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On the $\Delta^f_{\Lambda^2}$-Statistical Convergence on Product Time Scale

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Abstract

In this paper, we first define a new density of a Δ-measurable subset of a product time scale Λ^2 with respect to an unbounded modulus function. Then, by using this definition, we introduce the concepts of $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy for a Δ-measurable real-valued function defined on product time scale Λ^2 and also obtain some results about these new concepts. Finally, we present the definition of strong $\Delta^f_{\Lambda^2}$-Cesaro summability on Λ^2 and investigate the connections between these new concepts.

1. Introduction

The idea of statistical convergence of number sequences was formally introduced by Fast [1] and also independently Steinhaus [2]. This concept is a generalization of classical convergence and has a close relation with the concept of density of the subset of natural numbers $\mathbb{N}$. The natural density of $K \subseteq \mathbb{N}$ is defined by $\delta(K) = \lim_n n^{-1} |\{k \le n : k \in K\}|$ if the limit exists, where and throughout the paper $|K|$ denotes the cardinality of K. A sequence $x = (x_k)$ is said to be statistically convergent to L if, for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0,$$

and we denote this by $st - \lim x = L$. In later years, statistical convergence has taken a very important place in mathematical analysis and has been studied by many researchers, see [3–12]. Another notion that can be of importance is modulus function which was first given by Nakano [13]. The readers can consult the works [14–16] for more on this function. We remind here that a modulus $f : [0,\infty) \to [0,\infty)$ is a function which satisfies

i) $f(x) = 0$ if and only if $x = 0$,
ii) $f(x+y) \le f(x) + f(y)$ for every $x \ge 0$, $y \ge 0$,
iii) f is increasing,
iv) f is continuous from right at 0.

We can easily see that a modulus function f is continuous everywhere on $[0,\infty)$ from above properties (ii) and (iv). A modulus function may be bounded or unbounded. As in example, $f(x) = \frac{x}{1+x}$ is bounded, while $f(x) = x^p$ is unbounded where $0 < p \le 1$.

In [17], by means of an unbounded modulus function, Aizpuru et al. firstly presented a new idea of density for the subset of $\mathbb{N}$. With this way, they also defined a new convergence idea with the name f-statistical convergence and show that it is between classical convergence and statistical convergence. The readers can found further works using this concept in the references [18, 19].

A time scale is an arbitrary closed subset of the real numbers $\mathbb{R}$ and it is denoted by the symbol $\mathbb{T}$. We here suppose that it has the subspace topology which is inherited from $\mathbb{R}$ with the standart topology. The calculus of time scales was constructed by Hilger [20], and it allows to the unification of continuous and discrete cases. After that, this theory has received much attention [21–26] as it has tremendous potential for applications. Moreover, the idea of statistical convergence has been studied on time scales in [27] and [28], independently. Later, by inspiring from these works, various researchers have done many studies using the time scale on the summability theory in the literature, see [29–39]. Let's now remember some necessary concepts about the time scale calculus before proceeding further.

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For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. Here we take $\inf \emptyset = \sup \mathbb{T}$, where $\emptyset$ is an empty set. For $a \leq b$, a closed interval in $\mathbb{T}$ is defined by $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Similarly, half-open intervals or open intervals can be defined on time scales. Let F_1 denote the family of all intervals of $\mathbb{T}$ having the form $[a,b)_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t < b\}$ with $a,b \in \mathbb{T}$ and $a \leq b$. Then the set function $m_1 : F_1 \to [0,\infty)$ define as $m_1([a,b)_{\mathbb{T}}) = b-a$ is a countably additive measure on F_1. The Caratheodory extension of the set function m_1 associated with family F_1 is said to be the Lebesgue $\Delta-$measure on $\mathbb{T}$ and also this is denoted by μ_Δ, see [23]. Also from the work [23] by Guseinov, one knows that if $a \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta-$measurable and $\mu_\Delta(\{a\}) = \sigma(a) - a$. If $a,b \in \mathbb{T}$ and $a \leq b$, then $\mu_\Delta([a,b)_{\mathbb{T}}) = b-a$ and $\mu_\Delta((a,b)_{\mathbb{T}}) = b - \sigma(a)$. If $a,b \in \mathbb{T} \setminus \{\max \mathbb{T}\}$ and $a \leq b$, then $\mu_\Delta((a,b]_{\mathbb{T}}) = \sigma(b) - \sigma(a)$ and $\mu_\Delta([a,b]_{\mathbb{T}}) = \sigma(b) - a$.
Turan and Başarır [36] gave Δ_f-convergence by combining the ideas of Seyyidoğlu and Tan [27], Turan and Duman [28], and Aizpuru et al. [17] as in the following:

Definition 1.1. *[36] Let $\mathbb{T}$ be a time scale such that $\inf \mathbb{T} = \alpha > 0$ and $\sup \mathbb{T} = \infty$ and let f be a modulus function. A $\Delta-$measurable function $g : \mathbb{T} \to \mathbb{R}$ is Δ_f- convergent to a number L on $\mathbb{T}$, if for every $\varepsilon > 0$*

$$\lim_{t \to \infty} \frac{f(\mu_\Delta(\{s \in [\alpha,t]_{\mathbb{T}} : |g(s) - L| \geqslant \varepsilon\}))}{f(\mu_\Delta([\alpha,t]_{\mathbb{T}}))} = 0,$$

which is denoted by $\Delta_f - \lim_{t \to \infty} g(t) = L$

Quite recently, Çınar et al. [32] carried statistical convergence and its related concepts which are given on 1-dimensional time scales to an arbitrary product time scales. Before remembering these definitions, let's give some necessary concepts and notations that we will use throughout this study. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be a time scale. Consider the Cartesian product

$$\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2 = \{t = (t_1,t_2) : t_1 \in \mathbb{T}_1 \text{ and } t_2 \in \mathbb{T}_2\}.$$

Then Λ^2 is called an 2-dimensional time scale or product time scale. Here, we are interested in a product time scale $\Lambda^2 = \mathbb{T}_1 \times \mathbb{T}_2$ such that $\inf \mathbb{T}_1 = t_0$ and $\sup \mathbb{T}_1 = \infty$; $\inf \mathbb{T}_2 = r_0$ and $\sup \mathbb{T}_2 = \infty$. For convenience, we denote $A := \left\{[t_0,t]_{\mathbb{T}_1} \times [r_0,r]_{\mathbb{T}_2}\right\}$ for $(t,r) \in \Lambda^2$. Thanks to the work [25] given by Bohner and Guseinov, it is clear that $\mu_\Delta(A) = \mu_\Delta\left([t_0,t]_{\mathbb{T}_1}\right).\mu_\Delta\left([r_0,r]_{\mathbb{T}_2}\right)$.

Definition 1.2. *[32] Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function. Then g is said to be statistically convergent to L on Λ^2 , if for every $\varepsilon > 0$,*

$$\lim_{(t,r) \to \infty} \frac{\mu_\Delta(\{(s,u) \in A : |g(s,u) - L| \geq \varepsilon\})}{\mu_\Delta(A)} = 0,$$

which is denoted by $st_{\Lambda^2} - \lim_{(t,r) \to \infty} g(t,r) = L$.

Definition 1.3. *[32] Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function and $0 < p < \infty$. Then we say that g is strongly p-double Cesaro summable to L on Λ^2, if*

$$\lim_{(t,r) \to \infty} \frac{1}{\mu_\Delta(A)} \iint_A |g(s,u) - L|^p \Delta s \Delta u = 0.$$

We write $[w_p]_{\Lambda^2}$ for the set of all strongly p-double Cesaro summable functions on Λ^2.

The aim of this study is to extend the concept of f-statistical convergence and its related notions to any product time scale, in light of works Aizpuru et al. [17], Turan and Başarır [36] and Çınar et al. [32].
This paper has the following order. In Section 2, we introduce the new notions such as $\Delta^f_{\Lambda^2}$-density, $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy on product time scales, where f is any unbounded modulus. We also establish some results related to these new concepts. In Section 3, the definition of strong $\Delta^f_{\Lambda^2}$-Cesaro summability on any product time scale is presented, and we examine the connections between strong $\Delta^f_{\Lambda^2}$-Cesaro summability and $\Delta^f_{\Lambda^2}$-statistical convergence, Cesaro summability.

2. $\Delta^f_{\Lambda^2}$-Density, $\Delta^f_{\Lambda^2}$-Statistical Convergence and $\Delta^f_{\Lambda^2}$-Statistical Cauchy on Product Time Scale

We first define a new type of density on a product time scale Λ^2, namely $\Delta^f_{\Lambda^2}$-density, by using the idea of Aizpuru et al. [17]. Then, with the aid of this definition, the new concepts such as $\Delta^f_{\Lambda^2}$-statistical convergence and $\Delta^f_{\Lambda^2}$-statistical Cauchy on any product time scale are introduced. Throughout the paper let f be an unbounded modulus function.

Definition 2.1. *Let Ω be a Δ-measurable subset of Λ^2. Then, the $\Delta^f_{\Lambda^2}$-density of Ω on Λ^2 is defined by*

$$\delta^f_{\Lambda^2}(\Omega) = \lim_{(t,r) \to \infty} \frac{f(\mu_\Delta(\Omega(t,r)))}{f(\mu_\Delta(A))}$$

if this limit exists, where $\Omega(t,r) = \{(s,u) \in A : (s,u) \in \Omega\}$ for $(t,r) \in \Lambda^2$.

Definition 2.2. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we say that g is $\Delta^f_{\Lambda^2}$-statistically convergent to L on Λ^2, if for every $\varepsilon>0$,*

$$\delta^f_{\Lambda^2}\left(\left\{(t,r)\in\Lambda^2:|g(t,r)-L|\geq\varepsilon\right\}\right)=0$$

holds, i.e.,

$$\lim_{(t,r)\to\infty}\frac{f\left(\mu_\Delta\left(\{(s,u)\in A:|g(s,u)-L|\geq\varepsilon\}\right)\right)}{f\left(\mu_\Delta(A)\right)}=0,$$

which is denoted by $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. Also, we denote the set of all $\Delta^f_{\Lambda^2}$-statistically convergent functions on Λ^2 by $S^f_{\Lambda^2}$.

Remark 2.3. *If we choose $f(x)=x$ in Definition 2.2, then $\Delta^f_{\Lambda^2}$-statistical convergence is reduced to statistical convergence given in Definition 1.2.*

Proposition 2.4. *If $g:\Lambda^2\to\mathbb{R}$ is $\Delta^f_{\Lambda^2}$-statistically convergent function, then its limit is unique.*

Proof. The proof can be carried out by using similar techniques to Proposition 2.4 in [32]. □

Proposition 2.5. *Let $g,h:\Lambda^2\to\mathbb{R}$ be Δ-measurable functions with $st^f_{\Lambda^2}-\lim g(t,r)=L_1$ and $st^f_{\Lambda^2}-\lim h(t,r)=L_2$. Then, we have:*
i) $st^f_{\Lambda^2}-\lim(g(t,r)+h(t,r))=L_1+L_2$,
ii) $st^f_{\Lambda^2}-\lim(cg(t,r))=cL_1$ for any $c\in\mathbb{R}$.

Proof. The proof can be carried out by using similar techniques to Proposition 2.5 in [32]. □

Theorem 2.6. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. If $\lim_{(t,r)\to\infty}g(t,r)=L$, then $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$.*

Proof. Suppose that $\lim_{(t,r)\to\infty}g(t,r)=L$. Then, the set $\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}$ is bounded, for each $\varepsilon>0$. Since

$$\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\subset\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}$$

and modulus function f is increasing, we get

$$\frac{f\left(\mu_\Delta\left(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\right)\right)}{f\left(\mu_\Delta(A)\right)}\leqslant\frac{f\left(\mu_\Delta\left(\left\{(s,u)\in\Lambda^2:|g(s,u)-L|\geqslant\varepsilon\right\}\right)\right)}{f\left(\mu_\Delta(A)\right)}.$$

Taking limit as $(t,r)\to\infty$ in here, we obtain

$$\lim_{(t,r)\to\infty}\frac{f\left(\mu_\Delta\left(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\right)\right)}{f\left(\mu_\Delta(A)\right)}=0,$$

which means that $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. □

Theorem 2.7. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$ implies $st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$.*

Proof. Suppose that $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. Then, using the limit definition and also properties of subadditivity of the modulus function f, for every $p\in\mathbb{N}$, for sufficiently large $(t,r)\in\Lambda^2$, we have

$$f\left(\mu_\Delta\left(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\right)\right)\leqslant\frac{1}{p}f\left(\mu_\Delta(A)\right)\leqslant\frac{1}{p}pf\left(\frac{\mu_\Delta(A)}{p}\right)=f\left(\frac{\mu_\Delta(A)}{p}\right).$$

Also, since f is increasing, we get

$$\frac{\mu_\Delta\left(\{(s,u)\in A:|g(s,u)-L|\geqslant\varepsilon\}\right)}{\mu_\Delta(A)}\leqslant\frac{1}{p},$$

which means that $st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$. □

Corollary 2.8. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we have*

$$\lim_{(t,r)\to\infty}g(t,r)=L\Rightarrow st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L\Rightarrow st_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L.$$

Theorem 2.9. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function and $h:\mathbb{R}\to\mathbb{R}$ be a continuous function at L. If $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}g(t,r)=L$, then $st^f_{\Lambda^2}-\lim_{(t,r)\to\infty}h(g(t,r))=h(L)$.*

Proof. Using techniques similar to Lemma 3.11 in [28], the proof can be carried out easily and is therefore omitted. □

Definition 2.10. *A Δ-measurable function $g:\Lambda^2\to\mathbb{R}$ is $\Delta^f_{\Lambda^2}$-statistical Cauchy on Λ^2, if for every $\varepsilon>0$, there exist some numbers $t_1>t_0$ and $r_1>r_0$ such that $\delta^f_{\Lambda^2}\left(\left\{(t,r)\in\Lambda^2:\left|g(t,r)-g(t_1,r_1)\right|\geq\varepsilon\right\}\right)=0$.*

Theorem 2.11. *Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, the following statements are equivalent:*
i) g is $\Delta^f_{\Lambda^2}$-statistical convergent on Λ^2,
ii) g is $\Delta^f_{\Lambda^2}$-statistical Cauchy on Λ^2.

Proof. Using techniques similar to Theorem 3 in [27], the proof can be carried out easily and is therefore omitted. □

3. Strong $\Delta^f_{\Lambda^2}$-Cesaro Summability on Product Time Scale

We begin in here by presenting the last new definition, namely, strong $\Delta^f_{\Lambda^2}$-Cesaro summability on Λ^2.

Definition 3.1. *Let f be a modulus function and $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function. Then, we say that g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L on Λ^2, if*

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A f\left(\left|g(s,u)-L\right|\right)\Delta s\Delta u=0.$$

We also denote the set of all strongly $\Delta^f_{\Lambda^2}$-Cesaro summable functions on Λ^2 by $[w]^f_{\Lambda^2}$.

Lemma 3.2. *[15] Let f be any modulus function and let $0<\delta<1$. Then, for each $x\geqslant\delta$, we have $f(x)\leqslant 2f(1)\delta^{-1}x$.*

Lemma 3.3. *[16] Let f be any modulus function. Then $\lim\limits_{t\to\infty}\frac{f(t)}{t}$ exists.*

The next theorem gives us the connection between the concepts of strong $\Delta^f_{\Lambda^2}$-Cesaro summability and strong double Cesaro summability given in Definition 1.3.

Theorem 3.4. ***i)*** *For any modulus function f, we have $[w]_{\Lambda^2}\subset[w]^f_{\Lambda^2}$.*
ii) *Let f be any modulus function. If $\lim\limits_{t\to\infty}\frac{f(t)}{t}>0$, then we have $[w]^f_{\Lambda^2}\subset[w]_{\Lambda^2}$.*

Proof. **i)** Let $g\in[w]_{\Lambda^2}$ with the limit L. Then, we have

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A \left|g(s,u)-L\right|\Delta s\Delta u=0.$$

Since modulus f is continuous, for any given $\varepsilon>0$, we may choose δ with $0<\delta<1$ such that $f(t)<\varepsilon$ for every t with $0\leqslant t\leqslant\delta$. Then, by Lemma 3.2, we write

$$\begin{aligned}\frac{1}{\mu_\Delta(A)}\iint\limits_A f\left(\left|g(s,u)-L\right|\right)\Delta s\Delta u&=\frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|<\delta}} f\left(\left|g(s,u)-L\right|\right)\Delta s\Delta u+\frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|\geqslant\delta}} f\left(\left|g(s,u)-L\right|\right)\Delta s\Delta u\\&\leqslant\varepsilon+2f(1)\delta^{-1}\frac{1}{\mu_\Delta(A)}\iint\limits_A\left|g(s,u)-L\right|\Delta s\Delta u.\end{aligned}$$

Taking limit as $(t,r)\to\infty$ in here, because $\varepsilon>0$ is arbitrary, we obtain that $g\in[w]^f_{\Lambda^2}$.
ii) From the proof of Proposition 1 of [16], one has $\beta=\lim\limits_{t\to\infty}\frac{f(t)}{t}=\inf\left\{\frac{f(t)}{t}:t>0\right\}$. Then, we get $f(t)\geqslant\beta t$ for all $t\geqslant0$. Now let $g\in[w]^f_{\Lambda^2}$ with the limit L. Since $\beta>0$, we get

$$\lim_{(t,r)\to\infty}\frac{1}{\mu_\Delta(A)}\iint\limits_A f\left(\left|g(s,u)-L\right|\right)\Delta s\Delta u\geqslant\lim_{(t,r)\to\infty}\frac{\beta}{\mu_\Delta(A)}\iint\limits_A\left|g(s,u)-L\right|\Delta s\Delta u.$$

It follows that $g\in[w]_{\Lambda^2}$ and so the proof is completed. □

Before giving the last theorem of this study, we give some lemmas that will be used in the its proof.

Lemma 3.5. *[32] Let $g:\Lambda^2\to\mathbb{R}$ be a Δ-measurable function and let*

$$\Omega(t,r)=\left\{(s,u)\in A:\left|g(s,u)-L\right|\geqslant\varepsilon\right\}$$

for $\varepsilon>0$. Then, we have

$$\mu_\Delta(\Omega(t,r))\leqslant\frac{1}{\varepsilon}\iint\limits_{\Omega(t,r)}\left|g(s,u)-L\right|\Delta s\Delta u\leqslant\frac{1}{\varepsilon}\iint\limits_A\left|g(s,u)-L\right|\Delta s\Delta u.$$

Lemma 3.6. *Let $t_1,t_2 \in \mathbb{T}_1$, $r_1,r_2 \in \mathbb{T}_2$ and $c,d \in \mathbb{R}$ and $D = \left\{[t_1,t_2]_{\mathbb{T}_1} \times [r_1,r_2]_{\mathbb{T}_2}\right\}$. If $\phi : D \to (c,d)$ is Δ-integrable and $F : (c,d) \to \mathbb{R}$ is convex, then*

$$F\left(\frac{\iint_D \phi(s,u)\,\Delta s\Delta u}{\mu_\Delta(D)}\right) \leqslant \frac{\iint_D F(\phi(s,u))\,\Delta s\Delta u}{\mu_\Delta(D)}.$$

Proof. It can be proved by considering a similar way in the proof of Theorem 4.1 of [22]. □

Now, we construct a connection between $\Delta^f_{\Lambda^2}$-statistical convergence and strong $\Delta^f_{\Lambda^2}$-Cesaro summability in the next theorem.

Theorem 3.7. *Let $g : \Lambda^2 \to \mathbb{R}$ be a Δ-measurable function. Then, we have*
i) Let f be a convex, modulus function such that there exists a positive constant c such that $f(xy) \geq cf(x)f(y)$ for all $x \geq 0$, $y \geq 0$, and $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$ exist. If g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, then $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$.
ii) If $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$ and g is a bounded function, then g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, for any modulus f.

Proof. i) Let g be strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L. Using the lemmas 3.5 and 3.6, for any given $\varepsilon > 0$, we obtain that

$$\begin{aligned}
\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\,\Delta s\Delta u &\geqslant \frac{\mu_\Delta(A)}{\mu_\Delta(A)} f\left(\frac{\iint_A f(|g(s,u)-L|)\,\Delta s\Delta u}{\mu_\Delta(A)}\right),\\
&\geqslant f\left(\frac{\iint\limits_{\substack{A\\ |g(s,u)-L|\geqslant\varepsilon}} f(|g(s,u)-L|)\,\Delta s\Delta u}{\mu_\Delta(A)}\right),\\
&\geqslant f\left(\frac{\mu_\Delta(\{(s,u)\in A : |g(s,u)-L| \geqslant \varepsilon\})}{\mu_\Delta(A)}\varepsilon\right),\\
&\geqslant cf(\mu_\Delta(\{(s,u)\in A : |g(s,u)-L| \geqslant \varepsilon\}))\,f\left(\frac{\varepsilon}{\mu_\Delta(A)}\right),\\
&= c\varepsilon \frac{f(\mu_\Delta(A))}{\mu_\Delta(A)} \frac{f(\mu_\Delta(\{(s,u)\in A : |g(s,u)-L| \geqslant \varepsilon\}))}{f(\mu_\Delta(A))} \frac{f\left(\frac{\varepsilon}{\mu_\Delta(A)}\right)}{\frac{\varepsilon}{\mu_\Delta(A)}}.
\end{aligned}$$

Also, by using $\lim_{t\to\infty} \frac{f(t)}{t} > 0$ and $\lim_{t\to\infty} \frac{f(1/t)}{1/t} > 0$, since g is strongly $\Delta^f_{\Lambda^2}$-Cesaro summable to L, we get $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$.
ii) Let g be bounded and $st^f_{\Lambda^2} - \lim_{(t,r)\to\infty} g(t,r) = L$. Then, there exists a positive number M such that $|g(s,u)-L| \leq M$ for all $(s,u) \in \Lambda^2$. For any given $\varepsilon > 0$, we get

$$\begin{aligned}
\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\,\Delta s\Delta u &= \frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|\geqslant\varepsilon}} f(|g(s,u)-L|)\,\Delta s\Delta u + \frac{1}{\mu_\Delta(A)}\iint\limits_{\substack{A\\ |g(s,u)-L|<\varepsilon}} f(|g(s,u)-L|)\,\Delta s\Delta u,\\
&\leqslant \frac{\mu_\Delta(\{(s,u)\in A : |g(s,u)-L| \geqslant \varepsilon\})}{\mu_\Delta(A)} f(M) + \frac{\mu_\Delta(A)}{\mu_\Delta(A)} f(\varepsilon).
\end{aligned}$$

Hence, letting $(t,r) \to \infty$ on both sides in here and then $\varepsilon \to 0$, by means of Theorem 2.7, we get

$$\frac{1}{\mu_\Delta(A)}\iint_A f(|g(s,u)-L|)\,\Delta s\Delta u = 0,$$

which completes the proof. □

Remark 3.8. *If we take $f(x) = x$ in Theorem 3.7, we get Theorem 2.10 of [32] for the special case $p = 1$.*

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