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Common Fixed Point Theorem for Hybrid Pair of Mappings in a Generalised (F, ξ, η)-contraction in weak Partial b- Metric Spaces with some Application

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Abstract

In the present paper, we proved a common fixed-point theorem for two-hybrid pair of non-self mappings satisfying a generalized (F, ξ, η)- contraction condition under joint common limit range property in weak partial b- metric spaces. Our result is a generalization of many works available in metric space settings. An example and application to the integral equation are given to support the results proved in this paper.

Keywords: Common fixed point; weak partial b - metric space; joint common limit range property; non-self mappings.

1. Introduction

In 1993, Czerwik [13] introduced b-metric space by weakening the triangle inequality and generalized Banach's contraction principle to this space. This research influenced many other potential researchers to perform and analyze contraction condition variants by using single and multi-valued maps in b-metric space. One ca see [4, 10, 22, 26, 35]. In 1994, Matthews [27] introduced a generalization of the metric space called the partial metric space as a part of the study of denotational semantics of dataflow networks in computer programming. Recently, Shukla [36] introduced the notion of partial b-metric spaces by combining partial metric spaces and b-metric spaces. He generalized the Banach contraction principle [7] and proved the Kannan type fixed point theorem in partial b-metric spaces. Furthermore, Mustafa *et al.* [28] introduced a modified version of partial b-metric space and proved the fixed point results. In 2019, Ameer *et al.* [2] proved fixed

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point theorem for hybrid multi-valued type contraction mappings in αK-complete partial b-metric spaces and applications.

Wardowski [38] introduced a new contraction called F- contraction in metric spaces and proved fixed point results as a generalization of the Banach contraction principle. Wardowski and Van Dung [39] established weak F-contraction in metric space and proved fixed point results as an extension of the Banach contraction principle. Also, Cosentino *et al.* [12] improved the results due to Wardowski [38] by introducing the concept of b-metric space and proved some fixed point results. For more details, we refer the reader to [6, 23] and the references therein.

In 2018, Beg and Pathak [8] proved Nadler's theorem on weak partial metric spaces with application to homotopy result. Later, in 2019, Kanwal *et al.* [21] define the notion of weak partial b-metric spaces and weak partial Hausdorff b-metric spaces along with the topology of weak partial b-metric space. Moreover, they generalized Nadler's theorem using weak partial Hausdorff b-metric spaces in the context of a weak partial b-metric space.

Later, Sintunavarat and Kumam [37] initiated the concept of common limit range (CLR) property in order to exhibit its sharpness over the (EA) property due to Aamri and El Moutawakil [1]. Persuaded by the ideas of Sintunavarat and Kumam [37], Imdad *et al.* [19] introduced the notion of common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semi-metric) space. Besides this, Imdad *et al.* [18] established the joint common limit range notion and proved the common fixed point theorem for a pair of non-self mappings in metric space.

Naimpally *et al.* [29] generalized Goebel's [16] result to a hybrid of multi-valued and single-valued maps satisfying a contractive condition. Henceforth, several fixed point theorems for multi-valued maps are extended by Naimpally *et al.* [29].

The contributions of Aserkar and Gandhi in [3], Wardowski and Van Dung [39], Secelean [34], Joshi *et al.* [20], Nashine *et al.* [30, 31], upon this particular study has influenced us to prove a common fixed point theorem for two hybrid pairs of non-self mappings satisfying a generalized (F, ξ, η)-contraction condition under joint common limit range (JCLR) property in weak partial b-metric space with application to a non-linear hybrid ordinary differential equation. Our results generalize and improve several known works of the existing literature.

2. Preliminaries

We will require the following preliminary definitions and theorems for establishing our result.

Czerwik [13] gave a generalization of metric space to b-metric space as bellow;

Definition 2.1. *[13]* Let M be a non empty set and $s \geq 1$ be a given real number. A function $d : M \times M \to [0, \infty)$ is called a b- metric if for all $x, y, z \in M$ the following condition satisfied:

$(B1)$ $d(x, y) = 0$ iff $x = y$,

$(B2)$ $d(x, y) = d(y, x)$ and

$(B3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (M, d) is called a b-metric space. The number $s \geq 1$ is called the coefficient of (M, d).

Example 2.2. [9] Let $p \in (0, 1)$, and

$$X = l_p(\mathbb{R}) := \Big\{x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\Big\},$$

together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}.$$

where $x = x_n, y = y_n \in l^p(\mathbb{R})$. Then (M, d) is a b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$.

Definition 2.3. [27] A partial metric space is a pair (M, p) consisting of a non-empty set M together with a function $p : M \times M \to \mathbb{R}$, called the partial metric, such that for all $x, y, z \in M$ we have the following properties:

$(P1)$ $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,

$(P2)$ $p(x, x) \leq p(x, y)$,

$(P3)$ $p(x, y) = p(y, x)$ and

$(P4)$ $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

From $(P1)$ and $(P2)$ we have

$$p(x, y) = 0 \Rightarrow p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y.$$

As an example, the pair $(\mathbb{R}^+, p)$, where and $p : M \times M \to \mathbb{R}^+$ is defined as $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$, is a partial metric space.

Shukla [36] gave an extension by combining partial metric space and b-metric space to partial b-metric space.

Definition 2.4. *[36]* A partial b- metric on a non-empty set M is a function $b : M \times M \to \mathbb{R}^+$ such that for all $x, y, z \in M$:

$(Pb1)$ $x = y$ if and only if $b(x, x) = b(x, y) = b(y, y)$,

$(Pb2)$ $b(x, x) \leq b(x, y)$,

$(Pb3)$ $b(x, y) = b(y, x)$ and

$(Pb4)$ there exist a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)] - b(z, z)$.

A partial b-metric space is a pair (M, b) such that X is a non-empty set and b is a partial b- metric on M. The number $s \geq 1$ is called the coefficient of (M, b).

Mustafa *et al.* [28] gave an extension of partial b-metric space as follows;

Definition 2.5. [28] *Let M be a non empty set and $s \geq 1$ be a given real number. A function $p_b : M \times M \to \mathbb{R}^+$ is called a partial b- metric if for all $x, y, z \in M$ the following condition are satisfied:*

$(PB1)$ $x = y \iff p_b(x, x) = p_b(x, y) = p_b(y, y)$,

$(PB2)$ $p_b(x, x) \leq p_b(x, y)$,

$(PB3)$ $p_b(x, y) = p_b(y, x)$ *and*

$(PB4)$ $p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] + \frac{1-s}{2}[p_b(x, x) + p_b(y, y)]$.

The pair (M, p_b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (M, p_b).

Example 2.6. [36] Let $M = \mathbb{R}^+$, $q > 1$ be a constant and $p_b : M \times M \to \mathbb{R}^+$ be defined by

$$p_b(x, y) = [\max\{x, y\}]^q + |x - y|^q,$$

for all $x, y \in M$. Then, (M, p_b) is a partial b-metric space with the coefficient $s = 2^q > 1$, but it is neither a b-metric nor a partial metric space.

In 2018, Beg and Pathak [8] gave a generalized notion of weak partial metric space as follows:

Definition 2.7. [8] *Let M be a non empty set. A function $q : M \times M \to \mathbb{R}^+$ is called a weak partial metric on M if for all $x, y, z \in M$ the following conditions satisfied:*

$(WP1)$ $q(x,x) = q(x,y) \Longleftrightarrow x = y$,

$(WP2)$ $q(x,x) \leq q(x,y)$,

$(WP3)$ $q(x,y) = q(y,x)$ *and*

$(WP4)$ $q(x,y) \leq q(x,z) + q(z,y)$.

The pair (M, q) is called a weak partial metric space.
Some examples of weak partial metric spaces are the following.

Example 2.8. [8]

(1) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = |x - y| + 1,$$

for all $x, y \in \mathbb{R}^+$.

(2) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = \frac{1}{4}|x - y| + \max\{x, y\},$$

for all $x, y \in \mathbb{R}^+$.

(3) $(\mathbb{R}^+, q)$, *where* $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$q(x,y) = \max\{x, y\} + e^{|x-y|} + 1,$$

for all $x, y \in \mathbb{R}^+$.

In 2019, Kanwal *et al.* [21] gave a generalized concept from weak partial metric space to weak partial b-metric space as follows:

Definition 2.9. [21] *Let $M \neq \emptyset$ and $s \geq 1$, a function $\varrho_b : M \times M \to \mathbb{R}^+$ is called a weak partial b-metric on M if for all $x, y, z \in M$, following conditions are satisfied:*

$(WPB1)$ $\varrho_b(x,x) = \varrho_b(x,y) \Longleftrightarrow x = y$,

$(WPB2)$ $\varrho_b(x,x) \leq \varrho_b(x,y)$,

$(WPB3)$ $\varrho_b(x,y) = \varrho_b(y,x)$ *and*

$(WPB4)$ $\varrho_b(x,y) \leq s[\varrho_b(x,z) + \varrho_b(z,y)]$.

The pair (M, ϱ_b) is called a weak partial b- metric space.
Some of the examples of weak partial b-metric space are:

Example 2.10. [21]

(1) $(\mathbb{R}^+, \varrho_b)$, *where* $\varrho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$\varrho_b(x, y) = |x - y|^2 + 1,$$

for all $x, y \in \mathbb{R}^+$.

(2) $(\mathbb{R}^+, q)$, *where* $\varrho_b : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is defined as*

$$\varrho_b(x, y) = \frac{1}{2}|x - y|^2 + \max\{x, y\},$$

for all $x, y \in \mathbb{R}^+$.

Definition 2.11. [21] *A sequence* $\{x_n\}$ *in* (M, ϱ_b) *is said to converges a point* $x \in M$*, if and only if*

$$\varrho_b(x, x) = \lim_{n\to\infty} \varrho_b(x, x_n).$$

Definition 2.12. [21] *Let* (M, ϱ_b) *be a weak partial b-metric space. Then*

(*i*) *A Cauchy sequence in metric space* (M, ϱ_b^s) *is Cauchy sequence in* M.

(*ii*) *If the metric space* (M, ϱ_b^s) *is complete, so is weak partial b-metric space* (M, ϱ_b).

(*iii*) *If* ϱ_b *is a weak partial b-metric on* M*, the function* $\varrho_b^s : M \times M \to \mathbb{R}^+$ *given by*

$$\varrho_b^s(x, y) = \varrho_b(x, y) - \frac{1}{2}[\varrho_b(x, x) + \varrho_b(y, y)],$$

define a b metric on M*. Further, a sequence* $\{x_n,\}$ *in* (M, ϱ_b^s) *converges to a point* $x \in M$*, iff*

$$\lim_{n,m\to\infty} \varrho_b^s(x_n, x_m) = \lim_{n\to\infty} \varrho_b(s_n, s) = \varrho_b(s, s).$$

Motivated by Kanwal *et al.* [21] we define multivalued notion in weak partial b-metric space, which is an extension of the concept given by Aydi *et al.* [5].

Let (M, ϱ_b) be a weak partial b-metric space and $CB^{\varrho_b}(M)$ be class of all nonempty, closed and bounded subsets of (M, ϱ_b). For $A, B \in CB^{\varrho_b}(M)$ and $x \in M$, define:

$$\begin{aligned} \varrho_b(x, A) &= inf\{\varrho_b(x, a) : a \in A\}; \\ \delta_{\varrho_b}(A, B) &= \sup\{\varrho_b(a, B) : a \in A\}; \\ \delta_{\varrho_b}(B, A) &= \sup\{\varrho_b(b, A) : b \in B\}. \end{aligned}$$

Note that

$$\varrho_b(x, A) = 0 \implies \varrho_b^s(x, A) = 0, \tag{1}$$

where

$$\varrho_b^s(x, A) = \inf\{\varrho_b^s(x, A), x \in A\}.$$

Remark 2.13. [21] Let (M, ϱ_b) be a weak partial b-metric space and A a nonempty subset of M, then

$$a \in \bar{A} \iff \varrho_b(a, A) = \varrho_b(a, a).$$

Definition 2.14. [21] Let (M, ϱ_b) be a weak partial b-metric space. For $A, B \in CB^{\varrho_b(M)}$, the mapping $\mathcal{H}^+_{\varrho_b} : CB^{\varrho_b} \times CB^{\varrho_b} \to [0, \infty)$ defined by

$$\mathcal{H}^+_{\varrho_b}(A, B) = \frac{1}{2}\{\delta_{\varrho_b}(A, B) + \delta_{\varrho_b}(B, A)\},$$

is called $\mathcal{H}^+_{\varrho_b}$-type Hausdorff metric induced by ϱ_b.

The following explanations for developing the F-contraction definition are from Wardowski and Van Dung [39].

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying

($F1$) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;

($F2$) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

($F3$) there exists $k \in (0, 1)$ satisfying $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the family of all functions F satisfying conditions $(F1-F3)$ by Ω. Some examples of functions $F \in \Omega$ are:

(1) $F(a) = \ln a$,

(2) $F(a) = a + \ln a$.

(3) $F(a) = \ln(a^2 + a)$.

Motivated by Wardowski and Van Dug [39], we introduce the notion of F-weak partial b-metric space.

Definition 2.15. *Let (M, ϱ_b) be a weak partial b-metric space. A map $T : M \to M$ is said to be an F-weak contraction on (M, ϱ_b) if there exists $F \in \Omega$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $\varrho_b(fx, fy) > 0$, the following condition holds:*

$$\tau + F(\varrho_b(fx, fy)) \leq F\Big(\max\Big\{\varrho_b(x, y), \varrho_b(x, fx), \varrho_b(y, fy), \frac{\varrho_b(x, fy) + \varrho_b(y, fx)}{2}\Big\}\Big).$$

Motivated by Piri and Rahrovi [33], we establish the concept of multivalued F-weak partial b-metric space as follows:

Definition 2.16. [33] *Let (M, ϱ_b) be a weak partial b-metric space. A map $T : M \to CB^{\varrho_b}(M)$ is said to be multivalued F-weak contraction on (M, ϱ_b) if there exists $F \in \Omega$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $\mathcal{H}^+_{\varrho_b}(Tx, Ty) > 0$, the following holds:*

$$\tau + F(\mathcal{H}^+_{\varrho_b}(Tx, Ty)) \leq F(N(x, y)),$$

where,

$$N(x, y) = \max\left\{\varrho_b(x, y), \varrho_b(x, Tx), \varrho_b(y, Ty), \frac{\varrho_b(x, Ty) + \varrho_b(y, Tx)}{2}\right\}.$$

In 1984, Khan *et al.* [24] established an altering distances concept between the points in metric space as follows:

Definition 2.17. [24] $(\xi, \eta) \in \Psi$ iff ξ, η are continuous functions from $[0, \infty) \to [0, \infty)$ and $s \geq 1$ be a given real number are called an altering distance function if satisfies:

(i) ξ is continuous and non-decreasing.

(ii) $\xi(t) = 0$ if and only if $t = 0$.

(iii) $s\xi(t) \leq \xi(t) - \eta(t)$ if and only if $t = 0$.

Imdad *et al.* [18], have established the concept of joint common limit range property for two hybrid pairs of non-self mappings as follows:

Definition 2.18. Let (M,d) be a metric space whereas Y an arbitrary non-empty set with $F,G : Y \to CB(X)$ and $f,g : Y \to M$. Then the pairs of hybrid mappings (F,f) and (G,g) are said to have the $(JCLR)$ property, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in Y and $A,B \in CB(X)$ such that

$$\begin{aligned}\lim_{n\to\infty} Fx_n &= A,\ \lim_{n\to\infty} Gy_n = B,\\ \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = t \in A\cap B\cap f(Y)\cap g(Y),\end{aligned}$$

i.e., there exists $u,v \in Y$ such that $t = fu = gv \in A\cap B$.

Imdad *et al.* [17] defined that a map is said to be coincidentally idempotent if it satisfies the condition given in the following definition.

Definition 2.19. *[17]* Let (M,d) be a metric space whereas Y an arbitrary non-empty set with $T : Y \to CB(M)$ and $g : Y \to M$. The mapping g is said to be a coincidentally idempotent with respect to the mapping T, if $u \in M$, $gu \in Tu$ with $gu \in Y$ imply $ggu = gu$ that is, g is idempotent at coincidence point of the pair (T,g).

In 2020, Aserkar and Gandhi in [3] gave the following results in b-metric space for weakly compatible mappings in pairs that satisfy the common limit range property.

The theorem of Aserkar and Gandhi in [3] is as follows:

Theorem 2.20. [3] Let (M,d) be a b-metric space with $s \geq 1$ and $F,G,P,Q : M \to M$. Suppose that $\xi,\eta \in \psi$ and $L \geq 0$ such that

(*i*) (F,Q) satisfies CLR_P and (G,P) satisfies CLR_Q.

(*ii*) $s\xi(d(Fx,Gy)) \leq \xi(N_1(x,y)) - \eta(N_1(x,y)) + LN_2(x,y)$, where

$$\begin{aligned}N_1(x,y) = \max\Bigg\{ & d(Py,Qx), \frac{d(Qx,Fx)*d(Py,Gy)}{1+d(Fx,Gy)},\\ & \frac{(d(Py,Fx))^2+(d(Qx,Gy))^2}{d(Py,Fx)+d(Qx,Gy)},\\ & \frac{d(Qx,Fx)*d(Qx,Gy)+d(Py,Gy)*d(Py,Fx)}{d(Qx,Gy)+d(Py,Fx)}\Bigg\},\end{aligned}$$

and

$$N_2(x,y) = \min\Big\{d(Qx,Fx), d(Qx,Gy), d(Py,Fx), d(Py,Gy)\Big\},$$

for all $x,y \in M$.

(*iii*) The pair (F,Q) and (G,P) are weakly compatible.

Then F,G,P,Q have a unique common fixed point.

Motivated by the results obtained by Aserkar and Gandhi [3]. In the following section, we wish to establish the proof of common fixed point for two hybrid pairs of coincidentally idempotent non-self mappings in weakly partial b-metric space, which satisfies joint common limit range property in a generalized (F,ξ,η)-contraction. We provide an illustrative example to support the theorem proved. Also, an application for a hybrid differential equation will be provided to support the results.

3. Main Results

We commence by extending Definition 2.18 to weak partial b-metric space for non-self mappings as follows:

Definition 3.1. *Let (M, ϱ_b) be a weak partial b- metric space with $f, g : X \to M$ and $G, T : X \to CB^{\varrho_b}(M)$. Then the pairs of hybrid mappings (G, f) and (T, g) are said to have joint common limit range property, denoted by $(JCLR)$-property. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CB^{\varrho_b}(M)$ such that*

$$\begin{aligned}\lim_{n\to\infty} Gx_n &= A, \lim_{n\to\infty} Ty_n = B,\\ \lim_{n\to\infty} fx_n &= \lim_{n\to\infty} gy_n = t,\end{aligned}$$

with $t \in f(X) \cap g(X) \cap A \cap B$, that is, there exists $u, v \in X$ such that $t = fu = gv \in A \cap B$.

Next, we extend Definition 2.19 to weak partial b-metric space as follows:

Definition 3.2. *Let (M, ϱ_b) be a weak partial b- metric space with $f : X \to M$ and $G : X \to CB^{\varrho_b}(M)$. The mapping is said to be a coincidentally idempotent with respect to the mapping G, if $u \in M$, $fu \in Gu$ with $fu \in M$ imply $ffu = fu$ that is, f is idempotent at coincidence point of the pair (G, f).*

Now, we prove the following theorem which is an extended version of Theorem 2.20 and Definition 2.16 in weak partial b-metric space for two hybrid pairs of non-self mappings, which satisfies joint common limit range property.

Theorem 3.3. *Let $f, g : X \to M$ be two self mappings of a weak partial b-metric space (M, ϱ_b) with $s \geq 1$ and $G, T : X \to CB^{\varrho_b}(M)$ be two multivalued mappings from X into $CB^{\varrho_b}(M)$. Assume that $\xi, \eta \in \psi$ and $L \geq 0$ such that*

(i) the hybrid pair (G, f) and (T, g) satisfies JCLR property,

(ii) there exists $\tau > 0$ with $\mathcal{H}^+_{\varrho_b}(Gx, Ty) > 0$ such that

$$\tau + F(s\xi(\mathcal{H}^+_{\varrho_b}(Gx, Ty))) \leq F(\xi(N_1(x, y)) - \eta(N_1(x, y)) + LN_2(x, y)), \tag{2}$$

where

$$\begin{aligned}N_1(x, y) = \max\Bigg\{ & \varrho_b(gy, fx), \frac{\varrho_b(fx, Gx) * \varrho_b(gy, Ty)}{1 + \varrho_b(Gx, Ty)},\\ & \frac{(\varrho_b(gy, Gx))^2 + (\varrho_b(fx, Ty))^2}{\varrho_b(gy, Gx) + \varrho_b(fx, Ty)},\\ & \frac{\varrho_b(fx, Gx) * \varrho_b(fx, Ty) + \varrho_b(gy, Ty) * \varrho_b(gy, Gx)}{\varrho_b(fx, Ty) + \varrho_b(gy, Gx)}\Bigg\},\end{aligned}$$

and

$$N_2(x, y) = \min\Big\{\varrho_b(fx, Gx), \varrho_b(fx, Ty), \varrho_b(gy, Gx), \varrho_b(gy, Ty)\Big\},$$

for all $x, y \in M$,

(iii) if $X \subset M$ and the pairs (G, f) and (T, g) are coincidentally commuting and coincidentally idempotent.

Then the pair (G, f) and (T, g) have a common fixed point in $u \in M$ and $\varrho_b(u, u) = 0$.

Proof. Since the hybrid pairs (G, f) and (T, g) satisfies the $JCLR$ property, by Definition 3.1 there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CB^{\varrho_b}(M)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Gx_n, \lim_{n\to\infty} gy_n = t \in B = \lim_{n\to\infty} Ty_n,$$

for some $u, v \in X$ and $t = fv = gu \in A \cap B$. We assert that $gu \in Tu$. If not, then using $x = x_n$ and $y = u$ in (2), we get

$$\tau + F(s\xi(\mathcal{H}^+_{\varrho_b}(Gx_n, Tu))) \leq F(\xi(N_1(x_n, u)) - \eta(N_1(x_n, u)) + LN_2(x_n, u)), \tag{3}$$

where

$$\begin{aligned} N_1(x_n, u)) &= \max\Bigg\{\varrho_b(gu, fx_n), \frac{\varrho_b(fx_n, Gx_n) * \varrho_b(gu, Tu)}{1 + \varrho_b(Gx_n, Tu)}, \\ &\quad \frac{(\varrho_b(gu, Gx_n))^2 + (\varrho_b(fx_n, Tu))^2}{\varrho_b(gu, Gx_n) + \varrho_b(fx_n, Tu)}, \\ &\quad \frac{\varrho_b(fx_n, Gx_n) * \varrho_b(fx_n, Tu) + \varrho_b(gu, Tu) * \varrho_b(gu, Gx_n)}{\varrho_b(fx_n, Tu) + \varrho_b(gu, Gx_n)}\Bigg\}, \end{aligned} \tag{4}$$

Taking limit as $n \to \infty$ in (4), we get

$$\begin{aligned} &\leq \max\Bigg\{\varrho_b(gu, gu), \frac{\varrho_b(gu, A) * \varrho_b(gu, Tu)}{1 + \varrho_b(A, Tu)}, \\ &\quad \frac{(\varrho_b(gu, A))^2 + (\varrho_b(gu, Tu))^2}{\varrho_b(gu, A) + \varrho_b(gu, Tu)}, \\ &\quad \frac{\varrho_b(gu, A) * \varrho_b(gu, Tu) + \varrho_b(gu, Tu) * \varrho_b(gu, A)}{\varrho_b(gu, Tu) + \varrho_b(gu, A)}\Bigg\}, \\ &\leq \max\Bigg\{\varrho_b(t, t), \frac{\varrho_b(t, A) * \varrho_b(gu, Tu)}{1 + \varrho_b(A, Tu)}, \\ &\quad \frac{(\varrho_b(t, A))^2 + (\varrho_b(gu, Tu))^2}{\varrho_b(t, A) + \varrho_b(gu, Tu)}, \\ &\quad \frac{\varrho_b(t, A) * \varrho_b(gu, Tu) + \varrho_b(gu, Tu) * \varrho_b(t, A)}{\varrho_b(gu, Tu) + \varrho_b(t, A)}\Bigg\}, \end{aligned} \tag{5}$$

using Definition 2.12 and (1) in (5), we get

$$\begin{aligned} &\leq \max\Bigg\{0, \frac{0 * \varrho_b(gu, Tu)}{1 + \varrho_b(A, Tu)}, \frac{(0)^2 + (\varrho_b(gu, Tu))^2}{0 + \varrho_b(gu, Tu)}, \\ &\quad \frac{0 * \varrho_b(gu, Tu) + \varrho_b(gu, Tu) * 0}{\varrho_b(gu, Tu) + 0}\Bigg\}, \\ &\leq \max\Bigg\{0, 0, \frac{\varrho_b(gu, Tu))^2}{\varrho_b(gu, Tu)}, 0\Bigg\}, \\ &\leq \max\Big\{0, 0, \varrho_b(gu, Tu), 0\Big\}, \\ &= \varrho_b(gu, Tu). \end{aligned} \tag{6}$$

Consequently, we have

$$\begin{aligned} N_2(x_n,u)) &= \min\Big\{\varrho_b(fx_n,Gx_n),\varrho_b(fx_n,Tu),\varrho_b(gu,Gx_n),\varrho_b(gu,Tu)\Big\},\\ &\le \min\Big\{\varrho_b(gu,A),\varrho_b(gu,Tu),\varrho_b(gu,A),\varrho_b(gu,Tu)\Big\},\\ &\le \min\Big\{\varrho_b(t,A),\varrho_b(gu,Tu),\varrho_b(t,A),\varrho_b(gu,Tu)\Big\},\\ &\le \min\Big\{0,\varrho_b(gu,Tu),0,\varrho_b(gu,Tu)\Big\}\\ &= 0. \end{aligned} \tag{7}$$

Using (7) and (6) in (3), one obtains

$$\tau+F(s\xi\mathcal{H}^+_{\varrho_b}(A,Tu))\le F(\xi\varrho_b(gu,Tu)-\eta\varrho_b(gu,Tu)+L(0)),$$
$$\tau+F(s\xi\mathcal{H}^+_{\varrho_b}(A,Tu))\le F(\xi\varrho_b(gu,Tu)-\eta\varrho_b(gu,Tu)).$$

Since $\tau>0$, in viewing the properties of η,ξ, and F is strictly increasing, by $(F1)$ we have

$$\mathcal{H}^+_{\varrho_b}(A,Tu)<\varrho_b(gu,Tu)$$
$$s\xi\mathcal{H}^+_{\varrho_b}(A,Tu)\le(\xi-\eta)\varrho_b(gu,Tu)$$

As $t=fv=gu\in A\cap B$, it follows that

$$\mathcal{H}^+_{\varrho_b}(A,Tu))\le\frac{\xi-\eta}{s\xi}\Big\{\varrho_b(gu,Tu)\Big\}.$$

Thus,

$$\varrho_b(gu,Tu)<\mathcal{H}^+_{\varrho_b}(A,Tu)<\frac{\xi-\eta}{s\xi}\Big\{\varrho_b(gu,Tu)\Big\},$$

a contradiction. Hence $gu\in Tu$ which shows that the pair (T,g) has a coincidence point u in M.

Similar, we assert that $fv\in Gv$. Suppose that $fv\neq Gv$, then using $x=v$ and $y=y_n$ in (2), one gets

$$\tau+F(s\xi(\mathcal{H}^+_{\varrho_b}(Gv,Ty_n)))\le F(\xi(N_1(v,y_n))-\eta(N_1(v,y_n))+LN_2(v,y_n,)), \tag{8}$$

where

$$\begin{aligned} N_1(v,y_n)) &= \max\Bigg\{\varrho_b(gy_n,fv),\frac{\varrho_b(fv,Gv)*\varrho_b(gy_n,Ty_n)}{1+\varrho_b(Gv,Ty_n)},\\ &\frac{(\varrho_b(gy_n,Gv))^2+(\varrho_b(fv,Ty_n))^2}{\varrho_b(gy_n,Gv)+\varrho_b(fv,Ty_n)},\\ &\frac{\varrho_b(fv,Gv)*\varrho_b(fv,Ty_n)+\varrho_b(gy_n,Ty_n)*\varrho_b(gy_n,Gv)}{\varrho_b(fv,Ty_n)+\varrho_b(gy_n,Gv)}\Bigg\}, \end{aligned}$$

Taking limit as $n\to\infty$ in (9), we have

$$\begin{aligned} \le\ &\max\Bigg\{\varrho_b(fv,fv),\frac{\varrho_b(fv,Gv)*\varrho_b(fv,B)}{1+\varrho_b(Gv,B)},\\ &\frac{(\varrho_b(fv,Gv))^2+(\varrho_b(fv,B))^2}{\varrho_b(fv,Gv)+\varrho_b(fv,B)},\\ &\frac{\varrho_b(fv,Gv)*\varrho_b(fv,B)+\varrho_b(fv,B)*\varrho_b(fv,Gv)}{\varrho_b(fv,B)+\varrho_b(fv,Gv)}\Bigg\}, \end{aligned}$$

$$\leq \max\Big\{\varrho_b(t,t), \frac{\varrho_b(fv,Gv)*\varrho_b(t,B)}{1+\varrho_b(Gv,B)}, \frac{(\varrho_b(fv,Gv))^2+(\varrho_b(t,B))^2}{\varrho_b(fv,Gv)+\varrho_b(t,B)}, \frac{\varrho_b(fv,Gv)*\varrho_b(t,B)+\varrho_b(t,B)*\varrho_b(fv,Gv)}{\varrho_b(t,B)+\varrho_b(fv,Gv)}\Big\}, \tag{9}$$

using Definition 2.12 and (1) in (5), we get

$$\begin{aligned}
&\leq \max\Big\{0, \frac{\varrho_b(fv,Gv)*0}{1+\varrho_b(Gv,B)}, \frac{(\varrho_b(fv,Gv))^2+(0)^2}{\varrho_b(fv,Gv)+0}, \frac{\varrho_b(fv,Gv)*0+0*\varrho_b(fv,Gv)}{0+\varrho_b(fv,Gv)}\Big\},\\
&\leq \max\Big\{0,0,\frac{\varrho_b(fv,Gv))^2}{\varrho_b(fv,Gv)},0\Big\},\\
&\leq \max\Big\{0,0,\varrho_b(fv,Gv),0\Big\},\\
&= \varrho_b(fv,Gv).
\end{aligned} \tag{10}$$

Consequently, we have

$$\begin{aligned}
N_2(v,y_n,)) &= \min\Big\{\varrho_b(fv,Gv), \varrho_b(fv,Ty_n), \varrho_b(gy_n,Gv), \varrho_b(gy_n,Ty_n)\Big\},\\
&\leq \min\Big\{\varrho_b(fv,Gv), \varrho_b(fv,B), \varrho_b(fv,Gv), \varrho_b(fv,B)\Big\},\\
&\leq \min\Big\{\varrho_b(fv,Gv), \varrho_b(t,B), \varrho_b(fv,Gv), \varrho_b(t,B)\Big\},\\
&\leq \min\Big\{\varrho_b(fv,Gv), 0, \varrho_b(fv,Gv), 0\Big\}\\
&= 0.
\end{aligned} \tag{11}$$

Using (11) and (10) in (8), one obtains

$$\tau + F(s\xi\mathcal{H}^+_{\varrho_b}(Gv,B)) \leq F(\xi\varrho_b(fv,Gv) - \eta\varrho_b(fv,Gv) + L(0)),$$
$$\tau + F(s\xi\mathcal{H}^+_{\varrho_b}(Gv,B)) \leq F(\xi\varrho_b(fv,Gv) - \eta\varrho_b(fv,Gv)).$$

Since $\tau > 0$, in viewing the properties of η, ξ, and F is strictly increasing, by $(F1)$ we have

$$\mathcal{H}^+_{\varrho_b}(Gv,B) < \varrho_b(fv,Gv)$$
$$s\xi\mathcal{H}^+_{\varrho_b}(Gv,B) \leq (\xi-\eta)\varrho_b(fv,Gv)$$

As $t = fv = gu \in A \cap B$, it follows that

$$\mathcal{H}^+_{\varrho_b}(Gv,B)) \leq \frac{\xi-\eta}{s\xi}\Big\{\varrho_b(fv,Gv)\Big\}.$$

Thus,

$$\varrho_b(fv,Gv) < \mathcal{H}^+_{\varrho_b}(Gv,B) < \frac{\xi-\eta}{s\xi}\Big\{\varrho_b(fv,Gv)\Big\},$$

a contradiction. Hence $fv \in Gv$ which shows that the pair (G,f) has a coincidence point v in M.

Next we show that $gu \in Tu$ and $fv \in Gv$, if not, then using $x = u$ and $y = v$ in (2), we get

$$\tau + F(s\xi(\mathcal{H}^+_{\varrho_b}(Gu,Tv))) \leq F(\xi(N_1(u,v)) - \eta(N_1(u,v)) + LN_2(u,v)), \tag{12}$$

where

$$\begin{aligned} N_1(u,v)) &= \max\Bigg\{ \varrho_b(gv,fu), \frac{\varrho_b(fu,Gu) * \varrho_b(gv,Tu)}{1+\varrho_b(Gu,Tv)}, \\ &\quad \frac{(\varrho_b(gv,Gu))^2 + (\varrho_b(fu,Tv))^2}{\varrho_b(gv,Gu) + \varrho_b(fu,Tv)}, \\ &\quad \frac{\varrho_b(fu,Gu) * \varrho_b(fu,Tv) + \varrho_b(gv,Tv) * \varrho_b(gv,Gu)}{\varrho_b(fu,Tv) + \varrho_b(gv,Gu)} \Bigg\}, \end{aligned}$$

using (1), we have

$$\begin{aligned} &\leq \max\Big\{ \varrho_b(gv,fu), 0, 0, 0 \Big\}, \\ &= \varrho_b(gv,fu). \end{aligned} \tag{13}$$

and

$$\begin{aligned} N_2(u,v) &= \min\Big\{ \varrho_b(fu,Gu), \varrho_b(fu,Tv), \varrho_b(gv,Gu), \varrho_b(gv,Tv) \Big\}, \\ &\leq \min\Big\{ 0,0,0,0 \Big\}, \\ &= 0. \end{aligned} \tag{14}$$

Using (14) and (13) in (12), one gets

$$\tau + F(s\xi\mathcal{H}^+_{\varrho_b}(Gu,Tv)) \leq F(\xi\varrho_b(gv,fu)) - \eta\varrho_b(gv,fu) + L(0)), \tag{15}$$
$$\tau + F(s\xi\mathcal{H}^+_{\varrho_b}(Gu,Tv)) \leq F(\xi\varrho_b(gu,Gv)) - \eta\varrho_b(gu,Gv),$$

In viewing the properties of τ, η, ξ, and F is strictly increasing, by $(F1)$ we have

$$\mathcal{H}^+_{\varrho_b}(Gu,Tv) \leq \varrho_b(gv,fu) \tag{16}$$
$$\implies s\xi\mathcal{H}^+_{\varrho_b}(Gu,Tv) \leq (\xi-\eta)\varrho_b(gv,fu)$$

As $t = fv = gu \in A \cap B$, it follows that

$$\mathcal{H}^+_{\varrho_b}(Gu,Tv) \leq \frac{\xi-\eta}{s\xi}\varrho_b(gv,fu) \tag{17}$$

Thus,

$$\varrho_b(gv,fu) < \mathcal{H}^+_{\varrho_b}(Gu,Tv) < \frac{\xi-\eta}{s\xi}\varrho_b(gv,fu),$$

a contradiction. Hence $gu \in Tu$ and $fv \in Gv$ which shows that the pair (T,g), (G,f) has a coincidence point $u = v$ in M.

Suppose that $X \in M$. Since v is a coincidence point of the pair (G,f) which is coincidentally commuting and coincidentally idempotent. With respect to mapping G, we have $fv \in Gv$ and $ffv = fv$, therefore $fv = ffv \in f(Gv) \subset G(fv)$ which shows that fv is a common fixed point of the pair (G,f). Similarly, u is a coincidence point of the pair (T,g) which is coincidentally commuting and coincidentally idempotent concerning mapping T, one can easily show that gu is a common fixed point of the pair (T,g).

Moroever, if u and v are coincidence points which are coincidentally commuting and coincidentally idempontent, then there exists $u \in C(T,g)$ and $v \in C(G,f)$ such that $gu = Tu, fv = Gv$.

Hence $u = v = gu = fv$, consequently, u is a common fixed point of the two hybrid pairs of mappings (G,f) and (T,g) in M. □

Example 3.4. Let $X = [0,2] \subset [0,\infty) = M$ be a weak partial b-metric space equipped with metric $\varrho_b(x,y) = |x-y|^2+1$, for all $x,y \in M$. Let $G,T : X \to M$ be defined as

$$Gx = \begin{cases} [\frac{3}{5},\frac{3}{2}], \ if \quad 0 \le x \le 1, \\ \\ [\frac{1}{4},\frac{1}{2}], \ if \quad 0 \le x < 2. \end{cases}$$

$$Tx = \begin{cases} [\frac{3}{2},2], \ if \quad 0 \le x < 1, \\ \\ [\frac{1}{2},2], \ if \quad 1 \le x \le 2. \end{cases}$$

Suppose $f,g : X \to M$ be defined as

$$fx = \begin{cases} 1, \ if \quad 0 \le x \le 1, \\ \\ \frac{3x}{5}, \ if \quad 1 < x \le 2. \end{cases}$$

$$gx = \begin{cases} \frac{3x}{2}, \ if \quad 0 \le x < 1, \\ \\ 1, \ if \quad 1 \le x \le 2. \end{cases}$$

Let $F : \mathbb{R}^+ \to \mathbb{R}$ be defined by $F(a) = \ln a + a$ and $\xi, \psi : [0,\infty) \to [0,\infty)$ such that $\xi(t) = \frac{1}{10}t, \eta(t) = \frac{t+1}{2}, L = 5$, $s = 2$ and $\tau = 1$, then, Equation 2 takes the form

$$\frac{s\xi(\mathcal{H}^+_{\varrho_b}(Gx,Ty))}{\xi(N_1(x,y)) - \eta(N_1(x,y)) + LN_2(x,y)} e^{s\xi(\mathcal{H}^+_{\varrho_b}(Gx,Ty)) - [\xi(N_1(x,y)) - \eta(N_1(x,y)) + LN_2(x,y)]} \le e^{-\tau}. \tag{18}$$

Choosing two sequence $\{x_n\} = \{1 - \frac{1}{2n}\}$ and $\{y_n\} = \{1 + \frac{1}{2n}\}$ in X, one can see that the pairs (G,f) and (T,g) satisfies $(JCLR)$ property, i.e.

$$\begin{aligned} \lim_{n\to\infty} f\Big\{1 - \frac{1}{2n}\Big\} &= 1 \in \Big[\frac{3}{5},\frac{3}{2}\Big] = \lim_{n\to\infty} G\Big\{1 - \frac{1}{2n}\Big\}, \\ \lim_{n\to\infty} g\Big\{1 + \frac{1}{2n}\Big\} &= 1 \in \Big[\frac{1}{2},\frac{3}{2}\Big] = \lim_{n\to\infty} T\Big\{1 + \frac{1}{2n}\Big\}. \end{aligned}$$

Now to verify condition (2) we distinguish the following cases;

Case I

For $x \in [0,1]$, $y \in [1,2]$ and applying Definition 2.14, we have

$$\begin{aligned} \mathcal{H}^+_{\varrho_b}(Gx,Ty) &= \mathcal{H}^+_{\varrho_b}\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right) \\ &= \frac{1}{2}\left\{\sup\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right) + \sup\left(\left[\frac{1}{2},2\right],\left[\frac{3}{5},\frac{3}{2}\right]\right)\right\}. \end{aligned} \tag{19}$$

$$\begin{aligned} \sup\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right) &= \max\left\{\varrho_b\left(\frac{3}{5},\left[\frac{1}{2},2\right]\right),\varrho_b\left(\frac{3}{2},\left[\frac{1}{2},2\right]\right)\right\} \\ &= \max\Big\{1.01, 1.25\Big\} \\ &= 1.25. \end{aligned} \tag{20}$$

$$\begin{aligned}\sup\left(\left[\frac{1}{2},2\right],\left[\frac{3}{5},\frac{3}{2}\right]\right) &= \max\left\{\varrho_b\left(\frac{1}{2},\left[\frac{3}{5},\frac{3}{2}\right]\right),\varrho_b\left(2,\left[\frac{3}{5},\frac{3}{2}\right]\right)\right\}\\ &= \max\left\{1.01,1.25\right\}\\ &= 1.25. \end{aligned}\tag{21}$$

By applying (20) and (21) in (19) we get

$$\mathcal{H}^+_{\varrho_b}(Tx,Gy) = 1.25.$$

Similarly we calculate the following metric

$$\begin{aligned}\varrho_b(gy,fx) &= \varrho_b(1,1)=1,\\ varrho_b(fx,Gx) &= \varrho_b\left(1,\left[\frac{3}{5},\frac{3}{2}\right]\right)=1.16,\\ \varrho_b(gy,Ty) &= \varrho_b\left(1,\left[\frac{1}{2},2\right]\right)=1.25,\\ \varrho_b(Gx,Ty) &= \varrho_b\left(\left[\frac{3}{5},\frac{3}{2}\right],\left[\frac{1}{2},2\right]\right)=1.25,\\ \varrho_b(gy,Gx) &= \varrho_b\left(1,\left[\frac{3}{5},\frac{3}{2}\right]\right)=1.16,\\ \varrho_b(fx,Ty) &= \varrho_b\left(1,\left[\frac{1}{2},2\right]\right)=1.25.\end{aligned}$$

It follows that,

$$N_1(x,y)=\max\left\{1,\frac{1.16*1.25}{1+1.25},\frac{(1.16)^2+(1.25)^2}{1.16+1.25},\right.$$
$$\left.\frac{1.16*1.25+1.25*1.16}{1.25+1.16}\right\}=1.207,$$

and

$$N_2(x,y)=\min\left\{1.16,1.25,1.16,1.25\right\}=1.16.$$

Therefore, (13) reduces to

$$\begin{aligned}\frac{2\times 0.1\times 1.25}{0.1\times 1.207-1.1035+5\times 1.16}e^{2\times 0.1\times 1.25-[0.1\times 1.207-1.1035+5\times 1.16]} &\le e^{-\tau},\\ \frac{0.25}{4.8172}e^{0.25-4.8172} &\le e^{-1},\\ \frac{0.25}{4.5672}e^{-4.5672} &\le e^{-1},\end{aligned}$$

which is true.

Case II For $x\in[1,2]$, $y\in[0,1]$ and using Definition 2.14, we have

$$\begin{aligned}\mathcal{H}^+_{\varrho_b}(Gx,Ty) &= \mathcal{H}^+_{\varrho_b}\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)\\ &= \frac{1}{2}\left\{\sup\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)+\sup\left(\left[\frac{3}{2},2\right],\left[\frac{1}{4},\frac{1}{2}\right]\right)\right\}.\end{aligned}\tag{22}$$

$$\begin{aligned}\sup\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right) &= \max\left\{\varrho_b\left(\frac{1}{4},\left[\frac{3}{2},2\right]\right),\varrho_b\left(\frac{1}{2},\left[\frac{3}{2},2\right]\right)\right\}\\ &= \max\left\{2.5625,2\right\}\\ &= 2.5625. \end{aligned}\tag{23}$$

$$\begin{aligned}\sup\left(\left[\frac{3}{2},2\right],\left[\frac{1}{4},\frac{1}{2}\right]\right) &= \max\left\{\varrho_b\left(\frac{3}{2},\left[\frac{1}{4},\frac{1}{2}\right]\right),\varrho_b\left(2,\left[\frac{1}{4},\frac{1}{2}\right]\right)\right\}\\ &= \max\left\{2,3.25\right\}\\ &= 3.25. \end{aligned}\tag{24}$$

By applying (23) and (24) in (22) we get

$$\mathcal{H}^{+}_{\varrho_b}(Gx,Ty) = 2.90625,$$

Similarly we calculate the following metric

$$\begin{aligned}\varrho_b(gy,fx) &= \varrho_b\left(0,\frac{3}{5}\right)=1.36,\\ \varrho_b(fx,Gx) &= \varrho_b\left(\frac{3}{5},\left[\frac{1}{4},\frac{1}{2}\right]\right)=1.01,\\ \varrho_b(gy,Ty) &= \varrho_b\left(0,\left[\frac{3}{2},2\right]\right)=3.25,\\ \varrho_b(Gx,Ty) &= \varrho_b\left(\left[\frac{1}{4},\frac{1}{2}\right],\left[\frac{3}{2},2\right]\right)=2.90625,\\ \varrho_b(gy,Gx) &= \varrho_b\left(0,\left[\frac{1}{4},\frac{1}{2}\right]\right)=1.0625,\\ \varrho_b(fx,Ty) &= \varrho_b\left(\frac{6}{5},\left[\frac{3}{2},2\right]\right)=1.09.\end{aligned}$$

It follows that,

$$N_1(x,y)=\max\left\{1.36,\frac{1.01*3.25}{1+2.90625},\frac{(1.0625)^2+(1.09)^2}{1.0625+1.09},\right.\\ \left.\frac{1.01*1.09+3.25*1.0625}{1.09+1.0625}\right\}=2.115691057,$$

and

$$N_2(x,y)=\min\left\{1.01,1.09,1.0625,3.25\right\}=1.01.$$

Therefore, (13) reduces to

$$\frac{0.58125}{3.703723516}e^{0.58125-3.703723516}\leq e^{-\tau}.\\ \frac{0.58125}{3.703723516}e^{-3.122473516}\leq e^{-1}.$$

which is true.

Notice that for $x,y\in[0,1]$ and $x,y\in[1,2]$, Equation (13) is true. Thus, all conditions of Theorem 3.3 are satisfied, and the hybrid pairs (G,f) and (T,g) has the common fixed point in M. Consider $v=1$ be a coincidence point of the pair (G,f), then we have

(1) $f1=1\in G1=[\frac{3}{5},\frac{3}{2}]$,

(2) $ff1 = f1 = 1$,

(3) $f1 = ff1 \in f(G1) \subset G(f1)$ and

Similarly, if we consider $u = 1$ as a coincidence point of the pair (T, g), prove that $u = v = 1$ and 1 is a unique common fixed point for the two pairs of hybrid mappings (G, f) and (T, g).

4. Some Applications

In this section, we will discuss an approximation of a non-linear hybrid ordinary differential equation. Dhage [14] named it as a hybrid differential equation with a linear perturbation of first type (HDE), which will validate Theorem 3.3 for two pairs of hybrid mapping in weak partial b-metric space.

First, we will define some essential notions which will be useful in developing our results. One can see in [32] and the reference therein.

Assume that $\mathcal{J} = [t_0, t_0 + a]$ of a real line $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $t_0 \geq 0$, $a > 0$ be given.

Consider in the function space $C(\mathcal{J}, \mathbb{R})$ of continuous real valued functions defined on $\mathcal{J}$. Let us define a norm $\|.\|$ and order relation $\leq$ in $C(\mathcal{J}, \mathbb{R})$ by

$$\|x\| = \sup_{t \in \mathcal{J}} |x(t)|,$$

$x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in \mathcal{J}$. Then, we see that $C(\mathcal{J}, \mathbb{R})$ is a Banach space with respect to the partial order relation $\leq$.

The Hybrid differential equations have been investigated in different dimensions by several researchers one can see, [11, 14, 25] and the references therein.

Consider the initial value problem (IVP) of first order ordinary non-linear differential equation (HDE).

$$\begin{cases} x'(t) = f(t, x(t)) + g(t, x(t)), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{25}$$

for all $t \in \mathcal{J}$, where $f, g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Also, Consider (IVP) of (HDE).

$$\begin{cases} x'(t) + \lambda x(t) = \mu e^{-\lambda t} p(t, x(t)) + \tilde{f}(t, x(t)) + \tilde{g}(t, x(t)), \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{26}$$

for all $t \in \mathcal{J}$, where $\tilde{f}, \tilde{g} : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and

$$\begin{aligned} \tilde{f}(t, x) &= f(t, x) + \lambda x, \\ \tilde{g}(t, x) &= g(t, x) - \mu e^{-\lambda t} p(t, x), \end{aligned}$$

$\lambda \geq 0$ with $\mu \leq \frac{\lambda}{1 - e^{-a}}$.

Pathak [32] proved the following Lemma to satisfy HDE:

Lemma 4.1. [32] *A function $u \in C(\mathcal{J}, \mathbb{R})$ is a solution of HDE (25) if and only if it is a solution of a non-linear integral equation*

$$x(t) = x_0 \, e^{-\lambda(t - t_0)} + \mu e^{-\lambda t} \int_{t_0}^{t} p(s, x(s)) ds + e^{-\lambda t} \int_{t_0}^{t} e^{\lambda s} [\tilde{f}(s, x(s)) + \tilde{g}(s, x(s))] ds. \tag{27}$$

for all $t \in \mathcal{J}$.

By Lemma 4.1, the HDE (25) is equivalent to the operator equation

$$x(t) \quad = \quad Px(t) + Qx(t). \tag{28}$$

for all $t \in \mathcal{J}$, where

$$Px(t) \quad = \quad x_0 e^{-\lambda(t-t_0)} + \mu e^{-\lambda t} \int_{t_0}^{t} p(s, x(s))ds, \tag{29}$$

$$Qx(t) \quad = \quad e^{-\lambda(t)} \int_{t_0}^{t} e^{\lambda s}[\tilde{f}(s, x(s)) + \tilde{g}(s, x(s))]ds. \tag{30}$$

for all $t \in \mathcal{J}$.

Definition 4.2. [15] *An operator $T : E \to E$ is partially non-linear $\mathcal{D}$-contraction if there exists a $\mathcal{D}$-function ψ such that*

$$\|Tx - Ty\| \leq \psi(\|x - y\|),$$

for all comparable elements $x, y \in E$, where $0 < \psi(t) < t$ for $t > 0$.

From the continuity of integral, it follows that P and Q defines the maps $P, Q : E \to E$. The following applicable hybrid fixed point theorem proved in [14].

Theorem 4.3. [14] *Let $(E, \preceq, \|.\|)$ be a regular partial ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|.\|$ in E are compatible. Let $P, Q : E \to E$ be two nondencreasing operators such that*

(*i*) *P is partially bounded and partially non-linear $\mathfrak{D}$-contraction,*

(*ii*) *Q is partially Continuous and partially compact, and*

(*iii*) *there exists an element $x_0 \in E$ such that*

$$x(t) \quad \preceq \quad Px(t) + Qx(t).$$

Then the operator equation $x \preceq Px + Qx$ has a solution x^* in E and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations defined by

$$x_{n+1} \quad = \quad Px_n + Qx_n, \; n = 0, 1, 2 \ldots,$$

converge monotonically to x^*.

Consider in the function space $C(\mathcal{J}, \mathbb{R})$ of continuous real valued functions defined on $\mathcal{J}$. Let us define a norm $\|.\|$ of weak partial b-metric on M by

$$e_b(x, y) = \sup_{t \in \mathcal{J}} |x(t) - y(t)|^p + \alpha, \tag{31}$$

$\forall x, y \in C(\mathcal{J}, \mathcal{R}), p > 1$ and $\alpha > 0$.

We rewrite the integral equation (27) in the form of a fixed point problem

$$x(t) = Tx(t).$$

For a map T defined by

$$Tx(t) \quad = \quad x_0(t) + \int_{t_0}^{t} K(s, x(s))ds, \; t \in [\mathcal{J}, \mathbb{R}], \tag{32}$$

with

$$x_0(t) \;=\; x_0 e^{-\lambda(t-t_0)},$$

and

$$K(s,x(s)) \;=\; \mu e^{-\lambda t} p(s,x(s)) + e^{\lambda(s-t)}[\tilde{f}(s,x(s)) + \tilde{g}(s,x(s))].$$

Our main results of this section are as follows.

Theorem 4.4. *Let $(M, \preceq, \|.\|)$ be a weak partial b- ordered complete normed linear space such that the order relation $\preceq$ and the norm $\|.\|$ in M are coincidentally idempotent. Let $f, g : X \to M$ and $P, Q : X \to CB^{\varrho_b}(M)$ be two hybrid pairs of non-decreasing operators such that*

(i) *for any $x(t), y(t) \in C(\mathcal{J}, \mathbb{R})$ there exists a $\mathcal{D}$-contraction function that satisfy*

$$\|Tx(t) - Ty(t)\| \leq (\psi(t))^p \|x(t) - y(t)\|^p + \alpha. \tag{33}$$

where $0 \leq \psi(t) < 1$. Then Equation (27) has a fixed point $x \in M$.

Proof. Using equation (31) and (32) in (33) we obtain

$$\begin{aligned}
\|Tx(t) - Ty(t)\| &= \sup_{t\in\mathcal{J}} \left| \int_{t_0}^{t} [K(s,x(s)) - K(s,y(s))] ds \right|^p + \alpha, \\
&\leq \sup_{t\in\mathcal{J}} \left[\Big(\int_{t_0}^{t} ds \Big)^{\frac{1}{q}} \left(\int_{t_0}^{t} |K(s,x(s)) - K(s,y(s))|^p ds \right)^{\frac{1}{p}} \right]^p + \alpha, \\
&\leq \sup_{t\in\mathcal{J}} \Big(t - t_0\Big)^{\frac{p}{q}} \left(\int_{t_0}^{t} |K(s,x(s)) - K(s,y(s))|^p ds \right) + \alpha, \\
&\leq \sup_{t\in\mathcal{J}} \Big(t - t_0\Big)^{p-1} \left(\int_{t_0}^{t} \psi(t)^p |x(t) - y(t)|^p ds \right) + \alpha, \\
&\leq \Big(t - t_0\Big)^{p-1} (t - t_0) \Big(\psi(t)^p |x(t) - y(t)|^p \Big) + \alpha, \\
&\leq \Big(t - t_0\Big)^{p} \Big(\psi(t)^p |x(t) - y(t)|^p \Big) + \alpha, \\
&\leq \Big(\Big(t - t_0\Big) \psi(t) \Big)^p |x(t) - y(t)|^p + \alpha, \\
&= (\psi(t))^p |x(t) - y(t)|^p + \alpha.
\end{aligned}$$

Hence, the condition of hybrid differential equation (25) is satisfied and so Equation (27) has a solution. Therefore, the condition of Theorem (3.3) validated for two pairs of hybrid mappings which are coincidentally idempotent. □

5. Conclusion

The main contribution of this study to fixed point theory is the coincidence result given in Theorem 2.1. This theorem provides the coincidence conditions for a substantial class of non-self mappings on various abstract spaces. This paper, Motivated by the results obtained by Aserkar and Gandhi [3] in metric space.

We proved a fixed point theorem for common fixed point for two hybrid pairs of coincidentally idempotent non-self mappings in weakly partial b-metric space, which satisfies joint common limit range property in a generalized (F, ξ, η)-contraction, which generalizes some well-known results in the literature. These results have some applications in many areas of applied mathematics, especially in hybrid differential equations.

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