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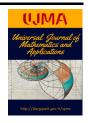
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A Bilocal Problem Associated to a Fractional Differential Inclusion of Caputo-Fabrizio Type

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Article Info

Abstract

A fractional differential inclusion defined by Caputo-Fabrizio fractional derivative with bilocal boundary conditions is studied. A nonlinear alternative of Leray-Schauder type, Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and Covitz-Nadler set-valued contraction principle are employed in order to obtain the existence of solutions when the set-valued map that define the problem has convex or non convex values.

1. Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order [1–3]. The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [4] allows to use Cauchy conditions which have physical meanings.

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [5]. The Caputo-Fabrizio operator is useful for modeling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model [6]. Another good property of this new definition is that using Laplace transform of the fractional derivative the fractional differential equation turns into a classical differential equation of integer order. Properties of this definition have been studied in [5–8]. Several recent papers are devoted to qualitative results for fractional differential equations and inclusions defined by Caputo-Fabrizio fractional derivative [9–12].

The aim of the present paper is to study the set-valued framework for problems defined by Caputo-Fabrizio operator. More exactly, we consider the following boundary value problem

$$D_{CF}^{\sigma} x(t) \in F(t, x(t)) \quad a.e. \ ([0,1]), \quad x(0) = x_0, \ x(1) = x_1, \tag{1.1}$$

where $F(.,.) : [0,1] \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ is a set-valued map, $x_0, x_1 \in \mathbf{R}$ and D_{CF}^{σ} denotes Caputo-Fabrizio's fractional derivative of order $\sigma \in (1,2)$. Our goal is to present several existence results for problem (1.1). The results are essentially based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Covitz and Nadler set-valued contraction principle. Even if we apply usual methods in the theory of existence of solutions for differential inclusions (e.g., [13]) the results obtained in the present paper are new in the framework of Caputo-Fabrizio fractional differential inclusions. As far as we know, in the literature there exists one paper dealing with fractional differential inclusions defined by Caputo-Fabrizio operator, namely [9]. In [9] it is considered a Cauchy problem, instead of a boundary value problem as in our approach.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main results.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later. Let (X,d) be a metric space with the corresponding norm $|.|$ and denote $I = [0,1]$. Denote by $\mathscr{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathscr{P}(X)$ the family of all nonempty subsets

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of X and by $\mathscr{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.): I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, where $d^*(A,B) = \sup\{d(a,B); a \in A\}$ and $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.): I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$, by $AC(I,X)$ the Banach space of all absolutely continuous functions $x(.): I \to X$ and by $L^p(I,X)$ the Banach space of all (Bochner) p-integrable functions $x(.): I \to X$; in particular, $L^1(I,X)$ is the Banach space of all (Bochner) integrable functions $x(.): I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| dt$. A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathscr{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $M: X \to \mathscr{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $M(.)$ if $x \in M(x)$. $M(.)$ is said to be bounded on bounded sets if $M(B) := \cup_{x \in B} M(x)$ is a bounded subset of X for all bounded sets B in X. $M(.)$ is said to be compact if $M(B)$ is relatively compact for any bounded sets B in X. $M(.)$ is said to be totally compact if $\overline{M(X)}$ is a compact subset of X. $M(.)$ is said to be upper semicontinuous if for any $x_0 \in X$, $M(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $M(x_0)$ there exists an open neighborhood V_0 of x_0 such that $M(V_0) \subset D$. Let E a Banach space, $Y \subset E$ a nonempty closed subset and $M(.): Y \to \mathscr{P}(E)$ a multifunction with nonempty closed values. $M(.)$ is said to be lower semicontinuous if for any open subset $D \subset E$, the set $\{y \in Y; M(y) \cap D \neq \emptyset\}$ is open. $M(.)$ is called completely continuous if it is upper semicontinuous and totally compact on X. It is well known that a compact set-valued map $M(.)$ with nonempty compact values is upper semicontinuous if and only if $M(.)$ has a closed graph (e.g., [14]).

The next results are key tools in the proof of our theorems. We recall, first, the following nonlinear alternative of Leray-Schauder type proved in [15] and its consequences.

Theorem 2.1. *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $M: \overline{D} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in M(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in M(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $M: \overline{B_r(0)} \to \mathscr{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in M(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in M(x)$ for some $\lambda > 1$.

Corollary 2.3. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $M: \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = M(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda M(x)$ for some $\lambda < 1$.

If $G(.,.): I \times X \to \mathscr{P}(X)$ is a set-valued map with compact values we define $S_G: C(I,X) \to \mathscr{P}(L^1(I,X))$ by $S_G(x) := \{g \in L^1(I,X); \quad g(t) \in G(t,x(t)) \quad a.e. \ (I)\}$. We say that $G(.,.)$ is of *lower semicontinuous type* if $S_G(.)$ is lower semicontinuous with nonempty closed and decomposable values. The next result is proved in [16].

Theorem 2.4. *Let S be a separable metric space and $G(.): S \to \mathscr{P}(L^1(I,X))$ be a lower semicontinuous set-valued map with closed decomposable values. Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.): S \to L^1(I,X)$ such that $g(s) \in G(s) \quad \forall s \in S$).*

A set-valued map $G: I \to \mathscr{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$ the function $t \to d(x, G(t))$ is measurable. A set-valued map $G(.,.): I \times X \to \mathscr{P}(X)$ is said to be *Carathéodory* if $t \to G(t,x)$ is measurable for any $x \in X$ and $x \to G(t,x)$ is upper semicontinuous for almost all $t \in I$. Moreover, $G(.,.)$ is said to be L^1-*Carathéodory* if for any $r > 0$ there exists $p_r(.) \in L^1(I, \mathbf{R})$ such that $\sup\{|v|; v \in G(t,x)\} \leq p_r(t)$ a.e. (I), $\forall x \in \overline{B_r(0)}$. The following theorem is proved in [17].

Theorem 2.5. *Let X be a Banach space, let $G(.,.): I \times X \to \mathscr{P}(X)$ be a L^1-Carathéodory set-valued map with $S_G(x) \neq \emptyset$ for all $x(.) \in C(I,X)$ and let $\Gamma: L^1(I,X) \to C(I,X)$ be a linear continuous mapping. Then the set-valued map $\Gamma \circ S_G: C(I,X) \to \mathscr{P}(C(I,X))$ defined by*

$$(\Gamma \circ S_G)(x) = \Gamma(S_G(x))$$

has compact convex values and has a closed graph in $C(I,X) \times C(I,X)$.

Note that if $dim X < \infty$, and $G(.,.)$ is as in Theorem 2.5, then $S_G(x) \neq \emptyset$ for any $x(.) \in C(I,X)$ (e.g., [17]).

The next definitions have been introduced by Caputo and Fabrizio in [5].

Definition 2.6. *a) Caputo-Fabrizio integral of order $\alpha \in (0,1)$ of a function $f \in AC_{loc}([0,\infty), \mathbf{R})$ (which means that $f'(.)$ is integrable on $[0,T]$ for any $T > 0$) is defined by*

$$I_{CF}^\alpha f(t) = (1-\alpha)f(t) + \alpha \int_0^t f(s)ds.$$

b) Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ of f is defined for $t \geq 0$ by

$$D_{CF}^\alpha f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s)ds.$$

c) Caputo-Fabrizio fractional derivative of order $\sigma = \alpha + n$, $\alpha \in (0,1)$ $n \in \mathbf{N}$ of f is defined by

$$D_{CF}^\sigma f(t) = D_{CF}^\alpha(D_{CF}^n f(t)).$$

In particular, if $\sigma = \alpha + 1$, $\alpha \in (0,1)$ $D_{CF}^\sigma f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s)ds$.

Definition 2.7. *A mapping $x(.) \in AC(I, \mathbf{R})$ is called a solution of problem* (1.1) *if there exists a function $f(.) \in L^1(I, \mathbf{R})$ such that $f(t) \in F(t, x(t))$ a.e. (I), $D_{CF}^{\alpha} x(t) = f(t)$, $t \in I$ and $x(0) = x_0$, $x(1) = x_1$.*

In order to prove our results we also need the next result proved in [11] (namely, Theorem 3.4).

Lemma 2.8. *For $\sigma = \alpha + 1$, $\alpha \in (0,1)$ and $f(.) \in L^1(I, \mathbf{R})$ the boundary value problem*

$$D_{CF}^{\sigma} x(t) = f(t), \quad x(0) = x_0, x(1) = x_1,$$

has a unique solution given by

$$x(t) = x_0 + (x_1 - x_0)t + (1-\alpha)(1-t)\int_0^t f(s)ds + \alpha(1-t)\int_0^t s f(s)ds - (1-\alpha)t\int_t^1 f(s)ds - \alpha t\int_t^1 (1-s)f(s)ds. \tag{2.1}$$

Remark 2.9. *If we define*

$$\mathscr{G}(t,s) = [(1-\alpha)(1-t) + \alpha(1-t)s]\chi_{[0,t]}(s) - [(1-\alpha)t + \alpha t(1-s)]\chi_{[t,1]}(s)$$

then the solution in (2.1) *may be written as*

$$x(t) = x_0 + (x_1 - x_0)t + \int_0^1 \mathscr{G}(t,s)f(s)ds.$$

Moreover, for any $s, t \in I$, $|\mathscr{G}(t,s)| \leq (1-\alpha) + \alpha + (1-\alpha) + \alpha = 2$.

3. The results

We present now the existence results for problem (1.1). We consider, first, the case when $F(.,.)$ is convex valued and is upper semicontinuous in the state variable.

Hypothesis 1. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.
ii) There exists $\varphi(.) \in L^1(I, \mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \leq \varphi(t)\psi(|x|) \quad a.e. \ (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis 1 is satisfied and there exists $r > 0$ such that*

$$r > |x_0| + |x_1 - x_0| + 2|\varphi|_1 \psi(r). \tag{3.1}$$

Then problem (1.1) *has at least one solution $x(.)$ such that $|x(.)|_C < r$.*

Proof. Consider $X = C(I, \mathbf{R})$ and let $r > 0$ be as in (3.1). From Definition 2.7 and Remark 2.9, the existence of solutions to problem (1.1) reduces to the existence of the solutions of the integral inclusion

$$x(t) \in x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)F(s,x(s))ds, \quad t \in I. \tag{3.2}$$

Defined the set-valued map $M : \overline{B_r(0)} \to \mathscr{P}(C(I, \mathbf{R}))$ by

$$M(x) := \{v(.) \in C(I, \mathbf{R}); v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds, f \in S_F(x)\}. \tag{3.3}$$

We show that $M(.)$ satisfies the hypotheses of Corollary 2.2. First, we show that $M(x) \subset C(I, \mathbf{R})$ is convex for any $x \in C(I, \mathbf{R})$. If $v_1, v_2 \in M(x)$ then there exist $f_1, f_2 \in S_F(x)$ such that for any $t \in I$ one has $v_i(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_i(s)ds$, $i = 1, 2$.
Let $0 \leq \alpha \leq 1$. Then for any $t \in I$ we have $(\alpha v_1 + (1-\alpha)v_2)(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds$. The values of $F(.,.)$ are convex; thus, $S_F(x)$ is a convex set and hence, $\alpha f_1 + (1-\alpha)f_2 \in M(x)$.
We show, secondly, that $M(.)$ is bounded on bounded sets of $C(I, \mathbf{R})$. Let $B \subset C(I, \mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \leq m \ \forall x \in B$. If $v \in M(x)$ there exists $f \in S_F(x)$ such that $v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$. One has for any $t \in I$

$$|v(t)| \leq |x_0| + |x_1 - x_0| + \int_0^t |\mathscr{G}(t,s)|.|f(s)|ds \leq |x_0| + |x_1 - x_0| + \int_0^t |\mathscr{G}(t,s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore, $|v|_C \leq |x_0| + |x_1 - x_0| + 2|\varphi|_1 \psi(m) \quad \forall v \in M(x)$, i.e., $M(B)$ is bounded.
Next we prove that $M(.)$ maps bounded sets into equi-continuous sets. Let $B \subset C(I, \mathbf{R})$ be a bounded set as before and $v \in M(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$|v(t) - v(\tau)| \leq |x_0 + (x_1 - x_0)t - a(\tau)| + |\int_0^t \mathscr{G}(t,s)f(s)ds - \int_0^\tau \mathscr{G}(\tau,s)f(s)ds| + |\int_\tau^t \mathscr{G}(t,s)f(s)ds| \leq |x_0 + (x_1 - x_0)t - a(\tau)| + 2\int_\tau^t \varphi(s)\psi(m)ds + \int_0^t |\mathscr{G}(t,s) - \mathscr{G}(\tau,s)|\varphi(s)\psi(m)ds.$$

Thus, $|v(t) - v(\tau)| \to 0$ as $\tau \to t$. It follows that $M(B)$ is an equi-continuous set in $C(I, \mathbf{R})$. It remains to apply Arzela-Ascoli's theorem to deduce that $M(.)$ is completely continuous on $C(I, \mathbf{R})$.
At the last step of the proof we prove that $M(.)$ has a closed graph. Let $x_n \in C(I, \mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in M(x_n) \ \forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in M(x^*)$. Since $v_n \in M(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_n(s)ds$. Define $\Gamma : L^1(I, \mathbf{R}) \to C(I, \mathbf{R})$ by $(\Gamma(f))(t) := \int_0^t \mathscr{G}(t,s)f(s)ds$. One has $\max_{t \in I} |v_n(t) - x_0 - (x_1 - x_0)t - (v^*(t) - x_0 - (x_1 - x_0)t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$. We apply Theorem 2.5 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we obtain $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f^*(s)ds$.
Therefore, $M(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.2 to deduce that either i) the inclusion $x \in M(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in M(x)$ for some $\lambda > 1$.
Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \leq |x_0| + |x_1 - x_0| + 2|\varphi|_1 \psi(r)$ which contradicts (3.1). Hence, only i) is valid and theorem is proved. $\square$

We consider, now, the case when $F(.,.)$ is not necessarily convex valued. In the first approach, $F(.,.)$ is lower semicontinuous in the state variable and, in this case, the existence result is based on the Leray-Schauder alternative for single valued maps and on Bressan-Colombo selection theorem.

Hypothesis 2. i) $F(.,.) : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathscr{L}(I) \otimes \mathscr{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.
ii) There exists $\varphi(.) \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \leq \varphi(t)\psi(|x|) \quad a.e. \ (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis 2 is satisfied and there exists $r > 0$ such that condition* (3.1) *is satisfied. Then problem* (1.1) *has at least one solution on I.*

Proof. We point out, first, that if Hypothesis 2 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [18]). Therefore, by Theorem 2.4 applied with $S = C(I,\mathbf{R})$ and $G(.) = S_F(.)$ we find a continuous mapping $f(.) : C(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x) \ \forall x \in C(I,\mathbf{R})$. Consider problem

$$x(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $X = C(I,\mathbf{R})$. By Definition 2.7 and Remark 2.9, if $x(.) \in C(I,\mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem (1.1). Let $r > 0$ that satisfies condition (3.1) and define $M : \overline{B_r(0)} \to C(I,\mathbf{R})$ by

$$(M(x))(t) := x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(x(s))ds.$$

The integral equation (3.4) is equivalent with the operator equation

$$x(t) = (M(x))(t), \quad t \in I. \tag{3.5}$$

We show, next, that $M(.)$ satisfies the hypotheses of Corollary 2.3. We note that $M(.)$ is continuous on $\overline{B_r(0)}$. By Hypotheses 2 ii) we have $|f(x(t))| \leq \varphi(t)\psi(|x(t)|)$ a.e. (I) for all $x(.) \in C(I,\mathbf{R})$. Consider $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then $|f(x_n(t))| \leq \varphi(t)\psi(r)$ a.e. (I). Using Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$, $\lim_{n\to\infty} \int_0^t \mathscr{G}(t,s)f(x_n(s))ds = \int_0^t \mathscr{G}(t,s)f(x(s))ds$ which provides the continuity of $M(.)$ on $\overline{B_r(0)}$.
As in the proof of Theorem 3.1, it follows that $M(.)$ is compact on $\overline{B_r(0)}$. With Corollary 2.3 we deduce that either i) the equation $x = M(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda M(x)$ for some $\lambda < 1$. Repeating the argument as in the proof of Theorem 3.1, if the statement ii) holds true, then we obtain a contradiction to (3.1). Thus, only the statement i) is true and problem (1.1) has a solution $x(.) \in C(I,\mathbf{R})$ with $|x(.)|_C < r$. $\qquad\square$

The second approach concerns the situation when the set-valued map is Lipschitz in the state variable. In order to obtain an existence result for problem (1.1) by using the set-valued contraction principle we introduce the following hypothesis on F.

Hypothesis 3. i) $F : I \times \mathbf{R} \to \mathscr{P}(\mathbf{R})$ has nonempty compact values is integrably bounded and for every $x \in \mathbf{R}$, $F(.,x)$ is measurable.
ii) There exists $l \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$,

$$d_H(F(t,x_1), F(t,x_2)) \leq l(t)|x_1 - x_2| \quad \forall \ x_1, x_2 \in \mathbf{R}.$$

iii) There exists $L \in L^1(I,\mathbf{R}_+)$ such that for almost all $t \in I$, $d(0,F(t,0)) \leq L(t)$.

Theorem 3.3. *Assume that Hypothesis 3 is satisfied and $2|l|_1 < 1$. Then problem* (1.1) *has a solution.*

Proof. We consider problem (1.1) as a fixed point problem. More precisely, define the set-valued map $M : C(I,\mathbf{R}) \to \mathscr{P}(C(I,\mathbf{R}))$ by

$$M(x) := \{v(.) \in C(I,\mathbf{R}); v(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds, f \in S_F(x)\}.$$

The multifunction $t \to F(t,x(t))$ is measurable; thus, with the measurable selection theorem it has a measurable selection $f : I \to \mathbf{R}$. At the same time, since F is integrably bounded, $f \in L^1(I,\mathbf{R})$. Hence, $S_F(x) \neq \emptyset$. The fixed points of M are solutions of problem (1.1). We show, next, that M verifies the assumptions of Covitz-Nadler contraction principle ([19]). Since $S_F(x) \neq \emptyset$, it follows that $M(x) \neq \emptyset$ for any $x \in C(I,\mathbf{R})$.
Now, we prove that $M(x)$ is closed for any $x \in C(I,\mathbf{R})$. Let $\{x_n\}_{n\geq 0} \in M(x)$ such that $x_n \to x^*$ in $C(I,\mathbf{R})$. Then $x^* \in C(I,\mathbf{R})$ and there exists $f_n \in S_F(x_n)$ such that $x_n(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_n(s)ds, t \in I$. From Hypothesis 3 and the fact that the values of F are compact, one may pass to a subsequence to obtain that f_n converges to $f \in L^1(I,\mathbf{R})$ in $L^1(I,\mathbf{R})$. In particular, $f \in S_F(x)$ and for any $t \in I$ we have $x_n(t) \to x^*(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f(s)ds$, i.e., $x^* \in M(x)$ and $M(x)$ is closed.
It remains to prove that M is a contraction on $C(I,\mathbf{R})$. Let $x_1, x_2 \in C(I,\mathbf{R})$ and $v_1 \in T(x_1)$. Then, there exists $f_1 \in S_F(x_1)$ such that $v_1(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_1(s)ds, t \in I$. Consider the multifunction

$$S(t) := F(t,x_2(t)) \cap \{x \in \mathbf{R}; |f_1(t) - x| \leq l(t)|x_1(t) - x_2(t)|\}, \quad t \in I.$$

Taking into account Hypothesis 3, one has

$$d_H(F(t,x_1(t)), F(t,x_2(t))) \leq l(t)|x_1(t) - x_2(t)|, \quad t \in I,$$

i.e., S has nonempty closed values. On the other hand, S is measurable; thus, there exists f_2 a measurable selection of S. It follows that $f_2 \in S_F(x_2)$ and for any $t \in I$, $|f_1(t) - f_2(t)| \le l(t)|x_1(t) - x_2(t)|$. Define

$$v_2(t) = x_0 + (x_1 - x_0)t + \int_0^t \mathscr{G}(t,s)f_2(s)ds, \quad t \in I.$$

One has $|v_1(t) - v_2(t)| \le \int_0^t |\mathscr{G}(t,s)||f_1(s) - f_2(s)|ds \le 2\int_0^t l(s)|x_1(s) - x_2(s)|ds \le 2|l|_1|x_1 - x_2|_C$. Therefore, $|v_1 - v_2|_C \le 2|l|_1|x_1 - x_2|_C$. By interchanging the roles of x_1 and x_2 we deduce

$$d_H(M(x_1), M(x_2)) \le 2|l|_1|x_1 - x_2|_C.$$

Thus, M has a fixed point which is a solution to problem (1.1). $\qquad\square$

4. Conclusions

In this paper we obtained several existence results for solutions of a bilocal problem associated to a fractional differential inclusion defined by Caputo-Fabrizio operator. In the case when the values of the set-valued map that defines the differential inclusion are convex and the set-valued map is upper semicontinuous in the state variable, the proof is based on a nonlinear alternative of Leray-Schauder type; in the situation when the values of the set-valued map are not necessarily convex and the set-valued map is lower semicontinuous in the state variable, the proof relies on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values. Also, if the multifunction has non convex values and is Lipischitz in the state variable an existence result is provided by applying Covitz and Nadler set-valued contraction principle. Such kind of results, that are new in the framework of Caputo-Fabrizio fractional differential inclusions, may be useful, afterwards, in order to obtain qualitative properties concerning the solutions of the problem considered.

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