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Set Inner Amenability for Semigroups

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Abstract

In this paper, we present a new concept of inner amenability for a non-empty arbitrary subset A of discrete semigroup S called A-inner amenability. This condition is considerably weaker than ordinary inner amenability. Further, we show some relationships between this version of inner amenability and Følner's condition.

Keywords: Set amenability, set inner amenability, semigroup. 2010 MSC: 43A07.

1. Introduction

Throughout this paper, S will denote a discrete semigroup. We shall use $\ell^{\infty}(S)$ to denote the Banach space of bounded real-valued functions on S with the supremum norm. For every subset A of S, let χ_A denote its characteristic function, that is

$$\chi_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$

A mean is a linear functional $m \in \ell^{\infty}(S)^*$ such that $m(\chi_S) = ||m|| = 1$. For each $s \in S$ and $f \in \ell^{\infty}(S)$ we define ${}_s f$ and f_s on S by $({}_s f)(t) = f(st)$ and $(f_s)(t) = f(ts)$ for all $t \in S$. We say that $m \in \ell^{\infty}(S)^*$ is invariant if $m({}_s f) = m(f) = m(f_s)$ for all $s \in S$ and $f \in \ell^{\infty}(S)$. A semigroup S is said to be amenable if it has an invariant mean m on $\ell^{\infty}(S)$. Also, let $\ell^1(S)$ denote the Banach space of all real-valued functions φ on S such that $\|\varphi\|_1 := \sum_{x \in S} |\varphi(x)| < \infty$. With pointwise addition and scalar multiplication, and with convolution

$$(\varphi * \psi)(x) = \sum_{st=x} \varphi(s)\psi(t) \quad (x \in S),$$

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as product, $\ell^1(S)$ is a Banach algebra.

We say that $m \in \ell^{\infty}(S)^*$ is inner invariant mean if

$$m(_sf) = m(f_s),$$

for all $s \in S$ and $f \in \ell^{\infty}(S)$. Following Ling[11], a semigroup S is said to be inner amenable if it has an inner invariant mean m on $\ell^{\infty}(S)$.

We will show that many results concerning inner amenability of semigoups have similar analogues for A-inner amenability. Finally, a number of equivalent conditions characterizing A-inner amenable semigroups is given.

2. Set inner amenability for semigroup

We start off with the following definition, which is the most important here.

Definition 2.1. Let S be a semigroup and $\phi \neq A \subseteq S$. We say that a mean m on $\ell^{\infty}(S)$, is an inner A-invariant mean if for all $a \in A$ and $f \in \ell^{\infty}(S)$ we have

$$m(_af) = m(f_a).$$

A semigroup S which admits inner A-invariant means is called A-inner amenable.

In other words, invarince of m is only required in the subsets of S. It follows immediately that every inner amenable semigroup is A-inner amenable for all subsets A of S. But the converse is not true in genaral. (see Examples 3.2 and 3.4)

For an arbitrary non-empty subset A of semigroup S, we denote by $\mathcal{H}(A)$, the real linear span of functions of the form $af - f_a$, where $a \in A$ and $f \in \ell^{\infty}(S)$. In the following theorem, a sequence of characterizations of A-inner amenable semigroup is given.

Theorem 2.2. Let S be a semigroup with non-empty subset A. Then the following proporties are equivalent:

- (a) S is an A-inner amenable semigroup.
- (b) for every $h \in \mathcal{H}(A)$, $\sup\{h(x) : x \in S\} \ge 0$.
- (c) $\inf\{\|1-h\|_{\infty}: h \in \mathcal{H}(A)\} = 1.$

Proof. $(a) \Rightarrow (b)$. Let *m* be an inner *A*-invariant mean on $\ell^{\infty}(S)$. If $h \in \mathcal{H}(A)$, then $\sup\{h(x) : x \in S\} \ge m(h) = 0$. Thus, the property (b) holds.

 $(b) \Rightarrow (c)$. For every $h \in \mathcal{H}(A)$, we have

$$0 \le \sup\{-h(x) : x \in S\} = -\inf\{h(x) : x \in S\}$$

This shows that, $\inf\{h(x) : x \in S\} \leq 0$. Hence, for any $\epsilon > 0$, there exists $x_0 \in S$ such that $h(x_0) < \epsilon$, and so $1 - h(x_0) > 1 - \epsilon$. Therefore, $||1 - h||_{\infty} \geq 1$ for any $h \in \mathcal{H}(A)$. But $0 \in \mathcal{H}(A)$, $\inf\{||1 - h||_{\infty} : h \in \mathcal{H}(A)\} \leq ||1 - 0||_{\infty} = 1$.

 $(c) \Rightarrow (a)$. Assume that the property (c) holds. Now by the Hahn-Banach theorem, there exists a linear functional m on $\ell^{\infty}(S)$ with norm one such that $m(\mathcal{H}(A)) = \{0\}$ and $m(1) = \inf\{\|1 - h\|_{\infty} : h \in \mathcal{H}(A)\}$. So m is an inner A-invariant mean on $\ell^{\infty}(S)$.

A non-empty subset A of S is said to act injectively on the left (right) of semigroup S, if ax = ay(xa = ya) implies x = y for every $a \in A, x, y \in S$. We say that A acts injectively on the semigroup S, if it acts on both left and right of S. In particular, if S is a cancellative semigroup, then every non-empty subset of S acts injectively on S. **Theorem 2.3.** Let A act injectively on the left of semigroup S. Then S is A-inner amenable if and only if $\mathcal{H}(A)$ is not norm dense in $\ell^{\infty}(S)$.

Proof. We suppose that m be a nonzero self-adjoint functional $m \in \ell^{\infty}(S)^*$ such that $m(\mathcal{H}(A)) = 0$. Consider the decomposition $m = m^+ - m^-$, such that

$$m^+(f) = \sup\{m(g) : 0 \le g \le f\}$$

and

$$m^{-}(f) = -\inf\{m(g) : 0 \le g \le f\}$$

for all $f \in \ell^{\infty}(S)$ with $f \geq 0$. A similar proof of Teorem 2 of [11], shows that m^+ and m^- are inner *A*-invariant mean on $\ell^{\infty}(S)$.

In the following proposition, we see that increasing union of a family of A-inner amenable semigroups is A-inner amenable.

Proposition 2.4. Let $\{S_{\alpha}\}_{\alpha \in I}$ be a family of subsemigroups of S such that for each $\alpha \in I$, S_{α} is A_{α} -inner amenable and $A = \bigcup_{\alpha \in I} A_{\alpha}$ with the following conditions:

(a) for each S_{α}, S_{β} that are A_{α} -inner amenable and A_{β} -inner amenable, respectively, there exists $S_{\gamma} \supseteq S_{\alpha} \cup S_{\beta}$ such that S_{γ} is A_{γ} -inner amenable with $A_{\gamma} \supseteq A_{\alpha} \cup A_{\beta}$.

(b) $S = \bigcup_{\alpha \in I} S_{\alpha}$. Then S is A-inner amenable.

Proof. Assume that $h = \sum_{k=1}^{n} (a_k(f_k) - (f_k)a_k)$ such that $f_k \in \ell^{\infty}(S)$, $a_k \in A$. By the assumption, there exists a S_{λ} such that $a_k \in A_{\lambda}$. Since S_{λ} is A_{λ} -inner amenable, it follows from Theorem 2.2, $\sup\{h(x) : x \in S_{\lambda}\} \ge 0$. In particular, $\sup\{h(x) : x \in S\} \ge 0$. Again by Theorem 2.2, S is A-inner amenable.

Remark 2.5. A subsemigroup of an A-inner amenable semigroup need not be A-inner amenable. As an example let S be any non A-inner amenable semigroup, and let S^o contain S and one new element o such that os = so = oo = o, and S is a subsemigroup of S^o. Then S^o has an inner A-invariant mean: m(f) = f(o), whereas S is not an A-inner amenable.

Theorem 2.6. Let T be a subsemigroup of S and $A \subseteq T$. Then T is an A-inner amenable if and only if S is an A-inner amenable with mean m such that $m(\chi_T) = 1$.

Proof. Let $\theta: T \longrightarrow S$ be the embedding map. Then it induces $\overline{\theta}: \ell^{\infty}(S) \longrightarrow \ell^{\infty}(T)$ by $\overline{\theta}(f) = f|_T$. It is easily that $\overline{\theta}$ is bounded and linear. Consider $\overline{\theta}^*: \ell^{\infty}(T)^* \longrightarrow \ell^{\infty}(S)^*$. Now suppose that $m \in \ell^{\infty}(T)^*$ is an inner A-invariant mean. Clearly $\overline{\theta}^*(m)$ is a mean on $\ell^{\infty}(S)$. Also, for any $f \in \ell^{\infty}(S)$, $a \in A$, it is easy to see that

$$\overline{\theta}(af) = (af)|_T = a(f|_T) = a(\overline{\theta}(f)),$$

and

$$\overline{\theta}(f_a) = (f_a)|_T = (f|_T)_a = (\overline{\theta}(f))_a.$$

Therefore, for all $f \in \ell^{\infty}(S)$, $a \in A$, we get

$$(\overline{\theta}^*(m))(af) = m(\overline{\theta}(af)) = m(a(\overline{\theta}(f))) = m((\overline{\theta}(f))_a) = (\overline{\theta}^*(m))(f_a).$$

This means that, $\overline{\theta}^*(m)$ is an inner A-invariant mean on $\ell^{\infty}(S)$. Also,

$$(\overline{\theta}^*(m))(\chi_T) = m(\overline{\theta}(\chi_T)) = m(\chi_T|_T) = m(1) = 1.$$

$$\varphi(f)(t) = \begin{cases} f(t) & t \in T \\ 0 & t \in S \backslash T \end{cases}$$

It is obvious that φ is a bounded and linear. Consider $\varphi^* : \ell^{\infty}(S)^* \longrightarrow \ell^{\infty}(T)^*$. For any $f \in \ell^{\infty}(T)$ with $f \ge 0$, we have $\varphi(f) \ge 0$. It is easy to see that $\varphi^*(m)$ is a mean on $\ell^{\infty}(T)$. Also, for any $f \in \ell^{\infty}(T)$, $a \in A$ and $t \in T$, we get

$$(\varphi(af) - a(\varphi(f)))(t) = (af)(t) - (\varphi(f))(at) = f(at) - f(at) = 0.$$

So, $(\varphi(af) - a(\varphi(f)))|_T = 0$, and

$$|\varphi(af) - a(\varphi(f))| \le ||\varphi(af) - a(\varphi(f))||_u \chi_{S \setminus T}.$$

This implies that $m(\varphi(_af) - _a(\varphi(f))) = 0$, or, $m(\varphi(_af)) = m(_a(\varphi(f)))$. Similarly, one can show that $m(\varphi(f_a)) = m((\varphi(f))_a)$.

Therefore,

$$\begin{aligned} (\varphi^*(m))(_af) &= m(\varphi(_af)) = m(_a(\varphi(f))) \\ &= m((\varphi(f))_a) = m(\varphi(f_a)) \\ &= (\varphi^*(m))(f_a). \end{aligned}$$

This shows that, $\varphi^*(m)$ is an inner A-invariant mean on $\ell^{\infty}(T)$.

Given semigroups S and T, a map $\varphi: S \longrightarrow T$ is called a homomorphism if it satisfies

$$\varphi(s_1s_2) = \varphi(s_1)\varphi(s_2) \ (s_1, s_2 \in S).$$

Theorem 2.7. Let S, T be semigroups and φ be a homomorphism of S onto T. If S is A-inner amenable, then T is $\varphi(A)$ -inner amenable.

Proof. Assume that m is an inner A-invariant mean on $\ell^{\infty}(S)$. Put $m_o(f) = m(f \circ \varphi)$ for each $f \in \ell^{\infty}(T)$. Now for every $s \in S$, $b \in B = \varphi(A)$ and $f \in \ell^{\infty}(T)$ we have

$${}_{b}fo\varphi(s) = f(b\varphi(s)) = f(\varphi(a)\varphi(s)) = f(\varphi(as)) = (fo\varphi)(as) = {}_{a}(fo\varphi)(s),$$

and

$$f_b o \varphi(s) = f(\varphi(s)b) = f(\varphi(s)\varphi(a)) = f(\varphi(sa)) = (f o \varphi)(sa) = (f o \varphi)_a(s),$$

where $a \in A$ is such that $\varphi(a) = b$. So, ${}_{b}fo\varphi = {}_{a}(fo\varphi)$ and $f_{b}o\varphi = (fo\varphi)_{a}$. It follows from this relations that

$$m_o(bf) = m(bfo\varphi) = m(a(fo\varphi)) = m((fo\varphi)_a) = m(f_bo\varphi) = m_o(f_b).$$

Thus m_o is an inner $\varphi(A)$ -invariant mean.

Let S and T be semigroups. Then $S \times T$ is a semigroup with the operation $(s_1, t_1)(s_2, t_2) = (s_1s_2, t_1t_2)$ for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Also we can consider $\ell^{\infty}(S \times T)$ as a Banach $S \times T$ -bimodule via

$$((s,t)f)(s',t') = f(ss',tt'), \text{ and } (f_{(s,t)})(s',t') = f(s's,t't),$$

for all $s, s' \in S$, $t, t' \in T$ and $f \in \ell^{\infty}(S \times T)$. The homomorphisms $\pi_S : S \times T \longrightarrow S$ and $\pi_T : S \times T \longrightarrow T$ with $\pi_S(s,t) = s, \pi_T(s,t) = t$, respectively, are called projection homomorphisms.

Theorem 2.8. Let S, T be semigroups such that $\ell^{\infty}(S \times T) = \ell^{\infty}(S) \times \ell^{\infty}(T)$. S and T are A-inner amenable and B-inner amenable, respectively if and only if $S \times T$ is $(A \times B)$ -inner amenable.

Proof. Suppose that m and n are inner A-invariant and inner B-invariant means for $\ell^{\infty}(S)$ and $\ell^{\infty}(T)$, respectively. Define the mean m_o on $\ell^{\infty}(S \times T)$ by $m_o(f,g) = m(f)n(g)$ for all $f \in \ell^{\infty}(S)$ and $g \in \ell^{\infty}(T)$. Then for each $(a,b) \in A \times B$

$$\begin{aligned} m_o(_{(a,b)}(f,g)) &= m_o(_af,_bg) = m(_af)n(_bg) \\ &= m(f_a)n(g_b) = m_o(f_a,g_b) \\ &= m_o((f,g)_{(a,b)}). \end{aligned}$$

This means that m_o is inner $(A \times B)$ -invariant mean.

Conversely, suppose that $S \times T$ is $(A \times B)$ -inner amenable. Then by projection homomorphism $\pi_S(A \times B) = A$ and Theorem 2.7, we obtain that S is A-inner amenable. Similarly, we conclude that T is B-inner amenable.

Theorem 2.9. Let S, T be two semigroups such that S and T are A-amenable and B-inner amenable, respectively. Then $S \times T$ is $(A \times B)$ -inner amenable.

Proof. Suppose that m be an A-invariant mean on $\ell^{\infty}(S)$ and n be an inner B-invariant mean on $\ell^{\infty}(T)$. For each $f \in \ell^{\infty}(S \times T)$ and $(s,t) \in S \times T$, we consider $f_T \in \ell^{\infty}(T)$ and $f_S^t \in \ell^{\infty}(S)$ by $f_S^t(s) = f(s,t)$ and $f_T(t) = m(f_S^t)$. Now, define the mean m_o on $\ell^{\infty}(S \times T)$ by

$$m_o(f) = n(f_T)$$
 for all $f \in \ell^\infty(S \times T)$.

For every $(a,b) \in A \times B$ it follows that $(_{(a,b)}f)_S^t = {}_a(f_S^{bt})$ and $(f_{(a,b)})_S^t = (f_S^{tb})_a$. Furthermore, for every $t \in T$

$$\begin{aligned} (_{(a,b)}f)_T(t) &= m((_{(a,b)}f)_T^t) = m(_a(f_S^{ot})) \\ &= m(f_S^{bt}) = (f_T(bt)) \\ &= _b(f_T)(t). \end{aligned}$$

That is, $(a,b)f_T = b(f_T)$. Similarly, one find that $(f_{(a,b)})_T = (f_T)_b$. For every $f \in \ell^{\infty}(S \times T)$ and $(a,b) \in A \times B$ we get

$$m_o(_{(a,b)}f) = n((_{(a,b)}f)_T) = n(_b(f_T))$$

= $n((f_T)_b) = n((f_{(a,b)})_T)$
= $m_o(f_{(a,b)}).$

It follows that m is an inner $(A \times B)$ -invariant mean.

3. Examples of A-inner amenability

Example 3.1. If there exists an element x in semigroup S that commutes with all $a \in A$, then the Dirac measure δ_x for all $f \in \ell^{\infty}(S)$ is an inner A-invariant mean on $\ell^{\infty}(S)$.

$$\delta_x(af) = f(ax) = f(xa) = \delta_x(f_a).$$

In the following examples, we study A-inner amenability over a left (right) zero semigroup, that is a semigroup whose multiplication is defined by st = s (st = t) for all $s, t \in S$. We denote the cardinal number of a set A by |A|.

Example 3.2. Let S be a left zero semigroup, then for any subset A of S:

(i) if |A| = 1, then S is A-inner amenable;

(ii) if $|A| \ge 2$, then S is not A-inner amenable.

Proof. (i) Assume that $A = \{a\}$. Define $m \in \ell^{\infty}(S)^*$ by m(f) = f(a) for every $f \in \ell^{\infty}(S)$. Then we obtain $m(af) = af(a) = f(aa) = f_a(a) = m(f_a)$. This shows that S is A-inner amenable. (ii) Clearly for each $a \in A$ we have

$$_{a}f = f(a)$$
 and $f_{a} = f(a)$

Now, if we suppose that S is A-inner amenable with an inner A-invariant mean m, then for every $a \in A$ and $f \in \ell^{\infty}(S)$, we have $m(af) = m(f_a)$. Therefore f(a) = m(f). Now if we consider $a \neq b \in A$ and $f = \chi_{\{a\}}$ then we obtain

$$1 = f(a) = m(f) = f(b) = 0.$$

This is a contradiction.

Example 3.3. Let \mathbb{F}_2 be free group on two generators a and b. If A is the set of elements of \mathbb{F}_2 that begin with a or a^{-1} when written as reduced words. Then \mathbb{F}_2 is not A-inner amenable.

Proof. We consider $f = \chi_A$ and

$$h = (({}_{ba^{-1}}f)_{ab^{-1}} - {}_{ab^{-1}}({}_{ba^{-1}}f)) + (({}_{b^{-1}a^{-1}}f)_{aba} - {}_{aba}({}_{b^{-1}a^{-1}}f)).$$

Clearly $h \in \mathcal{H}(A)$. Now by Theorem 2.2, it is enough to prove that the function h has the property that, sup{ $h(x) : x \in \mathbb{F}_2$ } < 0. For each $x \in \mathbb{F}_2$ we have

$$h(x) = f(ba^{-1}xab^{-1}) + f(b^{-1}a^{-1}xaba) - f(ax) - f(x).$$

Now the argument as in the proof of Theorem (17.16) of [7] shows $\sup\{h(x) : x \in \mathbb{F}_2\} \leq -1$.

By use of Theorem 2.2, in the following example, we study A-inner amenability over a right zero semigroup.

Example 3.4. Let S be a right zero semigroup, then for any subset A of S we have

- (i) if |A| = 1, then S is A-inner amenable.
- (ii) if $|A| \ge 2$, then S is not A-inner amenable.

Proof. (i) Assume that $A = \{a\}$. Since for every $h \in \mathcal{H}(A)$ and $x \in S$ we have

$$h(x) = \sum_{k=1}^{n} ((f_k)_a - a(f_k))(x)$$

=
$$\sum_{k=1}^{n} (f_k(xa) - f_k(ax))$$

=
$$\sum_{k=1}^{n} (f_k(a) - f_k(x)).$$

Then by set x = a we have $\sup\{h(x) : x \in S\} \ge 0$. This shows that S is A-inner amenable. (ii) For $a \ne b \in A$, we take $h = (a(\chi_{\{a\}}) - (\chi_{\{a\}})_a) + (b(\chi_{\{b\}}) - (\chi_{\{b\}})_b)$. Hence for each $x \in S$ we obtain

$$\begin{split} h(x) &= (a(\chi_{\{a\}}) - (\chi_{\{a\}})_a)(x) + (b(\chi_{\{b\}}) - (\chi_{\{b\}})_b)(x) \\ &= (\chi_{\{a\}}(ax) - \chi_{\{a\}}(xa)) + (\chi_{\{b\}}(bx) - \chi_{\{b\}}(xb)) \\ &= \chi_{\{a\}}(x) + \chi_{\{b\}}(x) - 2. \end{split}$$

and this implies that $\sup\{h(x): x \in S\} \leq -1$. Hence by theorem 2.2, S is not A-inner amenable.

4. Følner's condition

Before stating the following theorem, recall that a mean in $\ell^1(S)$ is called a finite mean if it is a convex combination of the Dirac measures. We shall use Φ denote the set of all finite means and δ_x denotes the Dirac measure at $x \in S$. It is obvious that Φ is convex subset of $\ell^1(S)$. In fact, Φ is convex hull of S.

Theorem 4.1. Let S be a semigroup and $A \subseteq S$. Then the following statements are equivalent:

- (a) S is A-inner amenable.
- (b) there is a net (φ_{α}) of finite means such that $\delta_a * \varphi_{\alpha} \varphi_{\alpha} * \delta_a \longrightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.
- (c) there is a net (ψ_{α}) of finite means such that $\|\delta_a * \psi_{\alpha} \psi_{\alpha} * \delta_a\|_1 \longrightarrow 0$ for every $a \in A$.

Proof. $(a) \Rightarrow (b)$. Let m be an inner A-invariant mean on $\ell^{\infty}(S)$. Since $m \in \ell^{\infty}(S)^*$, we can find a net (φ_{α}) of finite means such that $\lim_{\alpha} \varphi_{\alpha} = m$ in the weak* topology of $\ell^{\infty}(S)^*$. Then for all $f \in \ell^{\infty}(S)$ and $a \in A$,

$$f(\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a) = \varphi_\alpha(af) - \varphi_\alpha(f_a) \longrightarrow m(af) - m(f_a) = 0.$$

It follows that $\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a \longrightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.

 $(b) \Rightarrow (c)$. Let (φ_{β}) be a net as in (b). Using an idea of Ling [11], we define linear map $T : \ell^{1}(S) \longrightarrow \prod_{a \in A} \ell^{1}(S)$ by $(T(\varphi))(a) = \delta_{a} * \varphi - \varphi * \delta_{a}$ for every $\varphi \in \ell^{1}(S), a \in A$. Now by assumption, $(T(\varphi_{\beta}))(a) = \delta_{a} * \varphi_{\beta} - \varphi_{\beta} * \delta_{a} \longrightarrow 0$ weakly in $\ell^{1}(S)$, for every $a \in A$. This means that zero lies in the weak closure of $T(\Phi)$. Since $\prod_{a \in A} \ell^{1}(S)$ with product of the norm topology is a locally convex space and Φ is convex, the closure of $T(\Phi)$ in this topology contains 0. Thus, there exists a subnet $(\varphi_{\alpha}) \subseteq (\varphi_{\beta})$ such that $\|\delta_{a} * \psi_{\alpha} - \psi_{\alpha} * \delta_{a}\|_{1} \longrightarrow 0$ for every $a \in A$.

 $(c) \Rightarrow (b)$. Since convergence in norm implies convergence in weak topology, this implication is trivial.

 $(b) \Rightarrow (a)$. Let (φ_{α}) be a net satisfying the convergence in (b). By Alaoglu's theorem, it has a weak^{*} convergent subnet. By passing to such a subnet if necessary, there is a $m \in \ell^{\infty}(S)^*$ such that $\lim_{\alpha} \varphi_{\alpha} = m$ in the weak^{*} topology of $\ell^{\infty}(S)^*$. Therefore m is a mean on $\ell^{\infty}(S)$, and for all $a \in A$, $f \in \ell^{\infty}(S)$

$$m(_{a}f) - m(f_{a}) = \lim_{\alpha} (\varphi_{\alpha}(_{a}f) - \varphi_{\alpha}(f_{a})) = \lim_{\alpha} (\delta_{a} * \varphi_{a} - \varphi_{\alpha} * \delta_{a})(f) = 0.$$

For each $s \in S$ we put $s^{-1}A = \{t \in S : st \in A\}$ and $As^{-1} = \{t \in S : ts \in A\}$. We also note that $\frac{1}{|A|}\chi_A$ defines an element in $\ell^1(S)$.

Lemma 4.2. Let A acts injectively on the right of semigroup S, then for every $B \subseteq A$ and $a \in A$

$$||\chi_B * \delta_a - \delta_a * \chi_B||_1 = 2|Ba \backslash aB|.$$

Proof. For $a \in A$ and $B \subseteq A$, we get

$$(\delta_a * \chi_B)(x) = \sum_{as=x} \chi_B(s)$$
$$= \sum_{s \in a^{-1}\{x\}} \chi_B(s)$$
$$= |B \cap a^{-1}\{x\}|.$$

Similarly, we obtain $(\chi_B * \delta_a)(x) = |B \cap \{x\}a^{-1}|$. It is easy to see that

$$(\chi_B * \delta_a - \delta_a * \chi_B)(x) = \begin{cases} |B \cap \{x\}a^{-1}| & \text{if } x \in Ba \setminus aB \\ -|B \cap a^{-1}\{x\}| & \text{if } x \in aB \setminus Ba \\ |B \cap \{x\}a^{-1}| - |B \cap a^{-1}\{x\}| & \text{if } x \in aB \cap Ba \\ 0 & \text{if } x \notin aB \cup Ba \end{cases}$$

Since A acts injectively on the right of semigroup S, then for each $x \in Ba$ we obtain $|B \cap \{x\}a^{-1}| = 1$. This implies that

$$\begin{aligned} \|\chi_B * \delta_a - \delta_a * \chi_B\|_1 &= \sum_{x \in Ba \setminus aB} 1 + \sum_{x \in aB \setminus Ba} |B \cap a^{-1} \{x\}| + \sum_{x \in aB \cap Ba} (|B \cap a^{-1} \{x\}| - 1) \\ &= |Ba \setminus aB| + \sum_{x \in Ba} |B \cap a^{-1} \{x\}| - |aB \cap Ba| \\ &= |Ba \setminus aB| + |B| - |aB \cap Ba| \\ &= |Ba \setminus aB| + |Ba| - |aB \cap Ba| \\ &= |Ba \setminus aB| + |Ba \setminus aB| \\ &= 2|Ba \setminus aB| \end{aligned}$$

Theorem 4.3. Let A act injectively on the right of semigroup S. If for any finite set $F \subseteq A$ and any $\varepsilon > 0$, there exists a finite non-empty set $B \subseteq A$ such that $|Ba \setminus aB| < \epsilon |B|$ for all $a \in F$, then S is A-inner amenable.

Proof. By the assumption there exists a net of finite non-empty sets $B_{\alpha} \subseteq A$ such that

$$|B_{\alpha}a \setminus aB_{\alpha}|/|B_{\alpha}| \longrightarrow 0$$
 for all $a \in A$.

By Lemma 4.2, we have

$$||\chi_{B_{\alpha}} * \delta_a - \delta_a * \chi_{B_{\alpha}}||_1 = |B_{\alpha}a \backslash aB_{\alpha}|.$$

Set $\varphi_{\alpha} = |B_{\alpha}|^{-1} \chi_{B_{\alpha}}$. Then for α , and $a \in A$

$$||\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a||_1 \longrightarrow 0.$$

Now the proof is complete by Theorem 4.1.

Remark 4.4. The assumption of Theorem 4.3 'that A acts injectively on the right of semigroup S' is necessary. In fact, any right zero semigroup S is not A-inner amenable if A has at least two elements (see Example 3.2).

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