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Upper and Lower Solution method for Positive solution of generalized Caputo fractional differential equations

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Abstract

In this research paper, the nonlinear fractional relaxation equation involving the generalized Caputo derivative is reduced to an equivalent integral equation via the generalized Laplace transform. Moreover, the upper and lower solutions method combined with some fixed point theorems, and the properties of the Mittag-Leffler function are applied to investigate the existence and uniqueness of positive solutions for the problem at hand. At the end, to illustrate our results, we give an example.

Keywords: fractional differential equations, existence and upper and lower solutions, Mittag-Leffler function, generalized Laplace transform, fixed point theorems.
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1. Introduction

The fractional calculus (FC) is approximately 300 years old which is a generalization of classical calculus as it deals with the non-integer order. One can discover that there are many definitions of fractional derivatives that have been investigated in the literature. e.g., we refer here to the most well-known types such as Reimann-Liouville, Caputo, Hilfer, Hadamard, and Katugampola derivative, and many others. The best

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way to deal with an assortment of fractional operators is to understand the general forms of fractional operators that include other operators.

Fractional differential equations (FDEs) upspring in sundry areas of science and engineering. There have been many results on existence and uniqueness of solutions for nonlinear FDEs of the following type: evolution, functional, impulsive under various conditions can be found in the articles [4, 7, 14, 1, 16, 25] and the references cited therein.

On the other hand, there has been much more focus paid in developing the theory of existence and uniqueness of positive solutions for nonlinear FDEs have been investigated by using Leray-Schauder, coincidence degree theory, fixed point index theory, fixed point theorems in cones and so on, we refer the readers to [10, 11, 13, 12, 15, 23, 6, 22, 20, 30]. For instance, N. Li and C. Wang in [22] studied the existence and uniqueness of positive solution for nonlinear FDE

$$\begin{cases} D^{\theta}_{0+}\phi(t) = f(t,\phi(t)),\ 0<t<1, \\ \phi(0)=0, \end{cases} \tag{1}$$

where $0<\theta<1$, D^{θ}_{0+} is the standard Riemann Liouville fractional derivative of order θ, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous function.

In another paper, by using fixed point theorem on cones with the upper and lower solutions method, A. Chidouh et al. [15] considered the nonlinear fractional relaxation differential equation involving the standard Caputo fractional derivative

$$\begin{cases} {}^{C}D^{\theta}_{0+}\phi(t) + w\phi(t) = f(t,\phi(t)),\ 0<t\le 1,\ w>0 \\ \phi(0)=\phi_0>0, \end{cases} \tag{2}$$

where $0<\theta<1$, and $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous function.

For recent papers on ψ-fractional derivative of FDEs, can be found in [2, 3, 8, 9, 5, 21, 24, 27, 28, 29] and the references cited therein.

By motivating from the above papers, in this paper, we investigate the existence and uniqueness of positive solution of the following nonlinear fractional relaxation equation:

$$\begin{cases} {}^{C}D^{\theta,\psi}_{0+}\phi(t) + w\phi(t) = f(t,\phi(t)),\ 0<t\le 1, \\ \phi(0)=\phi_0>0, \end{cases} \tag{3}$$

where $0<\theta<1$ is a real number,w is a positive parameter, ${}^{C}D^{\theta,\psi}_{0+}$ is the generalized Caputo fractional derivative (so-called $\psi-$Caputo fractional derivative) of order θ, $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is a given continuous, and $\psi:[0,1]\to[0,1]$ is a strictly increasing such that $\psi\in C^1[0,1]$ with $\psi'(t)\neq 0$, for all $t\in[0,1]$. The positive solution which we consider in this work is such that $\phi(t)\ge 0,\ \ 0\le t\le 1$, $\phi\in C[0,1]$ and satisfies the problem (3).

To our knowledge, less work appears in the initial value problem (3) by using of upper and lower solution method. Our aim is to study the existence and uniqueness of positive solutions of the problem (3) through some properties of Mittag-Leffler function, generalized Laplace transform and fixed point theorems. Moreover, the result of existence obtained through constructing the upper and lower control functions of the nonlinear terms without any monotone requirement except for the continuity.

The paper is organized as follows: In section 2, we present some preliminaries concerning the assumpitions and several lemmas needed throughout this paper. In section 3, initial value problem (3) is reduced to an equivalent integral equation via generalized Laplace transform. Further, the existence and uniqueness of positive solutions of given problem are obtained by using the upper and lower solutions method combined with the fixed point theorems. An example is given in the last section.

2. Preliminary results

In this section we recall some basic definitions and lemmas related to fractional calculus, Generalized Laplace transform, Mittag-Leffler function, and fixed point theorems useful for our results. Let $C[0,1]$ be the Banach space endowed with the sup norm

$$\|\phi\| = \sup_{t\in[0,1]} |\phi(t)|,$$

Consider the classical cone E defined by

$$E = \{\phi \in C[0,1] : \phi(t) \geq 0, \quad 0 \leq t \leq 1\}.$$

Definition 2.1. *[19] Let $\theta > 0$, and $f : [a,b] \to \mathbb{R}$ be an integrable function. Then the generalized Riemann-Liouville fractional integral of order θ for a function f with respect to ψ is given by*

$$I^{\theta,\psi}_{a+} f(t) = \frac{1}{\Gamma(\theta)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\theta-1} f(s)ds.$$

where $\psi : [a,b] \to \mathbb{R}$ is a strictly increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$.

Definition 2.2. *[19] Let $n-1 < \theta < n$, and $f : [a,b] \to \mathbb{R}$ be an integrable function. Then the generalized Riemann-Liouville fractional derivative of order θ for a function f with respect to ψ is defined by*

$$D^{\theta,\psi}_{a+} f(t) = \Big[\frac{1}{\psi'(t)}\frac{d}{dt}\Big]^n I^{n-\theta,\psi}_{a+} f(t),$$

where $n = [\theta] + 1$ and ψ as in Definition 2.1.

Definition 2.3. *[19] Let $n-1 < \theta < n$, and $f \in C^{n-1}[a,b]$. Then the generalized Caputo fractional derivative of order θ for a function f with respect to ψ is given by*

$${}^C D^{\theta,\psi}_{a+} f(t) = D^{\theta,\psi}_{a+}\Big[h(t) - \sum_{k=0}^{n-1} \frac{f^{[k]}_{\psi}(a)}{k!}\Big(\psi(t) - \psi(a)\Big)^k\Big].$$

where $n = [\theta] + 1$ for $\theta \notin \mathbb{N}$, and $n = \theta$ for $\theta \in \mathbb{N}$, and $f^{[k]}_{\psi}(t) = \Big[\frac{1}{\psi'(t)}\frac{d}{dt}\Big]^k f(t)$. Moreover, if $f \in C^n[a,b]$, then ψ-Caputo fractional derivative can be written as

$$\begin{aligned} {}^C D^{\theta,\psi}_{a+} f(t) &= I^{n-\theta,\psi}_{a+}\Big[\frac{1}{\psi'(t)}\frac{d}{dt}\Big]^n f(t) \\ &= \frac{1}{\Gamma(n-\theta)} \int_a^t \psi'(\tau)(\psi(t) - \psi(\tau))^{n-\theta-1} f^{[n]}_{\psi}(\tau)d\tau. \end{aligned}$$

In paricular, if $\theta = n \in \mathbb{N}$, we have

$${}^C D^{\theta,\psi}_{a+} f(t) = f^{[n]}_{\psi}(t).$$

Lemma 2.1. *Let $\theta, \beta \in \mathbb{R}$ with $\beta > n$. If $g(t) = (\psi(t) - \psi(a))^{\beta-1}$. Then we have*

$$I^{\theta,\psi}_{a+} g(t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\theta)}(\psi(t) - \psi(a))^{\theta+\beta-1},$$

and

$${}^C D^{\theta,\psi}_{a+} g(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\theta)}(\psi(t) - \psi(a))^{\beta-\theta-1}.$$

In case, $g(t) = [\psi(t) - \psi(a)]^k$, then

$${}^C D^{\theta,\psi}_{a+} g(t) = 0, \quad \forall k \in \{0, 1, \ldots, n-1\}, \quad n \in \mathbb{N}.$$

Now, we give some concepts of generalized Laplace transform introduced by [18].

Definition 2.4. *The Laplace transform of a function f is defined by the improper integral*

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \tag{4}$$

provided that the integral in (4) exists for all s larger than or equal to some s_0, where f is defined for $t \geq 0$.

Definition 2.5. *Let $f, \psi : [a, \infty) \longrightarrow \mathbb{R}$ be real valued functions such that $\psi(t)$ is continuous and $\psi'(t) > 0$ on $[a, \infty)$. Then the generalized Laplace transform of f is defined by*

$$L_\psi\{f(t)\} = \int_a^\infty e^{-s[\psi(t)-\psi(a)]}\psi'(t)f(t)dt,$$

for all values of s. In particular, if $a = 0$, $\psi(0) = 0$, then we have

$$L_\psi\{f(t)\} = \int_0^\infty e^{-s\psi(t)}\psi'(t)f(t)dt.$$

Theorem 2.1. *Let $f, \psi : [a, \infty) \longrightarrow \mathbb{R}$ be real valued functions such that $\psi(t)$is continuous and $\psi'(t) > 0$ on $[a, \infty)$ and such that the generalized Laplace transform of f exists. Then*

$$L_\psi\{f(t)\} = L\left\{ f\Big(\psi^{-1}(t + \psi(a))\Big) \right\}, \tag{5}$$

where $L\{f\}$ is the usual Laplace transform of f.

Lemma 2.2. *Let $Re(\theta) > 0$ and $\left|\frac{\lambda}{s^\theta}\right| < 1$. Then*

$$L_\psi\{E_\theta(\lambda\,[\psi(t) - \psi(a)]^\theta\} = \frac{s^{\theta-1}}{s^\theta - \lambda}, \tag{6}$$

and

$$L_\psi\{[\psi(t) - \psi(a)]^{\beta-1}\, E_{\theta,\beta}(\lambda\,[\psi(t) - \psi(a)]^\theta\} = \frac{s^{\theta-\beta}}{s^\theta - \lambda}. \tag{7}$$

Theorem 2.2. *Assume that $\theta > 0$, $(n = [\theta] + 1)$ and $f(t)$, $D^{1;\psi} f(t), D^{2;\psi} f(t), ..., D^{n-1;\psi} f(t)$ are continuous function on each interval (a, ∞) and of $\psi(t)$-exponential order, while ${}^C D^{\theta;\psi}_{a^+} f(t)$ is piecewise continuous on $[a, t]$. Then*

$$L_\psi\left\{ {}^C D^{\theta;\psi}_{a^+} f(t) \right\} = s^\theta L_\psi\{f(t)\} - \sum_{k=0}^{n-1} s^{\theta-k-1} D^{k;\psi}\ f(a), \tag{8}$$

where $D^{j;\psi} = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^j$.

Definition 2.6. *The generalized convolution of f and g defined by*

$$(f *_\psi g)\,(t) = \int_a^t f(\tau) g\Big(\psi^{-1}[\psi(t) + \psi(a) - \psi(\tau)]\Big)\psi'(\tau)d\tau,$$

where f and g are piecewise continuous functions at each interval $[a, b]$ and of exponential order. Moreover, we have $L_\psi\{f *_\psi g\} = L_\psi\{f\}L_\psi\{g\}$.

Definition 2.7. *A function $u \in C[0, 1] \cap L[0, 1]$ is said to be a solution of (3) if u satisfies the equation ${}^C D^{\theta,\psi}_{0^+}\phi(t) + w\phi(t) = f(t, \phi(t))$, $0 < t \leq 1$, with the conditions $\phi(0) = \phi_0 > 0$.*

Definition 2.8. *A function $\phi \in C[0, 1]$ is called a positive solution of the problem (3) if $\phi(t) \geq 0$ for all $t \in [0, 1]$ and ϕ satisfies the problem (3).*

Definition 2.9. *The two-parameter function of the Mittag-Leffler is defined by the series expansion*

$$E_{\theta,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + \beta)}, \quad \theta > 0, \beta \in C, z \in C.$$

For $\beta = 1$, we obtain the Mittag-Leffler function one parameter,

$$E_{\theta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + 1)}, \quad \theta > 0, z \in C.$$

Lemma 2.3. *The generalized Mittag-Leffler function $E_{\theta,\beta}(-\phi)$ with $\phi \geq 0$ is completely monotonic if and only if $0 < \theta \leq 1$ and $\beta \geq \theta$. In other words, it yields*

$$(-1)^n \frac{d^n}{d\phi^n} E_{\theta,\beta}(-\phi) \geq 0, \quad \forall \quad n \in N.$$

$$\textit{Obiviously,} \quad 0 \leq E_{\theta,\beta}(-\phi) \leq \frac{1}{\Gamma(\beta)} \quad \textit{where} \quad \phi \geq 0, 0 < \theta \leq 1 \quad \textit{and} \quad \beta \geq \theta.$$

Lemma 2.4. *Let $\theta, \beta, \gamma > 0$ and $\lambda \in \mathbb{R}$. Then we have*

$$I_{0+}^{\theta} \left[(\psi(t) - \psi(0))^{\beta-1} E_{\gamma,\beta}(\lambda (\psi(t) - \psi(0))^{\gamma}) \right] = (\psi(t) - \psi(0))^{\theta+\beta-1} E_{\gamma,\theta+\beta}(\lambda (\psi(t) - \psi(0))^{\gamma}).$$

Moreover,

$$\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\theta-1} E_{\theta,\theta}(\lambda (\psi(t) - \psi(s))^{\theta}) ds = [\psi(t) - \psi(0)]^{\theta} E_{\theta,\theta+1}(\lambda[\psi(t) - \psi(0)]^{\theta}).$$

Proof. By the Definitions 2.1, 2.9 and Lemma 2.1, we get

$$\begin{aligned}
& I_{0+}^{\theta,\psi} \left[(\psi(t) - \psi(0))^{\beta-1} E_{\gamma,\beta}(\lambda (\psi(t) - \psi(0))^{\gamma}) \right] \\
= & \frac{1}{\Gamma(\theta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\theta-1} \left[(\psi(s) - \psi(0))^{\beta-1} E_{\gamma,\beta}(\lambda (\psi(s) - \psi(0))^{\gamma}) \right] ds \\
= & \frac{1}{\Gamma(\theta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\theta-1} \left[\sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\gamma n + \beta)} (\psi(s) - \psi(0))^{\gamma n + \beta - 1} \right] ds \\
= & \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\theta n + \beta)} I_{0+}^{\theta,\psi} (\psi(t) - \psi(0))^{\gamma n + \beta - 1} \\
= & \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma(\gamma n + \theta + \beta)} [\psi(t) - \psi(0)]^{\theta + \gamma n + \beta - 1} \\
= & [\psi(t) - \psi(0)]^{\theta+\beta-1} E_{\gamma,\theta+\beta}(\lambda (\psi(t) - \psi(0))^{\gamma}.
\end{aligned}$$

□

Lemma 2.5. *Let $\theta, \beta > 0$ be arbitrary. Then for any $c < 0$ and $\sigma_1, \sigma_2 \in [0, 1]$,*

$$E_{\theta,\theta+\beta}(c\sigma_2^{\theta}) \longrightarrow E_{\theta,\theta+\beta}(c\sigma_1^{\theta}) \quad \textit{as} \ \sigma_1 \longrightarrow \sigma_2.$$

Definition 2.10. *Let $(U, \|.\|)$ be a Banach space and $T : U \to U$. The operator T is a contraction operator if there is an $\kappa \in (0, 1)$ such that $u, v \in U$ imply*

$$\|Tu - Tv\| \leq \kappa \|u - v\|.$$

Theorem 2.3. *[31] (Banach fixed point theorem). Let (U, d) be a non-empty complete metric space with a contraction mapping $T : U \to U$. Then, T has a unique fixed-point u in U.*

Theorem 2.4. *[31] (Schauder fixed point theorem). Let U be a Banach space and let Ξ a closed convex, bounded subset of U. If $T : \Xi \longrightarrow \Xi$ is a continuous map such that the set $\{Tu : u \in \Xi\}$ is relatively compact in U. Then T has at least one fixed point.*

3. Main results

In this section, we shall prove the existence and uniqueness of positive solution for a $\psi-$Caputo problem (3). Before starting and proving the main results, we introduce the following lemma:

Lemma 3.1. *Let $0 < \theta < 1$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, and $\psi : [0,1] \to \mathbb{R}^+$ is a strictly increasing function such that $\psi'(t) \neq 0$, for all $t \in [0,1]$. Then the fractional integral equation*

$$\begin{aligned} \phi(t) &= \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &+ \int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta) f(\tau,\phi(\tau))\psi'(\tau)d\tau. \end{aligned} \tag{9}$$

is a solution of $\psi-$Caputo problem (3).

Proof. One can apply the generalized Laplace transform introduced by [18] to get the required formula (9). □

Remark 3.1. *In particular,*

1. *If $w = 0$, then the $\psi-$Caputo problem (3) has a unique solution defined by*

$$\phi(t) = \phi_0 + \int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} f(\tau,\phi(\tau))\psi'(\tau)d\tau.$$

2. *If $\psi(t) = t$, then the $\psi-$Caputo problem (3) reduces to problem (2) which has a unique solution defined by*

$$\begin{aligned} \phi(t) &= \phi_0 E_\theta(-wt^\theta) \\ &+ \int_0^t (t-\tau)^{\theta-1} E_{\theta,\theta}(-w(t-\tau)^\theta) f(\tau,\phi(\tau))d\tau. \end{aligned}$$

Transform the $\psi-$Caputo problem (3) into a fixed point equation as follows

$$\phi = \Pi\phi, \quad \phi \in C[0,1],$$

where the operator Π defined by

$$\begin{aligned} \Pi\phi(t) &= \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &+ \int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta) f(\tau,\phi(\tau))\psi'(\tau)d\tau. \end{aligned} \tag{10}$$

Now, we need the following axiom lemma to prove our results.

Lemma 3.2. *The operator $\Pi : E \to E$ is completely continuous.*

Proof. In view of Lemma 2.3, and taking into consideration that f is continuous and nonnegative function, we obtain that the operator $\Pi : E \to E$ is continuous.
Let us suppose that the function $f : [0,1] \times \mathcal{S}_\gamma \to \mathbb{R}^+$ be bounded by ζ, where $\mathcal{S}_\gamma = \{\phi \in E, \|\phi\| \leq \gamma\}$. Let $\phi \in \mathcal{S}_\gamma$. Then, for $t \in [0,1]$ and using Lemma 2.3, we have

$$\begin{aligned} |\Pi\phi(t)| &\leq \left|\phi_0 E_\theta(-w[\psi(s)-\psi(0)]^\theta)\right| \\ &+ \left|\int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta) f(\tau,\phi(\tau))\psi'(\tau)d\tau\right| \\ &\leq \phi_0 + \frac{1}{\Gamma(\theta)} \int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} |f(\tau,\phi(\tau))\psi'(\tau)|d\tau \\ &\leq \phi_0 + \frac{\zeta[\psi(t)-\psi(0)]^\theta}{\Gamma(\theta+1)}, \end{aligned}$$

which implies

$$\|\Pi\phi\| \leq \phi_0 + \frac{\zeta[\psi(1)-\psi(0)]^\theta}{\Gamma(\theta+1)}.$$

This proves that the family $\Pi(\mathcal{S}_\gamma) = \{\Pi\phi : \phi \in \mathcal{S}_\gamma\}$ is uniformly bounded. Now we shall show that the family $\Pi(\mathcal{S}_\gamma) = \{\Pi\phi : \phi \in \mathcal{S}_\gamma\}$ is an equicontinuous.

Consider $\phi \in \mathcal{S}_\gamma$. Then for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$, we get

$$\begin{aligned}
|\,|\Pi\phi(t_2) - \Pi\phi(t_1)| \leq\; & \Big|\phi_0 E_\theta(-w[\psi(t_2)-\psi(0)]^\theta) - \phi_0 E_\theta(-w[\psi(t_1)-\psi(0)]^\theta)\Big| \\
& + \Big|\int_0^{t_2} [\psi(t_2)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t_2)-\psi(\tau)]^\theta) f(\tau,\phi(\tau))\psi'(\tau)d\tau \\
& - \int_0^{t_1} [\psi(t_1)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t_1)-\psi(\tau)]^\theta) f(\tau,\phi(\tau))\psi'(\tau)d\tau\Big| \\
\leq\; & \Big|\phi_0 E_\theta(-w[\psi(t_2)-\psi(0)]^\theta) - \phi_0 E_\theta(-w[\psi(t_1)-\psi(0)]^\theta)\Big| \\
& + \frac{1}{\Gamma(\theta)}\int_0^{t_1} \Big|[\psi(t_1)-\psi(\tau)]^{\theta-1} - [\psi(t_2)-\psi(\tau)]^{\theta-1}\Big| f(\tau,\phi(\tau))\psi'(\tau)d\tau \\
& + \frac{1}{\Gamma(\theta)}\int_{t_1}^{t_2} \Big|[\psi(t_2)-\psi(\tau)]^{\theta-1}\Big| |f(\tau,\phi(\tau))\psi'(\tau)| d\tau \\
\leq\; & \Big|\phi_0 E_\theta(-w[\psi(t_2)-\psi(0)]^\theta) - \phi_0 E_\theta(-w[\psi(t_1)-\psi(0)]^\theta)\Big| \\
& + \frac{\zeta}{\Gamma(\theta)}\int_0^{t_1} \Big([\psi(t_1)-\psi(\tau)]^{\theta-1} - [\psi(t_2)-\psi(\tau)]^{\theta-1}\Big)\psi'(\tau)d\tau \\
& + \frac{\zeta}{\Gamma(\theta)}\int_{t_1}^{t_2} \Big|[\psi(t_2)-\psi(\tau)]^{\theta-1}\Big|\psi'(\tau)d\tau \\
\leq\; & \Big|\phi_0 E_\theta(-w[\psi(t_2)-\psi(0)]^\theta) - \phi_0 E_\theta(-w[\psi(t_1)-\psi(0)]^\theta)\Big| \\
& + \frac{2\zeta}{\Gamma(\theta+1)}\Big([\psi(t_2)-\psi(t_1)]^\theta\Big).
\end{aligned}$$

As $t_1 \to t_2$ and bearing in mind that the function $y(t) = \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta)$ is continuous on $[0,1]$, the right side of above inequality tends to zero. Which implies that $\Pi(\mathcal{S}_\gamma)$ is equicontinuous. As a consequence of Arzela–Ascoli theorem, we can conclude that Π is compact. □

Now will we define the lower and upper control functions as follows,

Definition 3.1. *Let $a, b \in \mathbb{R}^+ (b > a)$. Then for any $\phi \in [a,b] \subset \mathbb{R}^+$, we define the upper-control function $\overline{f}(t,\phi) = \sup_{a\leq\eta\leq\phi} f(t,\eta)$, and lower-control function $\underline{f}(t,\gamma) = \inf_{\phi\leq\eta\leq b} f(t,\eta)$. It is clear that functions $\overline{f}(t,\phi)$ and $\underline{f}(t,\phi)$ are non-decreasing on ϕ and satisfies*

$$\underline{f}(t,\phi) \leq f(t,\phi) \leq \overline{f}(t,\phi).$$

Definition 3.2. *Let $\overline{\phi}(t), \underline{\phi}(t) \in E$ and $a \leq \underline{\phi}(t) \leq \overline{\phi}(t) \leq b$ comply with*

$$\begin{aligned}
& {}^C D_{0+}^{\theta,\psi}\overline{\phi}(t) + w\overline{\phi}(t) \geq \overline{f}(t,\overline{\phi}(t)), \qquad 0 \leq t \leq 1, \\
& \overline{\phi}(0) \geq \phi_0,
\end{aligned}$$

or

$$\begin{aligned}
\overline{\phi}(t) \;\geq\; & \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\
& + \int_0^t [\psi(t)-\psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta) f(\tau,\overline{\phi}(\tau))\psi'(\tau)d\tau.
\end{aligned}$$

Then $\overline{\phi}(t)$ is called upper solution for the ψ-Caputo problem (3). On the other hand, we have

$$\begin{aligned} &{}^{C}D^{\theta,\psi}_{0+}\underline{\phi}(t) + w\underline{\phi}(t) \le \underline{f}(t, \underline{\phi}(t)), \qquad 0 \le t \le 1, \\ &\underline{\phi}(0) \le \phi_0, \end{aligned}$$

or

$$\begin{aligned} \underline{\phi}(t) \ \le\ & \phi_0 E_\theta(-w[\psi(t) - \psi(0)]^\theta) \\ & + \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta) f(\tau, \underline{\phi}(\tau))\psi'(\tau)d\tau. \end{aligned}$$

Then $\underline{\phi}(t)$ is also called lower solution for the ψ-Caputo problem (3).

Now, we are ready to give the main results of this paper.

Theorem 3.1. *Assume that $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous function, ψ as in Lemma 3.1, and $\overline{\phi}$, $\underline{\phi}$ are a pair of upper and lower solution of (3), respectively, then the ψ-Caputo problem (3) has at least one positive solution. Moreover,*

$$\underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t), \quad t \in [0, 1].$$

Proof. Define the set $\mathcal{K}$ as follows

$$\mathcal{K} := \{\phi \in E : \underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t), \quad t \in [0, 1]\}.$$

It is clear that $\mathcal{K}$ is a convex, bounded, and closed subset of the Banach space E, then taking into account Lemma 3.2, we have the operator $\Pi : \mathcal{K} \to \mathcal{K}$ is completely continuous due to $\mathcal{K} \subset E$. It is sufficient to show that $\Pi : \mathcal{K} \to \mathcal{K}$. By the Definitions 3.1,3.2 and for any $\phi(t) \in \mathcal{K}$, we have $\underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t)$, it follows that

$$\begin{aligned} \Pi\phi(t) \ =\ & \phi_0 E_\theta(-w[\psi(t) - \psi(0)]^\theta) \\ & + \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta) f(\tau, \phi(\tau))\psi'(\tau)d\tau \\ \le\ & \phi_0 E_\theta(-w[\psi(t) - \psi(0)]^\theta) \\ & + \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta) \overline{f}(\tau, \overline{\phi}(\tau))\psi'(\tau)d\tau \\ \le\ & \overline{\phi}(t). \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} \Pi\phi(t) \ =\ & \phi_0 E_\theta(-w[\psi(t) - \psi(0)]^\theta) \\ & + \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta) f(\tau, \phi(\tau))\psi'(\tau)d\tau \\ \ge\ & \phi_0 E_\theta(-w[\psi(t) - \psi(0)]^\theta) \\ & + \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta) \underline{f}(\tau, \underline{\phi}(\tau))\psi'(\tau)d\tau \\ \ge\ & \underline{\phi}(t) \end{aligned} \tag{12}$$

It follows from the equations (11) and (12) that

$$\underline{\phi}(t) \le \Pi\phi(t) \le \overline{\phi}(t), \quad 1 \ge t \ge 0,$$

which implies $\Pi\phi \in \mathcal{K}$, that proves that $\Pi : \mathcal{K} \to \mathcal{K}$ is compact. By means of fixed point theorem of Schauder, Π has a fixed point in $\mathcal{K}$. Hence the ψ-Caputo problem (3) has at least one positive solution $\phi(t)$ in $C[0, 1]$. □

Corollary 3.1. *Let $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous function, and there exist two constants $M_1, M_2\geq 0$ such that*

$$M_1\leq f(t,\varsigma)\leq M_2,\quad (t,\varsigma)\in[0,1]\times\mathbb{R}^+ \tag{13}$$

Then the ψ-Caputo problem (3) has at least one positive solution $\phi(t)\in C[0,1]$. Moreover, for each $t\in[0,1]$,

$$\begin{aligned}\phi(t) &\geq \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_1[\psi(t)-\psi(\tau)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta),\end{aligned} \tag{14}$$

and

$$\begin{aligned}\phi(t) &\leq \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_2[\psi(t)-\psi(\tau)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta).\end{aligned} \tag{15}$$

Proof. From the Definitions 3.1,3.2 and equation (13), we have

$$M_1\leq \underline{f}(t,\varsigma)\leq \overline{f}(t,\varsigma)\leq M_2. \tag{16}$$

Now, we consider the following ψ-Caputo problem

$$\begin{aligned}&{}^CD_{0+}^{\theta,\psi}\overline{\phi}(t)+w\overline{\phi}(t)=M_2,\qquad 0\leq t\leq 1,\\ &\overline{\phi}(0)=\phi_0>0\end{aligned} \tag{17}$$

Then, the ψ-Caputo problem (17) has a positive solution

$$\begin{aligned}\overline{\phi}(t) &= \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_2\int_0^t[\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\psi'(\tau)d\tau.\end{aligned}$$

Using Lemma 2.4, we get

$$\overline{\phi}(t)=\phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)+M_2[\psi(t)-\psi(0)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta).$$

On the other hand, by (16), we conclude that

$$\begin{aligned}\overline{\phi}(t) &= \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_2\int_0^t[\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\psi'(\tau)d\tau\\ &\geq \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +\int_0^t[\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\overline{f}(\tau,\overline{\phi}(\tau))\psi'(\tau)d\tau.\end{aligned}$$

Thus, the function $\overline{\phi}(t)$ is the upper solution of the ψ-Caputo problem (3).
Obviously, in the same way, the ψ-Caputo problem of the type

$$\begin{aligned}&{}^CD_{0+}^{\theta,\psi}\underline{\phi}(t)+w\underline{\phi}(t)=M_1,\qquad 0\leq t\leq 1,\\ &\underline{\phi}(0)=\phi_0>0,\end{aligned}$$

has also a positive solution

$$\begin{aligned}\underline{\phi}(t) &= \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_1\int_0^t[\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\psi'(\tau)d\tau\\ &= \phi_0E_\theta(-w[\psi(t)-\psi(0)]^\theta)\\ &\quad +M_1[\psi(t)-\psi(0)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta).\end{aligned}$$

On the opposite side, by (16), we get

$$\begin{aligned}\underline{\phi}(t) &= \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &\quad +M_1\int_0^t [\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\psi'(\tau)d\tau \\ &\le \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &\quad +\int_0^t [\psi(t)-\psi(\tau)]^{\theta-1}E_{\theta,\theta}(-w[\psi(t)-\psi(\tau)]^\theta)\underline{f}(\tau,\underline{\phi}(\tau))\psi'(\tau)d\tau.\end{aligned}$$

Thus, the function $\underline{\phi}(t)$ is the lower solution of the ψ-Caputo problem (3). By Theorem (3.1), we get that the ψ-Caputo problem (3) has at least one positive solution $\phi(t) \in C[0,1]$, which produces the inequalities (14) and (15). □

Corollary 3.2. *Assume that $f:[0,1]\times\mathbb{R}^+ \to [\sigma,\infty)$ is continuous function where $\sigma > 0$ such that*

$$0 < \lim_{\phi\to+\infty} f(t,\phi) < +\infty. \tag{18}$$

Then the ψ-Caputo problem (3) has at least one positive solution.

Proof. From the equation (18), suppose there exists positive constants m_1 and m_2 such that

$$f(t,\phi) \le m_1, \tag{19}$$

for any $\phi \ge m_2,\ t\in[0,1]$. Consider $\Theta = \max_{0\le t\le 1, 0\le\phi\le m_2} f(t,\phi)$. It follows from equation (19) that

$$\sigma \le f(t,\phi) \le m_1 + \Theta, \tag{20}$$

forany $\phi \ge 0,\ t\in[0,1]$. Therefore, according to Corollary 3.2, the ψ-Caputo problem (3) has at least one positive solution $\phi \in C[0,1]$. which verifies the subsequent inequalities

$$\begin{aligned}\phi(t) &\ge \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &\quad +\sigma[\psi(t)-\psi(0)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta).\end{aligned}$$

and

$$\begin{aligned}\phi(t) &\le \phi_0 E_\theta(-w[\psi(t)-\psi(0)]^\theta) \\ &\quad +(m_1+\Theta)[\psi(t)-\psi(0)]^\theta E_{\theta,\theta+1}(-w[\psi(t)-\psi(0)]^\theta).\end{aligned}$$

□

Corollary 3.3. *Assume that $f:[0,1]\times\mathbb{R}^+ \to [\sigma,\infty)$ is continuous function where $\sigma > 0$ and there exist two constants $r_1, r_2 > 0$, such that*

$$\max\{f(t,\phi) : (t,\phi)\in[0,1]\times[0,r_2]\} \le r_1\Gamma(\theta+1) - \phi_0. \tag{21}$$

Then the ψ-Caputo problem (3) has at least one positive solution $\phi \in C[0,1]$.

Proof. From the equation (21), we have

$$\sigma \le f(t,\phi) \le r_1\Gamma(\theta+1) - \phi_0,$$

for any $(t,\phi)\in[0,1]\times[0,r_2]$. In view of Corollary 3.1, we deduce directly that the ψ-Caputo problem (3) has at least one positive solution $\phi \in C[0,1]$. which obeying

$$0 \le \|\phi\| \le r_1.$$

□

The final result is based on the Banach fixed point theorem.

Theorem 3.2. *Assume that* $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and there exist* $\mathcal{A} > 0$ *such that*

$$\|f(t,y) - f(t,y^*)\| \leq \mathcal{A}\|y - y^*\|, \quad \text{for } t \in [0,1] \text{ and } y, y^* \in \mathbb{R}^+.$$

Then the ψ*-Caputo problem (3) has a unique positive solution provided that*

$$\mathcal{R}_{\psi,\theta,\mathcal{A}} := \frac{[\psi(1) - \psi(0)]^\theta}{\Gamma(\theta+1)}\mathcal{A} < 1, \tag{22}$$

where ψ *as in Lemma 3.1.*

Proof. Consider the operator Π defined by (10). Then we shall prove that this operator is a contraction in $C[0,1]$. Let $\phi, \phi^* \in C[0,1]$. Then by Lemmas 2.3,2.4 and for $t \in [0,1]$, we have

$$\begin{aligned}
\|\Pi(\phi) - \Pi(\phi^*)\| &= \sup_{t\in[0,1]} |\Pi(\phi)(t) - \Pi(\phi^*)(t)| \\
&\leq \sup_{t\in[0,1]} \int_0^t [\psi(t) - \psi(\tau)]^{\theta-1} E_{\theta,\theta}(-w[\psi(t) - \psi(\tau)]^\theta)|f(\tau,\phi(\tau)) - f(\tau,\phi^*(\tau))|\psi'(\tau)d\tau \\
&\leq \sup_{t\in[0,1]} [\psi(t) - \psi(0)]^\theta E_{\theta,\theta+1}(-w[\psi(t) - \psi(0)]^\theta)\|f(\cdot,\phi(\cdot)) - f(\cdot,\phi^*(\cdot))\| \\
&\leq \sup_{t\in[0,1]} \frac{[\psi(t) - \psi(0)]^\theta}{\Gamma(\theta+1)}\mathcal{A}\|\phi - \phi^*\| \\
&\leq \mathcal{R}_{\psi,\theta,\mathcal{A}}\|\phi_1 - \phi_2\|.
\end{aligned}$$

From the inequality (22), Π is contraction mapping. Hence, by Theorem 2.3, we can conclude that Π has a unique fixed point which is the unique positive solution of ψ-Caputo problem (3) on $[0,1]$. □

4. Example

In this section, we give two examples to illuminate our results.

Example 4.1. *Consider the fractional differential equation with integral boundary condition*

$$\begin{cases} {}^C D_{0^+}^{\frac{1}{2};\psi} u(t) + u(t) = \psi(t) - \psi(0) + \frac{0.5}{1+u(t)}, & 0 < t \leq 1, \\ u(0) = 1, \end{cases} \tag{23}$$

where $\theta = \frac{1}{2}$, $f(t,u) = \psi(t) - \psi(0) + \frac{0.5}{1+u}$.

i) It is easy to see that f is continuous and nonnegative function. It follows that,

$$\|f(t,u) - f(t,v)\| \leq \frac{1}{2}\|u - v\| = \mathcal{A}\|u - v\|,$$

for all $t \in [0,1]$ and $u, v \in [0,\infty)$. Set $\psi(t) := e^t$. Then we find that

$$\mathcal{R}_{\psi,\theta,\mathcal{A}} = \sqrt{\frac{e-1}{\pi}} < 1.$$

All assumptions of Theorem 3.2 hold. Therefore, Theorem 3.2 guarantees that (23) has a unique positive solution $u(t) \in C[0,1]$.

ii) For all $(t, \phi) \in [0, 1] \times \mathbb{R}^+$, we have

$$\frac{1}{2} \leq f(t, \phi) \leq e - \frac{1}{2}.$$

Thus, the condition (13) holds with $M_1 = \frac{1}{2}$ and $M_2 = e - \frac{1}{2}$. Hence by Corollary 3.1, the problem (23) has a positive solution which verifies $\underline{\phi}(t) \leq \phi(t) \leq \overline{\phi}(t)$ where

$$\overline{\phi}(t) = E_{\frac{1}{2}}(-\left(e^t - 1\right)^{\frac{1}{2}}) + \left(e - \frac{1}{2}\right)\left(e^t - 1\right)^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(-\left(e^t - 1\right)^{\frac{1}{2}}),$$

and

$$\underline{\phi}(t) = E_{\frac{1}{2}}(-\left(e^t - 1\right)^{\frac{1}{2}}) + \frac{1}{2}\left(e^t - 1\right)^{\frac{1}{2}} E_{\frac{1}{2},\frac{3}{2}}(-\left(e^t - 1\right)^{\frac{1}{2}})$$

are respectively the upper and lower solutions of the problem (23).

iii) For all $(t, \phi) \in [0, 1] \times \mathbb{R}^+$, we have

$$0 < \lim_{\phi \to +\infty} f(t, \phi) < e - 1.$$

Thus, the condition (18) holds with $M_1 = \frac{1}{2}$ and $M_2 = e - \frac{1}{2}$. Hence by Corollary 3.2, the problem (23) has a positive solution.

iv) Let $\sigma = \frac{1}{2}$ and $r_2 = \frac{1}{3}$, there exists $r_1 \in (0, +\infty)$ such that

$$\max\{f(t, \phi) : (t, \phi) \in [0, 1] \times [0, \frac{1}{3}]\} = e - 1 + \frac{3}{8} \leq r_1 \frac{\sqrt{\pi}}{2} - 1,$$

Thus, the condition (21) holds with $M_1 = \frac{1}{2}$ and $M_2 = e - \frac{1}{2}$. Hence by Corollary 3.3, the problem (23) has a positive solution ϕ satisfies $0 \leq \|\phi\| \leq r_1$.

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