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On Cayley Graphs with Constant Ricci Curvature

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Abstract. Understanding the geometry of graphs has become increasingly important. One approach utilizes the Ricci curvature introduced by Lin, Lu, and Yau, which offers a valuable isomorphism invariant for locally finite graphs. One of the key tools used in calculating curvatures is the matching condition. This paper exploits the matching condition to construct families of Cayley graphs exhibiting constant Ricci curvature.

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1. Introduction

Traditional Ricci curvature is a cornerstone of Riemannian geometry, quantifying the local bending of a smooth manifold. Generalizing this notion to graphs requires adaptation due to their discrete nature. One prominent approach is the Ollivier-Ricci curvature, which is related to the behavior of random walks on metric spaces, including graphs [10]. In this approach, the transportation distance between probability measures originating from two given points is compared to the distance between these points. This has led to numerous subsequent studies on the Ollivier-Ricci curvature on graphs [1–3,5,11], resulting in a wide range of applications in diverse fields such as network analysis, data analysis, machine learning, and even biology [6, 9, 12, 13]. Lin, Lu, and Yau [7] later modified Ollivier's definition to study the properties of the Ricci curvature of general graphs. Although both definitions have some similar properties, they vary in several aspects. Numerous subsequent studies have explored properties related to the modified definition.

The classification problem of circulant graphs with constant curvature within the Lin, Lu, and Yau framework was proposed by Smith [14] due to their applications in computer architecture. The matching condition introduced in the same work served as a crucial tool for constructing families of circulant graphs with constant Ricci curvature in [4]. One of these constructions utilized the factorization $n = p_1 \cdots p_m$ of the number of the vertices with coprime factors $p_i > 3$. It was also conjectured that their construction holds if one of the factors is 3. It is noteworthy to acknowledge [8], which focused on the Ricci curvature of Cayley graphs, including circulant graphs with certain 4-element jump sets.

In this paper, we build upon the work of [4, 15]. Specifically, we use certain subgroup union complements as the generating sets for Cayley graphs on direct products of cyclic groups to obtain graphs with constant Ricci curvature. This approach relaxes the requirement for the factors of n to be coprime. We also provide a proof of the conjecture of [4], and demonstrate that it remains applicable even when only one of the factors of n is 2.

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2. Preliminaries

Let $G = (V, E)$ be simple graph with a vertex set V and an edge set E. For any vertex $x \in V$, we denote its set of neighbors by $N(x)$, i.e., $N(x) = \{y \in V : xy \in E\}$. Let $\varepsilon < 1$ be a positive real number. The ε-ball centered at x is the probability distribution defined by

$$b_x^{\varepsilon}(v) = \begin{cases} 1-\varepsilon; & v = x, \\ \frac{\varepsilon}{|N(x)|}; & v \in N(x), \\ 0; & \text{otherwise.} \end{cases}$$

The Wasserstein Distance, also known as the Earth Mover's Distance, is a metric used to compare probability distributions on a metric space. It quantifies the minimum cost of transforming one distribution into another. More formally, consider an edge $xy \in E$. Then the Wasserstein distance $W(b_x^{\varepsilon}, b_y^{\varepsilon})$ represents the cost of the optimal transport from b_x^{ε} to b_y^{ε}. This cost is calculated by considering the movement of mass between vertices according to these distributions. For each unit of mass being moved from a source vertex to a target vertex, the distance between the vertices is multiplied by the probability weight, i.e., the amount of mass being moved. The total cost is then minimized over all possible transport plans that specify how much mass to move from each source vertex to each target vertex.

Definition 2.1. Let $G = (V, E)$ be a simple graph. The Ricci curvature $\kappa(x, y)$ of an edge $xy \in E$ is defined as

$$\kappa(x, y) = \lim_{\varepsilon \to 0} \frac{1 - W(b_x^{\varepsilon}, b_y^{\varepsilon})}{\varepsilon}.$$

The limit $\kappa(x, y)$ exists for all $xy \in E$. In other words, the Ricci curvature is defined on all edges [7].

The calculation of the Ricci curvature of an edge involves finding the Wasserstein distance between two probability distributions on a graph. This formulation defines a specific type of optimization problem, often referred to as the primal problem. In this problem, we aim to find the minimum of

$$\sum_{(u,v)\in V^2} A_{uv} d(u, v) \tag{2.1}$$

subject to the input constraint $\sum_{v\in V} A_{uv} = b_x^{\varepsilon}(u)$ and the output constraint $\sum_{u\in V} A_{uv} = b_y^{\varepsilon}(v)$, where A_{uv} represents the amount of mass being moved from the vertex u to the vertex v, and $d(u, v)$ denotes the distance between these vertices.

The primal problem has a dual formulation that seeks to maximize the sum of probability distribution differences weighted by a special function. More formally, the objective of the dual problem is to find the maximum of

$$\sum_{v\in V} \left(b_x^{\varepsilon}(v) - b_y^{\varepsilon}(v)\right) f(v) \tag{2.2}$$

subject to $|f(u) - f(v)| \le d(u, v)$. A function $f : V \to \mathbb{R}$ satisfying this condition for all $u, v \in V$ is called 1-Lipschitz.

To find the Wasserstein distance between two probability distributions, we either need to find the infimum of (2.1) over the set of all transport plans, or the supremum of (2.2) over the set of all 1-Lipschitz functions. It is noteworthy that each transportation plan provides an upper bound on the transportation cost (2.1), and each 1-Lipschitz function offers a lower bound on the sum of weighted differences (2.2). Hence, the optimal solution can be obtained by equating such upper and lower bounds. In other words, a transport plan solves the infimum problem and a 1-Lipschitz function solves the supremum problem if and only if the following equation holds:

$$\sum_{(u,v)\in V^2} A_{uv} d(u, v) = \sum_{v\in V} \left(b_x^{\varepsilon}(v) - b_y^{\varepsilon}(v)\right) f(v).$$

This observation is used along with the matching condition, defined below, to construct graphs with constant Ricci curvature [14].

Definition 2.2. Let $G = (V, E)$ be a simple graph. An edge $xy \in E$ is said to satisfy the local matching condition if there is perfect matching between the sets $N(x) \setminus (N(y) \cup \{y\})$ and $N(y) \setminus (N(x) \cup \{x\})$. If every edge of G satisfies the local matching condition, then G is said to satisfy the global matching condition.

Theorem 2.3. *[14, Teorem 6.3] In a graph $G = (V, E)$, suppose that xy is an edge satisfying the local matching condition. Then,*

$$\kappa = \frac{|N(x) \cap N(y)| + 2}{\delta} \tag{2.3}$$

where δ is the common degree of x and y.

Recall that a circulant graph, denoted by $C_n(J)$, is an undirected graph with a vertex set $V = \mathbb{Z}_n$ and an edge set defined by

$$E = \{\{i, i + j\}, i \in V, j \in J\}.$$

Here, $J \subseteq \{1, 2, \ldots, n-1\}$ is a non-empty subset of integers, called the jump set or the shift set.

Example 2.4. Consider the circulant graph $C_{10}(J)$ where $J = \{1, 2, 3, 5\}$. Let us calculate the curvature $\kappa(0, 1)$. Note that $N(0) = \{1, 2, 3, 5, 7, 8, 9\}$ and $N(1) = \{0, 2, 3, 4, 6, 8, 9\}$. Since $N(0) \cap N(1) = \{2, 3, 8, 9\}$, it follows that

$$N(0) \setminus (N(1) \cup \{1\}) = \{5, 7\} \quad \text{and} \quad N(1) \setminus (N(0) \cup \{0\}) = \{4, 6\}.$$

The subgraph induced on the vertices $\{4, 5, 6, 7\}$ is isomorphic to K_4, the complete graph on four vertices. Hence, there is a perfect matching between the sets $\{5, 7\}$ and $\{4, 6\}$. Therefore, the edge $\{0, 1\}$ satisfies the local matching condition. It follows from (2.3) that $\kappa(0, 1) = \frac{6}{7}$.

Several constructions of circulant graphs with constant Ricci curvature were presented in [4]. One such construction utilizes integer factorizations along with the matching condition, as stated below.

Theorem 2.5. *[4, Theorem 5.13] Consider a product $n = p_1 p_2 \cdots p_m$ of mutually coprime integer factors $p_i > 3$. Within the cyclic group $\mathbb{Z}_n$, let $S = \cup_{i=1}^{m} \langle p_i \rangle$ and $J = \mathbb{Z}_n \setminus S$. Then the circulant graph $C_n(J)$ has constant Ricci curvature*

$$\kappa = \frac{2 + \prod_{i=1}^{m}(p_i - 2)}{\prod_{i=1}^{m}(p_i - 1)}. \tag{2.4}$$

Example 2.6. Let $n = 20$, $p_1 = 4$, and $p_2 = 5$. Consider the union $S = \langle 4 \rangle \cup \langle 5 \rangle$ and let $J = \mathbb{Z}_{20} \setminus S$. Then the circulant graph $C_{20}(J)$ has Ricci curvature $\kappa = \frac{2}{3}$.

It was conjectured in [4] that the formula (2.4) remains valid if one of the factors of $n = p_1 p_2 \ldots p_m$ is 3. We will prove this conjecture in next section, and show that it holds even when the factors of n are not mutually coprime, or if one of the factors is 2.

3. Main Results

The Fundamental Theorem of Finite Abelian Groups states that every finite Abelian group is isomorphic to a direct product of cyclic groups of prime-power order. In this context, the cyclic group $\mathbb{Z}_n$ is isomorphic to $\bigoplus_{i=1}^{m} \mathbb{Z}_{p_i}$ for the product $n = p_1 p_2 \cdots p_m$ of mutually coprime integer factors. It is easier to display the elements of jump sets of circulant graphs in $\bigoplus_{i=1}^{m} \mathbb{Z}_{p_i}$ rather than in $\mathbb{Z}_n$ if the jump set is a complement of subgroup unions. Motivated by this observation, Dağlı et al. [4] investigated the Ricci curvature for mutually coprime integer factors $p_i > 3$ in [4]. In the following, we will show that graphs with constant curvature can be constructed with the same jump set, but without mutually coprime factors. But let us recall the definition of Cayley graphs first.

Definition 3.1. Given an additive group H and a generating set $S \subseteq H$ with $0 \notin S$ and $S = -S$, the Cayley graph $\mathrm{Cay}(H, S)$ of H is the simple graph with vertex set H and edge set defined such that two vertices h and h' are adjacent if and only if $h - h' \in S$.

Remark 3.2. An arbitrary edge of $\mathrm{Cay}(H, S)$ has the form $\{h, s+h\} = \{0, s\}+h$ for some $h \in H$ and $s \in S$. The function $R_+(h) : H \to H$ defined by $x \mapsto x + h$ is an automorphism of $\mathrm{Cay}(H, S)$ for any $h \in H$. Hence, $\kappa(h, s + h) = \kappa(0, s)$. This means that it is enough to calculate the curvature near the vertex corresponding to the identity element.

Example 3.3. Let p and q be integers strictly greater than 2 that are not necessarily coprime. Consider the Cayley graph $\mathrm{Cay}(H, S)$ where $H = \mathbb{Z}_p \oplus \mathbb{Z}_q$ and $S = \{(s_1, s_2) : s_1 \neq 0, s_2 \neq 0\} \subseteq H$. Note that S is the complement of the union of the subgroups $H_1 = \mathbb{Z}_p \oplus \{0\}$ and $H_2 = \{0\} \oplus \mathbb{Z}_q$, and it has the cardinality $|S| = (p-1)(q-1)$. Since $\langle S \rangle = H$, the Cayley graph $\mathrm{Cay}(H, S)$ is connected. Moreover, it can be shown that any edge satisfies the matching condition. Hence, $\mathrm{Cay}(H, S)$ has constant curvature $\kappa = \frac{(p-2)(q-2)+2}{(p-1)(q-1)}$.

The previous example shows that coprime factors in Theorem 2.5 are not necessary for constant curvature in certain Cayley graphs. The following theorem formalizes this relaxation. Its proof closely follows the proof of Theorem 2.5.

Theorem 3.4. *Consider the additive group $H = \bigoplus_{i=1}^{m} \mathbb{Z}_{p_i}$ with each $p_i > 3$. Let $S = \{(s_1, \ldots, s_m) : \forall i, s_i \neq 0\} \subset H$. Then the Cayley graph $\mathrm{Cay}(H, S)$ has constant Ricci curvature, given in (2.4).*

Proof. Let us denote the vertices $(0,\ldots,0)$ and $(s_1,\ldots,s_m)$ of $\mathrm{Cay}(H,S)$ by 0 and s, respectively. Then it can be seen that $N(0)=\{(a_1,\ldots,a_n):\forall i, a_i\neq 0\}$ and $N(s)=\{(a_1,\ldots,a_n):\forall i, a_i\neq s_i\}$. Hence,

$$N(0)\cap N(s)=\{(a_1,\ldots,a_n): a_i\neq 0, s_i\}.$$

Therefore, $|N(0)\cap N(s)|=(p_1-2)\cdots(p_m-2)$. To finish the proof, it is sufficient to show that the edge $\{0,s\}$ satisfies the matching condition. In other words, we need to establish a one-to-one correspondence between the sets

$$N(0)\smallsetminus(N(s)\cup\{s\})=\{(a_1,\ldots,a_m):\exists i, a_i=s_i;\ \forall j, a_j\neq 0\}\smallsetminus\{s\}, \tag{3.1}$$

and

$$N(s)\smallsetminus(N(0)\cup\{0\})=\{(b_1,\ldots,b_m):\exists i, b_i=0;\ \forall j, b_j\neq s_j\}\smallsetminus\{0\}, \tag{3.2}$$

where $1\le i,j\le m$. For a subset $\varnothing\subset I\subset\{1,\ldots,m\}$, define

$$X_I=\{(a_1,\ldots,a_m):\forall i\in I, a_i=s_i;\ \forall j\notin I, a_j\neq 0, s_j\}\smallsetminus\{s\},$$

and

$$Y_I=\{(b_1,\ldots,b_m):\forall i\in I, b_i=0;\ \forall j\notin I, b_j\neq 0, b_j\}\smallsetminus\{0\}.$$

Then, for any distinct subsets I_1 and I_2 of $\{1,\ldots,m\}$, we have $X_{I_1}\cap X_{I_2}=\varnothing$ and $Y_{I_1}\cap Y_{I_2}=\varnothing$. Moreover,

$$N(0)\smallsetminus(N(s)\cup\{s\})=\biguplus_{\varnothing\subset I\subset\{1,\ldots,m\}} X_I,$$

and

$$N(s)\smallsetminus(N(0)\cup\{0\})=\biguplus_{\varnothing\subset I\subset\{1,\ldots,m\}} Y_I.$$

Now, consider the subgraph of $\mathrm{Cay}(H,S)$ induced on $X_I\cup Y_I$. For any $(a_1,\ldots,a_m)\in X_I$ and $(b_1,\ldots,b_m)\in Y_I$, it can be seen that $|N(a_1,\ldots,a_m)|=|N(b_1,\ldots,b_m)|=\prod_{i\notin I}(p_i-3)$. Hence, the induced subgraph is a $\prod_{i\notin I}(p_i-3)$-regular bipartite graph. The proof then follows since regular bipartite graphs satisfy the matching condition. □

Next, we give a proof of the conjecture stated in Remark 5.14 of [4].

Theorem 3.5. *Let $H=\mathbb{Z}_3\bigoplus_{i=2}^m\mathbb{Z}_{p_i}$ with each $p_i>3$. Then, the Cayley graph $\mathrm{Cay}(H,S)$ has constant Ricci curvature for $S=\{(s_1,\ldots,s_m):\forall i, s_i\neq 0\}\subset H$.*

Proof. Let us denote the vertices $(0,\ldots,0)$ and $(s_1,\ldots,s_m)$ of $\mathrm{Cay}(H,S)$ by 0 and s, respectively. In the group of integers modulo 3, if s_1 is any nonzero element, then its additive inverse, $-s_1$, is the other nonzero element. Hence, we can write the disjoint unions $N(0)\smallsetminus(N(s)\cup\{s\})=A_1\uplus A_2\uplus A_3$ and $N(s)\smallsetminus(N(0)\cup\{0\})=B_1\uplus B_2\uplus B_3$, where

$$\begin{aligned}
A_1&=\{(s_1,a_2,\ldots,a_m):\forall i, a_i\neq 0, s_i\},\\
A_2&=\{(s_1,a_2,\ldots,a_m):\exists i, a_i=s_i;\ \forall j, a_j\neq 0\}\smallsetminus\{s\},\\
A_3&=\{(-s_1,a_2,\ldots,a_m):\exists i, a_i=s_i;\ \forall j, a_j\neq 0\},
\end{aligned}$$

and

$$\begin{aligned}
B_1&=\{(0,b_2,\ldots,b_m):\forall i, b_i\neq 0, s_i\},\\
B_2&=\{(0,b_2,\ldots,b_m):\exists i, b_i=0;\ \forall j, b_j\neq s_j\}\smallsetminus\{0\},\\
B_3&=\{(-s_1,b_2,\ldots,b_m):\exists i, b_i=0;\ \forall j, b_j\neq s_j\},
\end{aligned}$$

with $2\le i,j\le m$. Let us start with constructing a matching between A_2 and $B_3\smallsetminus\{(-s_1,0,\ldots,0)\}$. Consider the group $H'=\bigoplus_{i=2}^m\mathbb{Z}_{p_i}$ with each $p_i>3$, and the generating set $S'=\{(s_2,\ldots,s_m):\forall i, s_i\neq 0\}\subset H'$. Within $\mathrm{Cay}(H',S')$, there is a matching between the sets

$$\{(a_2,\ldots,a_m):\exists i, a_i=s_i;\ \forall j, a_j\neq 0\}\smallsetminus\{(s_2,\ldots,s_m)\}$$

and

$$\{(b_2,\ldots,b_m):\exists i, b_i=0;\ \forall j, b_j\neq s_j\}\smallsetminus\{(0,\ldots,0)\}$$

due to Theorem 3.4. Since the first entries of elements in A_2 are always s_1 and those in B_3 are always $-s_1$, this matching naturally induces a one to one correspondence between elements in A_2 and $B_3\smallsetminus\{(-s_1,0,\ldots,0)\}$. A similar matching can be constructed between elements in $A_3\smallsetminus\{(-s_1,s_2,\ldots,s_m)\}$ and B_2.

On the other hand, the subgraph induced on $A_1 \cup B_1$ in $\mathrm{Cay}(H, S)$ is a $(p_2 - 3)\cdots(p_m - 3)$ regular bipartite graph, and therefore it satisfies the matching condition. Let the vertex u of A_1 and the vertex v of B_1 be matched in this setting. Since $(-s_1, s_2, \ldots, s_m)$ is adjacent to every vertex in B_1, and $(-s_1, 0, \ldots, 0)$ is adjacent to every vertex in A_1, we can first remove the matching between u and v, and then match u with $(-s_1, 0, \ldots, 0)$, and v with $(-s_1, s_2, \ldots, s_m)$ to obtain a matching between $A_1 \cup \{(-s_1, s_2, \ldots, s_m)\}$ and $B_1 \cup \{(-s_1, 0, \ldots, 0)\}$.

In conclusion, the edge $\{0, s\}$ satisfies the matching condition, and therefore $\mathrm{Cay}(H, S)$ has constant curvature. □

In the following, we show that Theorem 3.4 holds even if only one of the factors of $n = p_1 p_2 \cdots p_m$ is 2.

Theorem 3.6. *Let $H = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \bigoplus_{i=3}^{m} \mathbb{Z}_{p_i}$ with each $p_i > 3$. Then, the Cayley graph $\mathrm{Cay}(H, S)$ has constant Ricci curvature for $S = \{(s_1, \ldots, s_m) : \forall i, s_i \neq 0\} \subset H$.*

Proof. Let us denote the vertices $(0, \ldots, 0)$ and $(s_1, \ldots, s_m)$ of $\mathrm{Cay}(H, S)$ by 0 and s, respectively. Then, we can write the disjoint unions $N(0) \setminus (N(s) \cup \{s\}) = A_1 \uplus A_2$ and $N(s) \setminus (N(0) \cup \{0\}) = B_1 \uplus B_2$, where

$$A_1 = \{(1, a_2, \ldots, a_m) : \forall i, a_i \neq 0, s_i\},$$
$$A_2 = \{(1, a_2, \ldots, a_m) : \exists i, a_i = s_i;\ \forall j, a_j \neq 0\} \setminus \{s\},$$

and

$$B_1 = \{(0, b_2, \ldots, b_m) : \forall i, b_i \neq 0, s_i\},$$
$$B_2 = \{(0, b_2, \ldots, b_m) : \exists i, b_i = 0;\ \forall j, b_j \neq s_j\} \setminus \{0\},$$

with $2 \leq i, j \leq m$. Note that the subgraph of $\mathrm{Cay}(H, S)$ induced on $A_1 \cup B_1$ is a $(p_2-3)\cdots(p_m-3)$ regular bipartite graph. Hence, we can perfectly match each vertex in A_1 with a unique vertex in B_1. To construct a matching between A_2 and B_2, consider the Cayley graph $\mathrm{Cay}(H', S')$ where $H' = \bigoplus_{i=2}^{m} \mathbb{Z}_{p_i}$ with each $p_i > 3$, and $S' = \{(s_2, \ldots, s_m) : \forall i, s_i \neq 0\} \subset H'$. There exists a matching between the sets

$$\{(a_2, \ldots, a_m) : \exists i, a_i = s_i;\ \forall j, a_j \neq 0\} \setminus \{(s_2, \ldots, s_m)\}$$

and

$$\{(b_2, \ldots, b_m) : \exists i, b_i = 0;\ \forall j, b_j \neq s_j\} \setminus \{(0, \ldots, 0)\}$$

within $\mathrm{Cay}(H', S')$ due to Theorem 3.4. Since the first entries of elements in A_2 are always 1 and those in B_2 are always 0, this naturally induces a matching between the vertices in A_2 and B_2.

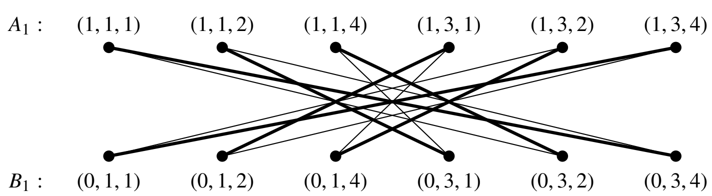
In conclusion, the edge $\{0, s\}$ satisfies the matching condition, and therefore $\mathrm{Cay}(H, S)$ has constant curvature. □

Remark 3.7. If two or more of the factors of n is 2, then the matching condition does not hold.

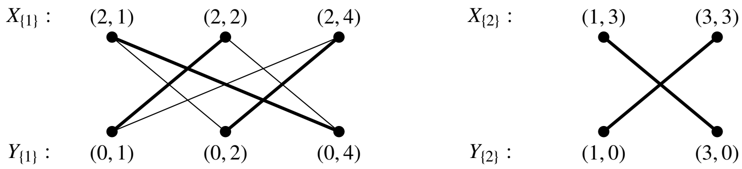
Example 3.8. Let us calculate the curvature $\kappa((0, 0, 0), (1, 2, 3))$ on the Cayley graph $\mathrm{Cay}(H, S)$ for $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5$ and $S = \{(s_1, s_2, s_3) : \forall i, s_i \neq 0\} \subset H$. First, we list the sets defined in the proof of Theorem 3.6:

$$A_1 = \{(1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 3, 1), (1, 3, 2), (1, 3, 4)\}$$
$$B_1 = \{(0, 1, 1), (0, 1, 2), (0, 1, 4), (0, 3, 1), (0, 3, 2), (0, 3, 4)\}$$
$$A_2 = \{(1, 1, 3), (1, 2, 1), (1, 2, 2), (1, 2, 4), (1, 3, 3)\}$$
$$B_2 = \{(0, 0, 1), (0, 0, 2), (0, 0, 4), (0, 1, 0), (0, 3, 0)\}.$$

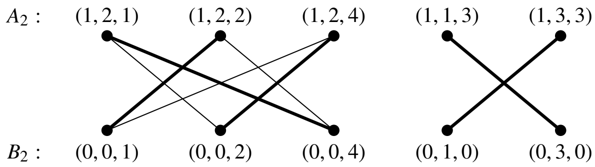
The subgraph induced on $A_1 \cup B_1$ in the Cayley graph $\mathrm{Cay}(H, S)$ is a 2-regular bipartite graph, as shown below. Hence it satisfies the matching condition. The bold edges indicate a sample matching between A_1 and A_2.



Now, consider the Cayley graph $\mathrm{Cay}(H', S')$ where $H' = \mathbb{Z}_4 \oplus \mathbb{Z}_5$, and $S' = \{(s_2, s_3) : \forall i, s_i \neq 0\} \subset H'$. Let us show the edge $\{(0, 0), (2, 3)\}$ satisfies the matching condition in $\mathrm{Cay}(H', S')$. According to (3.1) and (3.2), we can write the disjoint unions $N(0,0) \setminus (N(2,3) \cup \{(2,3)\}) = X_{\{1\}} \uplus X_{\{2\}}$ and $N(2,3) \setminus (N(0,0) \cup \{(0,0)\}) = Y_{\{1\}} \uplus Y_{\{2\}}$ where $X_{\{1\}} = \{(2,1),(2,2),(2,4)\}$, $X_{\{2\}} = \{(1,3),(3,3)\}$, $Y_{\{1\}} = \{(0,1),(0,2),(0,4)\}$, and $Y_{\{2\}} = \{(1,0),(3,0)\}$. In $\mathrm{Cay}(H', S')$, the subgraph induced on $X_{\{1\}} \cup Y_{\{1\}}$ is a 2-regular bipartite graph, and the subgraph induced on $X_{\{2\}} \cup Y_{\{2\}}$ is a 1-regular bipartite graph, as shown below. Sample matchings in both graph are indicated with bold edges.



Since the first entries of elements in A_2 are always 1 and those in B_2 are always 0, the above matching naturally induces between the vertices in A_2 and B_2 in $\mathrm{Cay}(H, S)$, as shown below.



In conclusion, the edge $\{(0,0,0),(1,2,3)\}$ satisfies the matching condition. Hence, its curvature is $\kappa = \frac{1}{6}$ by (2.3).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors have contributed equally. All authors have read and agreed to the published version of the manuscript.

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