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Asymptotically $\mathscr{I}$-Cesàro Equivalence of Sequences of Sets

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Abstract

In this paper, we defined concepts of asymptotically $\mathscr{I}$-Cesàro equivalence and investigate the relationships between the concepts of asymptotically strongly $\mathscr{I}$-Cesàro equivalence, asymptotically strongly $\mathscr{I}$-lacunary equivalence, asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence and asymptotically $\mathscr{I}$-statistical equivalence of sequences of sets.

1. Introduction

The concept of convergence of sequences of real numbers $\mathbb{R}$ has been transferred to statistical convergence by Fast [5] and independently by Schoenberg [16]. $\mathscr{I}$-convergence was first studied by Kostyrko et al. [9] in order to generalize of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [4] introduced new notions, namely $\mathscr{I}$-statistical convergence and $\mathscr{I}$-lacunary statistical convergence by using ideal.

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1], [3], [11], [21], [22]). The concepts of statistical convergence and lacunary statistical convergence of sequences of sets were studied in [11, 18] in Wijsman sense. Also, new convergence notions, for sequences of sets, which is called Wijsman $\mathscr{I}$-convergence, Wijsman $\mathscr{I}$-statistical convergence and Wijsman $\mathscr{I}$-Cesàro summability by using ideal were introduced in [7], [8], [20].

Marouf [10] peresented definitions for asymptotically equivalent and asymptotic regular matrices. This concepts was investigated in [12, 13, 14]. The concept of asymptotically equivalence of sequences of real numbers which is defined by Marouf [10] has been extended by Ulusu and Nuray [19] to concepts of Wijsman asymptotically equivalence of set sequences. Moreover, natural inclusion theorems are presented. Kişi et al. [8] introduced the concepts of Wijsman $\mathscr{I}$-asymptotically equivalence of sequences of sets.

2. Definitions and notations

Now, we recall the basic definitions and concepts (See [1, 2, 6, 7, 8, 9, 10, 11, 15, 19, 20]).

Let (Y,ρ) be a metric space. For any point $y\in Y$ and any non-empty subset U of Y, we define the distance from y to U by $d(y,U)=\inf\limits_{u\in U}\rho(y,u)$.

Let (Y,ρ) be a metric space and U,U_i be any non-empty closed subsets of Y. The sequence $\{U_i\}$ is Wijsman convergent to U if for each $y\in Y$,

$$\lim_{i\to\infty} d(y,U_i)=d(y,U).$$

Let (Y,ρ) be a metric space and U,U_i be any non-empty closed subsets of Y. The sequence $\{U_i\}$ is Wijsman statistical convergent to U if $\{d(y,U_i)\}$ is statistically convergent to $d(y,U)$; i.e., for every $\varepsilon>0$ and for each $y\in Y$,

$$\lim_{n\to\infty}\frac{1}{n}\Big|\big\{i\le n: |d(y,U_i)-d(y,U)|\ge\varepsilon\big\}\Big|=0.$$

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A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if (i) $\emptyset \in \mathscr{I}$, (ii) For each $U,V \in \mathscr{I}$ we have $U \cup V \in \mathscr{I}$, (iii) For each $U \in \mathscr{I}$ and each $V \subseteq U$ we have $V \in \mathscr{I}$.
An ideal is called non-trivial ideal if $\mathbb{N} \notin \mathscr{I}$ and non-trivial ideal is called admissible ideal if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if (i) $\emptyset \notin \mathscr{F}$, (ii) For each $U,V \in \mathscr{F}$ we have $U \cap V \in \mathscr{F}$, (iii) For each $U \in \mathscr{F}$ and each $V \supseteq U$ we have $V \in \mathscr{F}$.

Proposition 2.1. *([9]) $\mathscr{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if*

$$\mathscr{F}(\mathscr{I}) = \{E \subset \mathbb{N} : (\exists U \in \mathscr{I})(E = \mathbb{N} \backslash U)\}$$

is a filter in $\mathbb{N}$.

Throughout the paper, we let (Y,ρ) be a separable metric space, $\mathscr{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and U, U_i be any non-empty closed subsets of Y.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-convergent to U, if for every $\varepsilon > 0$ and for each $y \in Y$, $U(y,\varepsilon) = \left\{i \in \mathbb{N} : |d(y,U_i) - d(y,U)| \geq \varepsilon\right\}$ belongs to $\mathscr{I}$.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-statistical convergent to U, if for every $\varepsilon > 0$, $\delta > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\Big|\left\{i \leq n : |d(y,U_i) - d(y,U)| \geq \varepsilon\right\}\Big| \geq \delta\right\} \in \mathscr{I}$$

and we write $U_i \xrightarrow{S(\mathscr{I}_W)} U$.
The sequence $\{U_i\}$ is Wijsman $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \Big|\frac{1}{n}\sum_{i=1}^{n} d(y,U_i) - d(y,U)\Big| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \xrightarrow{C_1(\mathscr{I}_W)} U$.
The sequence $\{U_i\}$ is Wijsman strongly $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y,U_i) - d(y,U)| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \xrightarrow{C_1[\mathscr{I}_W]} U$.

The sequence $\{U_i\}$ is Wijsman p-strongly $\mathscr{I}$-Cesàro summable to U, if for every $\varepsilon > 0$, for each p positive real number and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y,U_i) - d(y,U)|^p \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \xrightarrow{C_p[\mathscr{I}_W]} U$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Let θ be a lacunary sequence. The sequence $\{U_i\}$ is Wijsman strongly $\mathscr{I}$-lacunary summable to U, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{i \in I_r} |d(y,U_i) - d(y,U)| \geq \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \xrightarrow{N_\theta[\mathscr{I}_W]} U$.

Two nonnegative sequences $a = (a_i)$ and $b = (b_i)$ are said to be asymptotically equivalent if

$$\lim_i \frac{a_i}{b_i} = 1$$

and denoted by $a \sim b$.

We define $d(y;U_i,V_i)$ as follows:

$$d(y;U_i,V_i) = \begin{cases} \dfrac{d(y,U_i)}{d(y,V_i)} & , \quad y \notin U_i \cup V_i \\ \\ \mathscr{L} & , \quad y \in U_i \cup V_i. \end{cases}$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically equivalent of multiple $\mathscr{L}$, if for each $y \in Y$,

$$\lim_{i \to \infty} d(y;U_i,V_i) = \mathscr{L}.$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically statistical equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{i \le n : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\}\right| = 0.$$

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically $\mathscr{I}$-equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and each $y \in Y$

$$\left\{i \in \mathbb{N} : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{\mathscr{I}_W^L}{\sim} V_i$ and simply Wijsman asymptotically $\mathscr{I}$-equivalent if $\mathscr{L} = 1$.

The sequences $\{U_i\}$ and $\{V_i\}$ are Wijsman asymptotically $\mathscr{I}$-statistical equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$, $\delta > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{i \le n : |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathscr{I}$$

and we write $U_i \overset{S(\mathscr{I}_W^L)}{\sim} V_i$ and simply Wijsman asymptotically $\mathscr{I}$-statistical equivalent if $\mathscr{L} = 1$.

Let θ be a lacunary sequence. The sequences $\{U_i\}$ and $\{V_i\}$ are said to be Wijsman asymptotically strongly $\mathscr{I}$-lacunary equivalent of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{N_\theta[\mathscr{I}_W^L]}{\sim} V_i$ and simply Wijsman asymptotically strongly $\mathscr{I}$-lacunary equivalent if $\mathscr{L} = 1$.

3. Main results

In this section, we defined notions of asymptotically $\mathscr{I}$-Cesàro equivalence of sequences of sets. Also, we investigate the relationships between the concepts of asymptotically strongly $\mathscr{I}$-Cesàro equivalence, asymptotically strongly $\mathscr{I}$-lacunary equivalence, asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence and asymptotically $\mathscr{I}$-statistical equivalence of sequences of sets.

Definition 3.1. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{C_1^L(\mathscr{I}_W)}{\sim} V_i$ and simply asymptotically $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Definition 3.2. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$, if for every $\varepsilon > 0$ and for each $y \in Y$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| \ge \varepsilon\right\} \in \mathscr{I}$$

and we write $U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i$ and simply asymptotically strongly $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Theorem 3.3. *Let θ be a lacunary sequence. If $\liminf_r q_r > 1$ then,*

$$U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i \Rightarrow U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i.$$

Proof. If $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r \ge 1+\delta$ for all $r \ge 1$. Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \le \frac{1+\delta}{\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} \le \frac{1}{\delta}.$$

Let $\varepsilon > 0$ and for each $y \in Y$, we define the set

$$S = \left\{k_r \in \mathbb{N} : \frac{1}{k_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \ < \varepsilon\right\}.$$

We can easily say that $S \in \mathscr{F}(\mathscr{I})$, which is a filter of the ideal $\mathscr{I}$, so we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| &= \frac{1}{h_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} |d(y;U_i,V_i) - \mathscr{L}| \\
&= \frac{k_r}{h_r}\cdot\frac{1}{k_r}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \\
&\quad - \frac{k_{r-1}}{h_r}\cdot\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} |d(y;U_i,V_i) - \mathscr{L}| \\
&\le \left(\frac{1+\delta}{\delta}\right)\varepsilon - \frac{1}{\delta}\varepsilon'
\end{aligned}$$

for each $y \in Y$ and for each $k_r \in S$. Choose $\eta = \left(\frac{1+\delta}{\delta}\right)\varepsilon + \frac{1}{\delta}\varepsilon'$. Therefore, for each $y \in Y$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| < \eta \right\} \in \mathscr{F}(\mathscr{I}).$$

Therefore, $U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i$. □

Theorem 3.4. *Let θ be a lacunary sequence. If $\limsup_r q_r < \infty$ then,*

$$U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i \Rightarrow U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i.$$

Proof. If $\limsup_r q_r < \infty$, then there exists $K > 0$ such that $q_r < K$ for all $r \geq 1$. Let $U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i$ and for each $y \in Y$, we define the sets T and R

$$T = \left\{ r \in \mathbb{N} : \frac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| < \varepsilon_1 \right\}$$

and

$$R = \left\{ n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| < \varepsilon_2 \right\}.$$

Let

$$a_j = \frac{1}{h_j}\sum_{i\in I_j} |d(y;U_i,V_i) - \mathscr{L}| < \varepsilon_1$$

for each $y \in Y$ and for all $j \in T$. It is obvious that $T \in \mathscr{F}(\mathscr{I})$. Choose n is any integer with $k_{r-1} < n < k_r$, where $r \in T$. Then, for each $y \in Y$ we have

$$\begin{aligned}
\tfrac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}| &\leq \tfrac{1}{k_{r-1}}\sum_{i=1}^{k_r} |d(y;U_i,V_i) - \mathscr{L}| \\
&= \tfrac{1}{k_{r-1}}\left(\sum_{i\in I_1} |d(y;U_i,V_i) - \mathscr{L}| + \sum_{i\in I_2} |d(y;U_i,V_i) - \mathscr{L}| \right. \\
&\quad \left. + \cdots + \sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&= \tfrac{k_1}{k_{r-1}}\left(\tfrac{1}{h_1}\sum_{i\in I_1} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&\quad + \tfrac{k_2-k_1}{k_{r-1}}\left(\tfrac{1}{h_2}\sum_{i\in I_2} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&\quad + \cdots + \tfrac{k_r-k_{r-1}}{k_{r-1}}\left(\tfrac{1}{h_r}\sum_{i\in I_r} |d(y;U_i,V_i) - \mathscr{L}| \right) \\
&= \tfrac{k_1}{k_{r-1}}a_1 + \tfrac{k_2-k_1}{k_{r-1}}a_2 + \cdots + \tfrac{k_r-k_{r-1}}{k_{r-1}}a_r \\
&\leq \left(\sup_{j\in T} a_j \right) \tfrac{k_r}{k_{r-1}} < \varepsilon_1 \cdot K.
\end{aligned}$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{K}$ and in view of the fact that

$$\bigcup \{ n : k_{r-1} < n < k_r,\ r \in T \} \subset R,$$

where $T \in \mathscr{F}(\mathscr{I})$, it follows from our assumption on θ that the set R also belongs to $\mathscr{F}(\mathscr{I})$ and therefore, $U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i$. □

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let θ be a lacunary sequence. If $1 < \liminf_r q_r < \limsup_r q_r < \infty$ then,*

$$U_i \overset{C_1^L[\mathscr{I}_W]}{\sim} V_i \Leftrightarrow U_i \overset{N_\theta^L[\mathscr{I}_W]}{\sim} V_i.$$

Definition 3.6. *The sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$ if for every $\varepsilon > 0$, for each p positive real number and for each $y \in Y$,*

$$\left\{ n \in \mathbb{N} : \frac{1}{n}\sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \varepsilon \right\} \in \mathscr{I}$$

and we write $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$ and simply asymptotically p-strongly $\mathscr{I}$-Cesàro equivalent if $\mathscr{L} = 1$.

Theorem 3.7. *If the sequences $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$ then, $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-statistical equivalence of multiple $\mathscr{L}$.*

Proof. Let $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$ and $\varepsilon > 0$ given. Then, for each $y \in Y$ we have

$$\begin{aligned} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p &\geq \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|\geq\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \\ &\geq \varepsilon^p \cdot |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| \end{aligned}$$

and so

$$\frac{1}{\varepsilon^p \cdot n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| .$$

Hence, for each $y \in Y$ and for a given $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \varepsilon^p \cdot \delta \right\} \in \mathscr{I}.$$

Therefore, $U_i \overset{S(\mathscr{I}_W)}{\sim} V_i$. □

Theorem 3.8. *Let $d(y,U_i) = \mathscr{O}\big(d(y,V_i)\big)$. If $\{U_i\}$ and $\{V_i\}$ are asymptotically $\mathscr{I}$-statistical equivalence of multiple $\mathscr{L}$ then, $\{U_i\}$ and $\{V_i\}$ are asymptotically p-strongly $\mathscr{I}$-Cesàro equivalence of multiple $\mathscr{L}$.*

Proof. Suppose that $d(y,U_i) = \mathscr{O}\big(d(y,V_i)\big)$ and $U_i \overset{S(\mathscr{I}_W)}{\sim} V_i$. Then, there is a $K > 0$ such that $|d(y;U_i,V_i) - \mathscr{L}| \leq K$, for all i and for each $y \in Y$. Given $\varepsilon > 0$ and for each $y \in Y$, we have

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p &= \frac{1}{n} \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|\geq\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p + \frac{1}{n} \sum_{\substack{i=1 \\ |d(y;U_i,V_i)-\mathscr{L}|<\varepsilon}}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \\ &\leq \frac{1}{n} K^p |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| + \frac{1}{n} \varepsilon^p |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| < \varepsilon\}| \\ &\leq \frac{K^p}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| + \varepsilon^p. \end{aligned}$$

Then, for any $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} |d(y;U_i,V_i) - \mathscr{L}|^p \geq \delta \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} |\{i \leq n : |d(y;U_i,V_i) - \mathscr{L}| \geq \varepsilon\}| \geq \frac{\delta^p}{K^p} \right\} \in \mathscr{I}.$$

Therefore, $U_i \overset{C_p^L[\mathscr{I}_W]}{\sim} V_i$. □

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