good curve in $\mathbf{E}^3$. Let a regular and unite speed curve $\alpha : \mathrm{I} \longrightarrow \mathbf{E}^3$ be parametrized by $s$ and the Frenet frame of the curve $\alpha$ be $\{T, N, B\}$. The derivative changes of these vector fields are given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $\kappa_\alpha$ and $\tau_\alpha$ are the curvature, and the torsion of the curve $\alpha$, respectively.

If we refer to the concepts in [2, 12], a canal surface appears as the envelope of a collection of 1-parameter spheres. Let these spheres have varying radii $r(t)$ and the centers on a curve $\alpha(t)$ which is named the spine curve. The circle where the canal surface is tangent to the spheres is named the characteristic circle denoted by $K(t)$. In this framework, with the help of Figure 1, a canal surface is parameterized as
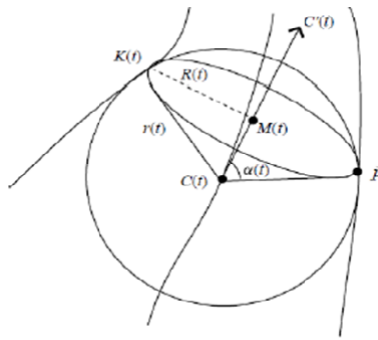
FIGURE 1. The part of a canal surface, [2]

$$
\begin{aligned}
K(t,\phi) \;=\;& \alpha(t) - r(t)r^{'}(t)\frac{\alpha^{'}(t)}{\|\alpha^{'}(t)\|} \\
&\pm \cos\phi\, r(t)\frac{\sqrt{\|\alpha^{'}(t)\|^2 - r^{'}(t)^2}}{\|\alpha^{'}(t)\|}N(t) \\
&\pm \sin\phi\, r(t)\frac{\sqrt{\|\alpha^{'}(t)\|^2 - r^{'}(t)^2}}{\|\alpha^{'}(t)\|}B(t).
\end{aligned}
$$

This surface is named a canal surface around the curve $\alpha$. The principal vector field $N$ and the binormal vector field $B$ of the curve $\alpha$ are in the plane with the characteristic circle. If the curve $\alpha$ is a unite speed curve by an arc length $\delta$, then the canal surface is reparameterized as

$$
\begin{aligned}
K(\delta,\phi) \;=\;& \alpha(\delta) - r(\delta)r^{'}(\delta)T(\delta) \\
&\pm \cos\phi\, r(\delta)\sqrt{1 - r^{'}(\delta)^2}N(\delta) \\
&\pm \sin\phi\, r(\delta)\sqrt{1 - r^{'}(\delta)^2}B(\delta).
\end{aligned}
$$

A canal surface is a tubular surface provided that the radius $r(\delta) = r$ is a constand, and in this case the equation will be as follows;

$$
L(\delta,\phi) = \alpha(\delta) + r\left(\cos\phi N(\delta) + \sin\phi B(\delta)\right), \tag{4.1}
$$

where $0 \le \phi \le 2\pi$.

If we take the focal curve $C_\alpha$ instead of the curve $\alpha$ in $\mathbf{E}^3$, (4.1) will be as follows;

$$
L(\delta,\phi) = C_\alpha(\delta) + r\left(\mp\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\cos\phi N(\delta) \pm \varepsilon_{\mathbf{n}}\sin\phi T(\delta)\right). \tag{4.2}
$$

Without loss of generality in (4.2) we can take $\varepsilon_{\mathbf{t}} = \varepsilon_{\mathbf{n}} = 1$. By such preference, (4.2) reads as

$$
L(\delta,\phi) = C_\alpha(\delta) + r\left(\sin\phi T(\delta) - \cos\phi N(\delta)\right). \tag{4.3}
$$

**Example 4.1.** Let $\beta$ be a circular helix defined by

$$
\beta(\delta) = \left(\frac{1}{\sqrt{2}}\cos(\delta), \frac{1}{\sqrt{2}}\sin(\delta), \frac{\delta}{\sqrt{2}}\right).
$$

Since $\|\beta'(\delta)\| = 1$, this curve is regular curve where $0 \le \delta \le 2\pi$.

The Frenet vectors $\left\{T_\beta, N_\beta, B_\beta\right\}$ of the curve $\beta$ are as follows;

$$
\begin{aligned}
T_\beta(\delta) \;&=\; \left(-\frac{1}{\sqrt{2}}\sin(\delta), \frac{1}{\sqrt{2}}\cos(s), \frac{1}{\sqrt{2}}\right), \\
N_\beta(\delta) \;&=\; \left(-\cos(\delta), -\sin(\delta), 0\right), \\
B_\beta(\delta) \;&=\; \left(\frac{1}{\sqrt{2}}\sin(\delta), -\frac{1}{\sqrt{2}}\cos(\delta), \frac{1}{\sqrt{2}}\right).
\end{aligned}
$$

The curvatures of the curve $\beta$ are found to be $\kappa_\beta = \frac{1}{\sqrt{2}}$, and $\tau_\beta = \frac{1}{\sqrt{2}}$ by using in (2.1). The focal coefficients of the curve $\beta$ can be computed from (3.2) as $c_1 = \sqrt{2}$, and $c_2 = 0$.

For this specific example, using (3.1) the focal curve $C_\beta$ of the curve $\beta$ may be computed as
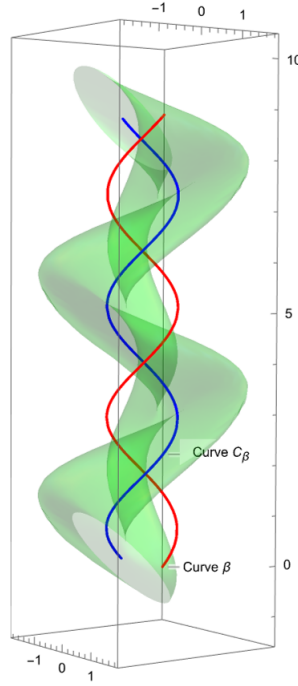
$$C_\beta = \beta + \sqrt{2}N. \tag{4.4}$$



FIGURE 2. For $0 \le \delta \le 4\pi$, $0 \le \phi \le 2\pi$ tubular surface around the focal curve $C_\beta$ of the curve $\beta$.

Using (4.3) and (4.4) with $r = \sqrt{2}$ lead to the components

$$
\begin{aligned}
x_\beta(\delta, \phi) &= \frac{1}{\sqrt{2}} \cos(\delta) - \sqrt{2}\cos(\delta) - \sin(\phi)\sin(\delta) - \sqrt{2}\cos(\phi)\cos(\delta), \\
y_\beta(\delta, \phi) &= \frac{1}{\sqrt{2}} \sin(\delta) - \sqrt{2}\sin(\delta) + \sin(\phi)\cos(\delta) - \sqrt{2}\cos(\phi)\sin(\delta), \\
z_\beta(\delta, \phi) &= \frac{\delta}{\sqrt{2}} + \sin(\phi)
\end{aligned}
$$

of the tubular surface $L_\beta(\delta, \phi) = \big(x_\beta(\delta, \phi), y_\beta(\delta, \phi), z_\beta(\delta, \phi)\big)$. According to the focal curve $C_\beta$ the tubular surface $L_\beta(\delta, \phi)$ is given in Figure 2.

**Example 4.2.** Let $\gamma$ be the Salkowski curve $\gamma$ [6] defined by

$$
\gamma(\delta) = \begin{pmatrix}
\frac{8}{\sqrt{65}}\left( -\frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)} \sin\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)} \sin\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1}{2}\sin\delta \right), \\
\frac{8}{\sqrt{65}}\left( \frac{1-\frac{1}{\sqrt{65}}}{4\left(1+\frac{2}{\sqrt{65}}\right)} \cos\left(\left(1+\frac{2}{\sqrt{65}}\right)\delta\right) + \frac{1+\frac{1}{\sqrt{65}}}{4\left(1-\frac{2}{\sqrt{65}}\right)} \cos\left(\left(1-\frac{2}{\sqrt{65}}\right)\delta\right) - \frac{1}{2}\cos\delta \right), \\
\frac{16}{\sqrt{65}} \cos\left(\frac{2}{\sqrt{65}}\delta\right)
\end{pmatrix} .
$$

Since $\|\gamma'(\delta)\| = \dfrac{8}{\sqrt{65}} \cos\left(\dfrac{\delta}{\sqrt{65}}\right)$, this curve is regular curve in $\left(-\dfrac{\pi\sqrt{65}}{2}, \dfrac{\pi\sqrt{65}}{2}\right)$, where $0 \le \delta \le \dfrac{\pi\sqrt{65}}{2}$ is shown in Figure 3.

The Frenet vectors $\left\{T_\gamma, N_\gamma, B_\gamma\right\}$ of the curve $\gamma$ are

$$T_\gamma(\delta) = -\begin{pmatrix} \cos(\delta)\cos(\frac{\delta}{\sqrt{65}}) + \frac{1}{\sqrt{65}}\sin(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ \cos(\frac{\delta}{\sqrt{65}})\sin(\delta) - \frac{1}{\sqrt{65}}\cos(t)\sin(\frac{\delta}{\sqrt{65}}), \\ \frac{1}{\sqrt{65}}\sin(\frac{\delta}{\sqrt{65}}) \end{pmatrix},$$

$$N_\gamma(\delta) = \left(\frac{1}{\sqrt{65}}8\sin(\delta), -\frac{1}{\sqrt{65}}8\cos(\delta), -\frac{1}{\sqrt{65}}\right),$$

$$B_\gamma(\delta) = \begin{pmatrix} \frac{1}{\sqrt{65}}\cos(\frac{\delta}{\sqrt{65}})\sin(\delta) - \cos(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ -\frac{1}{\sqrt{65}}\cos(\delta)\cos(\frac{\delta}{\sqrt{65}}) - \sin(\delta)\sin(\frac{\delta}{\sqrt{65}}), \\ \frac{1}{\sqrt{65}}\cos(\frac{\delta}{\sqrt{65}}) \end{pmatrix}.$$

The curvatures of $\gamma$ are found to be $\kappa_\gamma = 1$, and $\tau_\gamma = \tan\left(\dfrac{\delta}{\sqrt{65}}\right)$ using in (2.1). The focal coefficients of the curve $\gamma$ can be computed from (3.2) as $c_1 = 1$, and $c_2 = 0$. For this specific example, using (3.1) the focal curve $C_\gamma$ of the curve $\gamma$ may be computed as
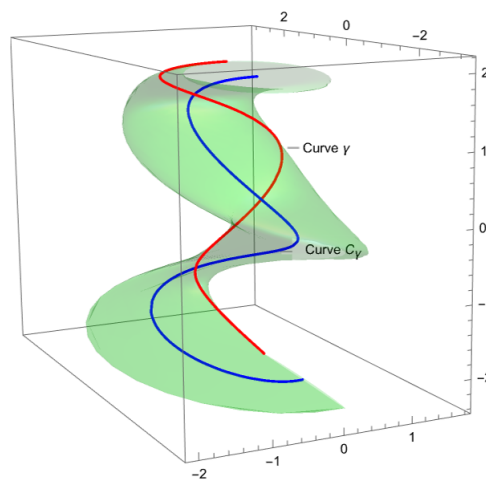
$$C_\gamma = \gamma + N_\gamma. \tag{4.5}$$



FIGURE 3. For $0 \le \delta \le \dfrac{\pi\sqrt{65}}{2}$, $0 \le \phi \le 2\pi$ tubular surface around focal curve $C_\gamma$ of the curve $\gamma$.

Using (4.3) and (4.5) with $r = 1$ lead to the components

$$x_\gamma(\delta, \phi) = -\frac{1 - \frac{1}{\sqrt{65}}}{4\left(1 + \frac{2}{\sqrt{65}}\right)} \frac{8}{\sqrt{65}} \sin\left(\left(1 + \frac{2}{\sqrt{65}}\right)\delta\right)$$

$$-\frac{1 + \frac{1}{\sqrt{65}}}{4\left(1 - \frac{2}{\sqrt{65}}\right)} \frac{8}{\sqrt{65}} \sin\left(\left(1 - \frac{2}{\sqrt{65}}\right)\delta\right)$$

$$-\frac{1}{2}\frac{8}{\sqrt{65}}\sin(\delta) + \frac{1}{\sqrt{65}}\frac{8}{\sqrt{65}} 8 \sin(\delta)$$

$$+ \sin(\phi)\cos(\delta)\cos(\frac{\delta}{\sqrt{65}}) + \frac{1}{\sqrt{65}}\sin(\phi)\sin(\delta)\sin(\frac{\delta}{\sqrt{65}})$$

$$-\frac{1}{\sqrt{65}} 8 \cos(\phi)\sin(\delta),$$

$$y_\gamma(\delta, \phi) = \frac{1 - \frac{1}{\sqrt{65}}}{4\left(1 + \frac{2}{\sqrt{65}}\right)} \frac{8}{\sqrt{65}} \cos\left(\left(1 + \frac{2}{\sqrt{65}}\right)\delta\right)$$

$$+\frac{1 + \frac{1}{\sqrt{65}}}{4\left(1 - \frac{2}{\sqrt{65}}\right)} \frac{8}{\sqrt{65}} \cos\left(\left(1 - \frac{2}{\sqrt{65}}\right)\delta\right)$$

$$-\frac{1}{2}\frac{8}{\sqrt{65}}\cos(\delta) - \frac{1}{\sqrt{65}}\frac{8}{\sqrt{65}} 8 \cos(\delta)$$

$$+ \sin(\phi)\cos(\frac{\delta}{\sqrt{65}})\sin(\delta) - \frac{1}{\sqrt{65}}\sin(\phi)\cos(\delta)\sin(\frac{\delta}{\sqrt{65}})$$

$$+\frac{1}{\sqrt{65}} 8 \cos(\phi)\cos(\delta),$$

$$z_\gamma(\delta, \phi) = \frac{16}{\sqrt{65}}\cos\left(\frac{2}{\sqrt{65}}\delta\right) - \frac{1}{\sqrt{65}}$$

$$+\frac{1}{\sqrt{65}}\sin(\phi)\sin(\frac{\delta}{\sqrt{65}})\frac{1}{\sqrt{65}}\sin(\phi)\sin(\frac{\delta}{\sqrt{65}})$$

$$+\frac{1}{\sqrt{65}}\cos(\phi)$$

of the tubular surface $L_\gamma(\delta, \phi) = \left(x_\gamma(\delta, \phi), y_\gamma(\delta, \phi), z_\gamma(\delta, \phi)\right)$. According to the focal curve $C_\gamma$ the tubular surface $L_\gamma(\delta, \phi)$ is given in Figure 3.

## 5. Fundamental Forms of the Tubular Surfaces

Let $\alpha = \alpha(\delta) : \mathrm{I} \longrightarrow \mathbf{E}^3$ be any unit speed curve. A parametrization of the tubular surface $L(s, \phi)$ around focal curve $C_\alpha(\delta)$ is presented (4.3). The partial derivatives of $L$ concerning the surface parameters $\delta$ , and $\phi$ can be written as follows

$$L_\delta = \cos\phi T + \sin\phi N + r\tau(1 - \cos\phi) B,$$
$$L_\phi = r\cos\phi T + r\sin\phi N,$$
$$L_{\delta\delta} = -\kappa\sin\phi T + \left[\kappa\cos\phi - r\tau^2(1 - \cos\phi)\right] N + \left[\tau\sin\phi + r\tau'(1 - \cos\phi)\right] B,$$
$$L_{\delta\phi} = -\sin\phi T + \cos\phi N + r\tau\sin\phi B,$$
$$L_{\phi\phi} = -r\sin\phi T + r\cos\phi N.$$

and

$$\mathbf{N} = \frac{L_\delta \times L_\phi}{\left\| L_\delta \times L_\phi \right\|} = -\sin\phi T + \cos\phi N,$$

where $\mathbf{N}$ is a unit normal vector field of the tubular surface $L(\delta, \phi)$. We can easily obtain

$$\left\| L_\delta \times L_\phi \right\|^2 = EG - F^2 = r^4 \tau^2 (1 - \cos\phi)^2. \tag{5.1}$$

The first fundamental form I of the tubular surface $L$ is defined as

$$\mathbf{I} = \mathbf{E}d\delta^2 + 2\mathbf{F}d\delta d\phi + \mathbf{G}d\phi^2,$$

where

$$\mathbf{E} = g(L_\delta, L_\delta) = 1 + r^2\tau^2 (1 - \cos\phi)^2,$$
$$\mathbf{F} = g(L_\delta, L_\phi) = r,$$
$$\mathbf{G} = g(L_\phi, L_\phi) = r^2.$$

The second fundamental form II of the tubular surface $L$ is defined as

$$\mathbf{II} = \mathbf{e}d\delta^2 + 2\mathbf{f}d\delta d\phi + \mathbf{g}d\phi^2,$$

where

$$\mathbf{e} = g(\mathbf{N}, L_{\delta\delta}) = \kappa - r\tau^2 \cos\phi(1 - \cos\phi),$$
$$\mathbf{f} = g(\mathbf{N}, L_{s\phi}) = 1,$$
$$\mathbf{g} = g(\mathbf{N}, L_{\phi\phi}) = r.$$

**Definition 5.1.** [2] Let $M$ be any surface, and the set $\{\mathbf{E}, \mathbf{F}, \mathbf{G}\}$ be the coefficients of its first fundamental form. $M$ is named a regular surface if $\mathbf{EG} - \mathbf{F}^2 \neq 0$.

**Lemma 5.2.** $L(\delta, \phi)$ is a regular tubular surface, when $\cos\phi \neq 1$, $\kappa \neq 0$, and $\tau \neq 0$.

*Proof.* If we use (5.1) and consider definition 5.1, the proof ends. □

**Theorem 5.3.** *For the tubular surface $L(\delta, \phi)$, Gaussian, and the mean curvatures are*

$$K = \frac{eg - f^2}{EG - F^2} = \frac{(1 - 2\cos\phi)}{2r^2 (1 - \cos\phi)},$$

*and*

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)} = -\frac{\cos\phi}{r^2 (1 - \cos\phi)^2}$$

*respectively.*

Let us now give the following theorems about parameter curves on a tubular surface.

**Theorem 5.4.** [7] *Let a curve $\gamma$ be on a regular surface. The curve $\gamma$ is an asymptotic curve provided that its acceleration vector $\gamma''$ is orthogonal to the normal vector $\mathbf{N}$ of the aforementioned surface. Namely, $g(\mathbf{N}, \gamma'') = e = 0$.*

**Theorem 5.5.** *Let $L(\delta, \phi)$ be a tubular surface and the curve $\gamma(\delta)$ be its focal curve.*
*(1) If $\delta$−parameter curves $L_\delta$ are asymptotic, then $\kappa = r\tau^2 \cos\phi(1 - \cos\phi)$, and $\cos\phi \neq 0$*
*(2) $\phi$−parameter curves $L_\phi$ are not asymptotic.*

*Proof.* The first coefficient $\mathbf{e}$ of the second fundamental form $\mathbf{II}$ as follows

$$\mathbf{e} = g(\mathbf{N}, L_{\delta\delta}) = \kappa - r\tau^2 \cos\phi(1 - \cos\phi).$$

If $\delta$−parameter curves $L_\delta$ are asymptotic, then $\kappa = r\tau^2 \cos\phi(1 - \cos\phi)$. Due to $\kappa \neq 0$, then $\cos\phi \neq 0$. Similarly, the third coefficient $g$ of the second fundamental form $\mathbf{II}$ as follows

$$\mathbf{g} = g(\mathbf{N}, L_{\phi\phi}) = r \neq 0.$$

If so, the $\phi$−parameter curves $L_\phi$ are not asymptotic curves. □

**Theorem 5.6** ( [4]). *Let a curve $\gamma$ be on a regular surface. The curve $\gamma$ is a geodesic curve provided its acceleration vector $\gamma''$ and the normal vector $\mathbf{N}$ of the surface is linearly dependent. Namely, $\mathbf{N} \times \gamma'' = 0$.*

**Theorem 5.7.** *Let $L(\delta, \phi)$ be a tubular surface and the curve $\gamma(\delta)$ be its focal curve.*
*(1) The $\delta-$parameter curves $L_\delta$ can not be geodesic,*
*(2) The $\phi-$parameter curves $L_\phi$ curves are geodesic.*

*Proof.* For any $\delta$ and $\phi$, $r^2 \tau (1 - \cos \phi) \neq 0$ since $L(\delta, \phi)$ is a regular tubular surface (see (5.1)). Therefore,

$$
\begin{aligned}
\mathbf{N} \times L_{\delta\delta} \quad = \quad & \cos \phi \left[ \tau \sin \phi + r\tau' (1 - \cos \phi) \right] T \\
& + \sin \phi \left[ \tau \sin \phi + r\tau' (1 - \cos \phi) \right] N \\
& + r\tau^2 \sin \phi (1 - \cos \phi) B \neq 0.
\end{aligned}
\tag{5.2}
$$

On the other hand, since

$$
\mathbf{N} \times L_{\phi\phi} = \mathbf{N} \times r\mathbf{N} = 0
$$

the $\phi-$parameter curves $L_\phi$ are geodesics. The opposite is also true. □

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

## Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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