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The existence and Ulam-Hyers stability results for generalized Hilfer fractional integro-differential equations with nonlocal integral boundary conditions

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Abstract

In this paper, we study the existence and uniqueness of mild solutions for nonlinear fractional integro-differential equations (FIDEs) subject to nonlocal integral boundary conditions (nonlocal IBC) in the frame of a ξ-Hilfer fractional derivative (FDs). Further, we discuss different kinds of stability of Ulam-Hyers (UH) for mild solutions to the given problem. Using the fixed point theorems (FPT's) together with generalized Gronwall inequality the desired outcomes are proven. Examples are given which illustrate the effectiveness of the theoretical results.

Keywords: ξ-Hilfer fractional integro-differential equation, Existence, Uniqueness, Ulam-Hyers stability, Fixed point theorems.
2010 MSC: 34A08, 34A12, 34B15, 47H10.

1. Introduction

In latest years, fractional differential equations (FDEs) theory has received very broad regard in the fields of pure and applied mathematics, see [5, 6, 11, 14, 15, 16, 20] and emerge naturally in diverse scopes of science, with many applications, e.g. [9, 10, 13, 18, 29].

In 1999 Hilfer introduced the generalization of Riemann-Liouville and Caputo (FDs) see [13]. The fundamental work on the theory of FDEs with Hilfer derivative can be found in [12]. The Boundary value

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problem (BVP) for FIDEs involving Hilfer derivative has been researched in [3, 4, 28]. Sousa and Oliveira in [26] presented the so-called Ψ-Hilfer FD with respect to another function, to unify in one fractional operator a large number of FD's and thus, window open to new applications.

One of the crucial and interesting areas of research in the theory of functional equations is devoted to the stability analysis. Stability analysis is the fundamental property of the mathematical analysis which has got paramount importance in many fields of engineering and science. In the existing literature, there are stabilities such as Mittag Leffler, h-stability, exponential, Lyapunov stability and so on. In the nineteenth-century, Ulam and Hyers presented an interesting type of stability called Ulam-Hyers stability, which, nowadays has been picked up a great deal of consideration due to a wide range of applications in many fields of science such as optimization and mathematical modeling.

The stability of the Ulam can be viewed as a special kind of data dependence which was initiated by the Ulam in [22]. Rassias in [21] extended the concept of UH stability. Many authors subsequently discussed different UH stability problem for various types of fractional integral (FI), FDEs, fuzzy dynamic and fuzzy difference equations utilizing various techniques, see [1, 2, 3, 4, 8, 17, 19, 24, 27, 31, 32] and the references therein.

Recently, the authors in [28] investigated the stability of Ulam-Hyers-Rassias (UHR) and UH of the following FIDE

$$\begin{cases} {}^{H}D_{0+}^{r_1,r_2;\psi}\varkappa(\mathfrak{t}) = f(\mathfrak{t},\varkappa(\mathfrak{t})) + \int_0^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi, \ \mathfrak{t}\in[0,T], \\ I_{0+}^{1-\mathfrak{v};\psi}\varkappa(0) = c, \end{cases}$$

where ${}^{H}D_{0+}^{r_1,r_2;\psi}$ is the generalized ψ-Hilfer FD, and $r_1\in(0,1)$ and type $r_2\in[0,1]$, $I_{0+}^{1-\mathfrak{v};\psi}$ is the ψ-Riemann-Liouville FI of order $\mathfrak{v}\in[0,1)$.

In [3] Abdo and Panchal discussed existence, uniqueness, and UH stability of the following nonlinear FIDE

$$\begin{cases} {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\Psi}\varkappa(\mathfrak{t}) = f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_0^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right), \ \mathfrak{t}>\mathfrak{a}, \\ I_{\mathfrak{a}+}^{1-\mathfrak{v};\Psi}\varkappa(0) = \varkappa_{\mathfrak{a}}, \ \mathfrak{v} = r_1+r_2-r_1r_2, \end{cases}$$

where $r_1\in(0,1)$ and $r_2\in[0,1]$, f and h are given continuous functions.

In [7], Asawasamrit et al. have started the study of Hilfer FDEs with nonlocal IBCs of the type

$$\begin{cases} {}^{H}D^{r_1,r_2}\varkappa(\mathfrak{t}) = f(\mathfrak{t},\varkappa(\mathfrak{t})), \ \mathfrak{t}\in[\mathfrak{a},b], \\ \varkappa(\mathfrak{a}) = 0, \ \varkappa(b) = \sum\limits_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i} x(\delta_i), \ \delta_i\in[\mathfrak{a},b], \end{cases} \tag{1}$$

where $1<r_1<2$, $0\le r_2\le 1$, $\eta_i>0$, $\theta_i\in\mathbb{R}$, ${}^{H}D^{r_1,r_2}$ is the Hilfer FD of order r_1 and type r_2, $I_{\mathfrak{a}+}^{\eta_i}$ is the Riemann-Liouville FI of order η_i. The authors in [19] have investigated the existence and stability results of implicit problem for FDEs (1) involving ψ-Hilfer FD.

Motivated by the aforementioned works, we study existence, uniqueness and Ulam stability of the following FIDE involving ξ-Hilfer FD with nonlocal IBCs

$$\begin{cases} {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\varkappa(\mathfrak{t}) = f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right), \ \mathfrak{t}\in(\mathfrak{a},b), \\ \varkappa(\mathfrak{a}) = 0, \ I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\varkappa(b) = \sum\limits_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi}\varkappa(\delta_i), \end{cases} \tag{2}$$

where ${}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}$ is the ξ-Hilfer FD of order $r_1\in(1,2)$ and type $r_2\in[0,1]$, $I^{2-\mathfrak{v};\xi}$ and $I^{\eta_i;\xi}$ are the ξ-RL FI of orders $2-\mathfrak{v}$, $\eta_i>0$ respectively, $\mathfrak{v}=r_1+r_2(2-r_1)\in(1,2)$, $\infty<\mathfrak{a}<b<\infty$, $\theta_i\in\mathbb{R}$, $i=1,2,...,m$, $0\le\mathfrak{a}\le\delta_1<\delta_2<\delta_3<...<\delta_m\le b$, $f:[\mathfrak{a},b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h:[\mathfrak{a},b]\times[\mathfrak{a},b]\times\mathbb{R}\to\mathbb{R}$ are given continuous functions.

Remark 1.1. ***i)*** *The FIDE (2) involving ξ-Hilfer FD is the more wide category of BVPs that combine the FDI involving Riemann-Liouville FD (for $r_2=0$, $\xi(\mathfrak{t})=\mathfrak{t}$) and Hadamard FD (for $r_2=0$, $\xi(\mathfrak{t})=\log\mathfrak{t}$).*

ii) *For various of r_2 and ξ, our problem reduce to FIDEs involving the FDs like Hilfer, Katugampola, Erdély-Kober, and many other FDs.*

The organization of the rest of the paper divided of four sections. In Section 2, some notations, definitions of fractional calculus (FC) and FPT's are presented. In Section 3, Some useful results about the existence, uniqueness and Ulam stability of nonlinear FIDE are obtained. In Section 4, we present some particular cases of the nonlinear FIDE which illustrates the effectiveness of the theoretical results.

2. Preliminaries

In this part, we give some essential ideas of FC, definitions of various types of Ulam stability and results of nonlinear analysis (FPT's and generalized Gronwall's inequality) that prerequisite in our analysis.

Let $J = [\mathfrak{a}, b]$, $r_1 \in (1,2)$, $r_2 \in [0,1]$. By $\mathcal{C} = C(J,\mathbb{R})$ we denote the Banach space of all continuous functions $\varkappa : J \to \mathbb{R}$ with norm

$$\|\varkappa\| = \sup\{|\varkappa(\mathfrak{t})| : \mathfrak{t} \in J\},$$

and $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue integrable functions $\varkappa : J \to \mathbb{R}$ with norm

$$\|\varkappa\|_{L^1} = \int_J |\varkappa(\mathfrak{t})|\, d\mathfrak{t}.$$

Let $\varkappa : J \to \mathbb{R}$ be an integrable function and $\xi \in \mathcal{C}^n(J,\mathbb{R})$ an increasing function such that $\xi'(\mathfrak{t}) \neq 0$, for any $\mathfrak{t} \in J$.

Definition 2.1 ([14]). *The ξ-Riemann-Liouville FI of a function $\varkappa$ of order r_1 is described by*

$$I_{\mathfrak{a}+}^{r_1;\xi}\varkappa(\mathfrak{t}) = \frac{1}{\Gamma(\alpha)}\int_{\mathfrak{a}}^{\mathfrak{t}} \xi'(\varsigma)\,(\xi(\mathfrak{t}) - \xi(\varsigma))^{r_1-1}\,\varkappa(\varsigma)\,d\varsigma.$$

Definition 2.2 ([14]). *The ξ-Riemann-Liouville FD of a function h of order r_1 is described by*

$$D_{\mathfrak{a}+}^{r_1;\xi}\varkappa(\mathfrak{t}) = \left(\frac{1}{\xi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^n I_{\mathfrak{a}+}^{(n-r_1);\xi}\varkappa(\mathfrak{t}),$$

where $n = [r_1] + 1$, $n \in \mathbb{N}$.

Definition 2.3 ([26]). *The ξ-Hilfer FD of a function $\varkappa$ of order r_1 and type r_2 is described by*

$${}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\varkappa(\mathfrak{t}) = I_{\mathfrak{a}+}^{r_2(n-r_1);\xi}\; D_{\xi}^{[n]}\; I_{\mathfrak{a}+}^{(1-r_2)(n-r_1);\xi}\varkappa(\mathfrak{t}),$$

where $D_{\xi}^{[n]} = \left(\frac{1}{\xi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^n$.

Lemma 2.4 ([14, 26]). *Let $r_1, r_2, \mu > 0$. Then*

1) $I_{\mathfrak{a}+}^{r_1;\xi} I_{\mathfrak{a}+}^{r_2;\xi}\varkappa(\mathfrak{t}) = I_{\mathfrak{a}+}^{r_1+r_2;\xi}\varkappa(\mathfrak{t})$.

2) $I_{\mathfrak{a}+}^{r_1;\xi}(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mu-1} = \frac{\Gamma(\mu)}{\Gamma(r_1+\mu)}(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{r_1+\mu-1}$.

Lemma 2.5 ([26]). *If $\varkappa \in C^n(J,\mathbb{R})$, $r_1 \in (n-1,n)$ and $r_2 \in (0,1)$, then*

1) $I_{\mathfrak{a}+}^{r_1;\xi}\;{}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\;\varkappa(\mathfrak{t}) = \varkappa(\mathfrak{t}) - \sum\limits_{k=1}^{n} \frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-k}}{\Gamma(\mathfrak{v}-k+1)}\left(\frac{1}{\xi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^{n-k} I_{\mathfrak{a}+}^{(1-r_2)(n-r_1);\xi}\varkappa(\mathfrak{a})$.

2) ${}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\; I_{\mathfrak{a}+}^{r_1;\xi}\varkappa(\mathfrak{t}) = \varkappa(\mathfrak{t})$.

To define Ulam's stability, we consider the following FIDE

$${}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\varkappa(\mathfrak{t}) = f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right),\ \mathfrak{t} \in J. \tag{3}$$

Definition 2.6 ([23]). *The equation (Eq) (3) is said to be UH stable if there is a number $k \in \mathbb{R}^*$ such that for each $\epsilon > 0$ and for each $\tilde{\varkappa} \in \mathcal{C}$ solution of the inequality*

$$\left| {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi} \tilde{\varkappa}(\mathrm{t}) - f\left(\mathrm{t}, \tilde{\varkappa}(\mathrm{t}), \int_{\mathfrak{a}}^{\mathrm{t}} \mathrm{h}(\mathrm{t}, \varphi, \tilde{\varkappa}(\varphi)) \, d\varphi\right) \right| \leq \epsilon, \ \mathrm{t} \in J, \tag{4}$$

there is a solution $\varkappa \in \mathcal{C}$ of the Eq (3) with

$$|\tilde{\varkappa}(\mathrm{t}) - \varkappa(\mathrm{t})| \leq k_f \epsilon, \ \mathrm{t} \in J.$$

Definition 2.7 ([23]). *Assume that $\tilde{\varkappa} \in \mathcal{C}$ satisfies the inequality in (4) and $\varkappa \in \mathcal{C}$ is a solution of the Eq (3). If there is a function $\phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\phi_f(0) = 0$ satisfying*

$$|\tilde{\varkappa}(\mathrm{t}) - \varkappa(\mathrm{t})| \leq \phi_f(\epsilon), \ \mathrm{t} \in J.$$

Then the Eq (3) is said to be generalized Ulam-Hyres (GUH) stable.

Definition 2.8 ([23]). *The Eq (3) is said to be UHR stable with respect to $\phi_f \in C(J, \mathbb{R}^+)$ if there is a number $k \in \mathbb{R}^*$ such that for each $\epsilon > 0$ and for each $\tilde{\varkappa} \in \mathcal{C}$ solution of the inequality*

$$\left| {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi} \tilde{\varkappa}(\mathrm{t}) - f\left(\mathrm{t}, \tilde{\varkappa}(\mathrm{t}), \int_{\mathfrak{a}}^{\mathrm{t}} \mathrm{h}(\mathrm{t}, \varphi, \tilde{\varkappa}(\varphi)) \, d\varphi\right) \right| \leq \epsilon \phi_f(\mathrm{t}), \ \mathrm{t} \in [0,1], \tag{5}$$

there is a solution $\varkappa \in \mathcal{C}$ of the Eq (3) with

$$|\tilde{\varkappa}(\mathrm{t}) - \varkappa(\mathrm{t})| \leq k_{\phi,f} \phi_f(\mathrm{t}) \epsilon, \ \mathrm{t} \in J.$$

Definition 2.9 ([23]). *Assume that $\tilde{\varkappa} \in \mathcal{C}$ satisfies the inequality in (5) and $\varkappa \in \mathcal{C}$ is a solution of the Eq (3). If there is a constant $k_{\phi,f} > 0$ such that*

$$|\tilde{\varkappa}(\mathrm{t}) - \varkappa(\mathrm{t})| \leq k_{\phi,f} \phi_f(\mathrm{t}), \ \mathrm{t} \in J.$$

Then the Eq (3) is said to be generalized Ulam-Hyres-Rassias (GUHR) stable.

Remark 2.10. *If there is a function $v \in \mathcal{C}$ (dependent on $\tilde{\varkappa}$), such that*
1) $|v(\mathrm{t})| \leq \epsilon$, for all $\mathrm{t} \in J$,
2) ${}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi} \tilde{\varkappa}(\mathrm{t}) = f\left(\mathrm{t}, \tilde{\varkappa}(\mathrm{t}), \int_{\mathfrak{a}}^{\mathrm{t}} \mathrm{h}(\mathrm{t}, \varphi, \tilde{\varkappa}(\varphi)) \, d\varphi\right) + v(\mathrm{t})$, $\mathrm{t} \in J$.
Then the function $\tilde{\varkappa} \in \mathcal{C}$ is a solution of the inequality (4).

We state the following generalization of Gronwall's Lemma.

Lemma 2.11 ([30]). *Let u and v be two integrable functions, z be continuous with domain $[\mathfrak{a}, b]$ and ξ is defined at the beginning. Suppose that*
1) u and v are nonnegative,
2) z is nonnegative and nondecreasing.
If

$$u(\mathrm{t}) \leq v(\mathrm{t}) + z(\mathrm{t}) \int_{\mathfrak{a}}^{\mathrm{t}} \xi'(\varsigma) (\xi(\mathrm{t}) - \xi(\varsigma))^{r_1 - 1} u(\varsigma) \, d\varsigma,$$

then

$$u(\mathrm{t}) \leq v(\mathrm{t}) + \int_{\mathfrak{a}}^{\mathrm{t}} \sum_{k=1}^{\infty} \frac{[z(\mathrm{t}) \Gamma(r_1)]^k}{\Gamma(kr_2)} \xi'(\varsigma) (\xi(\mathrm{t}) - \xi(\varsigma))^{kr_1 - 1} v(\varsigma) \, d\varsigma.$$

Corollary 2.12 ([30]). *Under the hypotheses of Lemma 2.11, assume further that $v(\mathfrak{t})$ is nondecreasing function for $\mathfrak{t} \in [\mathfrak{a}, b]$. Then*

$$u(\mathfrak{t}) \leq v(\mathfrak{t}) E_{r_1}\left(z(\mathfrak{t}) \Gamma(r_1) (\xi(\mathfrak{t}) - \xi(\varsigma))^{r_1}\right),$$

where $E_{r_1}(.)$ is the Mittag-Leffler function of one parameter, defined as

$$E_{r_1}(\mathfrak{t}) = \sum_{k=0}^{\infty} \frac{\mathfrak{t}^k}{\Gamma(kr_1 + 1)}.$$

Theorem 2.13 (Banach FPT [25]). *Let $\Omega \neq \emptyset$ be a closed subset of a Banach space $(\mathcal{X}, \|.\|)$. If $\tilde{\mathcal{S}} : \Omega \to \Omega$ is a contraction mapping. Then, $\tilde{\mathcal{S}}$ admits a unique fixed point.*

Theorem 2.14 (Schauder FPT [25]). *Let $\Omega \neq \emptyset$ be a bounded closed convex subset of a Banach space $\mathcal{X}$. If $\tilde{\mathcal{S}} : \Omega \to \Omega$ be a continuous compact operator. Then, $\tilde{\mathcal{S}}$ has a fixed point in Ω.*

To obtain our results, we need the following lemma.

Lemma 2.15. *Let*

$$\varpi = \frac{(\xi(b) - \xi(\mathfrak{a}))}{\Gamma(2)} - \sum_{\mathrm{i}=1}^{\mathfrak{m}} \frac{\theta_i}{\Gamma(\mathfrak{v} + \eta_i)} (\xi(\delta_i) - \xi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} \neq 0, \tag{6}$$

and for any $p \in \mathcal{C}$, then the nonlocal BVP

$$\begin{cases} {}^H D^{r_1,r_2;\xi} \varkappa(\mathfrak{t}) = p(\mathfrak{t}), \ \mathfrak{t} \in (\mathfrak{a}, b), \\ \varkappa(\mathfrak{a}) = 0, \ I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi} \varkappa(b) = \sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi} \varkappa(\delta_i), \end{cases} \tag{7}$$

has a unique mild solution given by

$$\varkappa(\mathfrak{t}) = \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} \left(\sum_{\mathrm{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} p(\delta_i) - I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} p(b) \right) + I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}). \tag{8}$$

Proof. Taking ξ-FI $I_{\mathfrak{a}+}^{r_1;\xi}$ to the first Eq of (7), and from Lemma 2.5, we obtain

$$\varkappa(\mathfrak{t}) - \sum_{k=1}^{2} \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-k}}{\Gamma(\mathfrak{v} - k + 1)} h_\xi^{[2-k]} I_{\mathfrak{a}+}^{(1-r_2)(2-r_1);\xi} \varkappa(\mathfrak{a}) = I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}), \ \mathfrak{t} \in J. \tag{9}$$

We have $(1 - r_2)(2 - r_1) = 2 - \mathfrak{v}$. Therefore

$$\begin{aligned} \varkappa(\mathfrak{t}) &= \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} \left(\frac{1}{\xi'(\mathfrak{t})} \frac{d}{d\mathfrak{t}} \right) I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=\mathfrak{a}} \\ &+ \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-2}}{\Gamma(\mathfrak{v} - 1)} I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=\mathfrak{a}} + I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}) \\ &= \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} D^{\mathfrak{v}-1;\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=\mathfrak{a}} + \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\gamma-2}}{\Gamma(\mathfrak{v} - 1)} I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=a} + I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}). \end{aligned}$$

Put

$$c_1 = D^{\mathfrak{v}-1;\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=\mathfrak{a}} \text{ and } c_2 = I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi} \varkappa(\mathfrak{t}) \Big|_{\mathfrak{t}=\mathfrak{a}}, \ \mathfrak{t} \in J.$$

Then

$$\varkappa(\mathfrak{t}) = \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} c_1 + \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-2}}{\Gamma(\mathfrak{v} - 1)} c_2 + I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}).$$

Because $\lim\limits_{\mathfrak{t}=\mathfrak{a}} (\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-2} = \infty$, in the view of boundary conditions $\varkappa(\mathfrak{a}) = 0$, we must have

$$c_2 = 0.$$

Replacing c_2 by their value in (9), we get

$$\varkappa(\mathfrak{t}) = \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\Gamma(\mathfrak{v})} c_1 + I_{\mathfrak{a}+}^{r_1;\xi} p(\mathfrak{t}). \tag{10}$$

Next, we use the second boundary condition to determine the constant c_1. Applying $I_{\mathfrak{a}+}^{\eta_i;\xi}$ on both side of Eq (10), we get

$$I_{\mathfrak{a}+}^{\eta_i;\xi} \varkappa(\mathfrak{t}) = \frac{c_1}{\Gamma(\mathfrak{v} + \eta_i)} (\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} p(\mathfrak{t}). \tag{11}$$

From the condition $\varkappa(b) = \sum\limits_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi} x(\delta_i)$ and (11), we have

$$\begin{aligned} \varkappa(b) &= \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi} x(\delta_i) \\ &= c_1 \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \frac{\theta_i}{\Gamma(\mathfrak{v} + \eta_i)} (\xi(\delta_i) - \xi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} p(\delta_i). \end{aligned} \tag{12}$$

From Eq's (10) and (12), we have

$$\begin{aligned} I_{a+}^{2-\gamma;\xi} \varkappa(b) &= \frac{(\xi(b) - \xi(\mathfrak{a}))}{\Gamma(2)} c_1 + I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} q(b) \\ &= c_1 \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \frac{\theta_i}{\Gamma(\mathfrak{v} + \eta_i)} (\xi(\delta_i) - \xi(\mathfrak{a}))^{\mathfrak{v}+\eta_i-1} + \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} p(\delta_i). \end{aligned}$$

Thus, we find

$$c_1 = \frac{1}{\varpi} \left(\sum_{i=1}^{m} \theta_i I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} p(\delta_i) - I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} p(b) \right).$$

Replacing the value of c_1 into (10), we obtain (8). □

3. Existence results

In what follows, we apply some FPT's to demonstrate the existence and uniqueness results for problem (2).

To obtain our findings, We need the following assumptions

(As1) There is a constants $l_i > 0$, $i = 1, 2, 3$ such that

$$\begin{aligned} |f(\mathfrak{t}, \varkappa_1, \mathfrak{y}_1) - f(\mathfrak{t}, \varkappa_2, \mathfrak{y}_2)| &\le l_1 |\varkappa_1 - \varkappa_2| + l_2 |\mathfrak{y}_1 - \mathfrak{y}_2|, \\ |h(\mathfrak{t}, \xi, \varkappa_1) - h(\mathfrak{t}, \xi, \varkappa_2)| &\le l_3 |\varkappa_1 - \varkappa_2|, \ \forall (\mathfrak{t}, \xi, \varkappa_j, \mathfrak{y}_j) \in J^2 \times \mathbb{R}^2, \ j = 1, 2. \end{aligned}$$

(As2) There is a function $w \in C(J, \mathbb{R}^+)$ such that

$$|f(\mathfrak{t}, \varkappa, \mathfrak{y})| \le w(\mathfrak{t}), \ \forall (\mathfrak{t}, \varkappa, \mathfrak{y}) \in J \times \mathbb{R} \times \mathbb{R}.$$

For the sake of convenience, we put

$$\begin{aligned}
k_1 &= \sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i| \frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1+\eta_i+\mathfrak{v}-1}}{|\varpi|\,\Gamma(\mathfrak{v})\,\Gamma(r_1+\eta_i+1)}, \\
k_2 &= \frac{(\xi(b)-\xi(\mathfrak{a}))^{1+r_1}}{|\varpi|\,\Gamma(\mathfrak{v})\,\Gamma(3+r_1-\mathfrak{v})}, \\
k_3 &= \frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}, \\
A_{\varkappa} &= \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i|\; I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=\delta_i} \right. \\
&\quad \left. - I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=b} \right). \qquad (13)
\end{aligned}$$

3.1. Existence and uniqueness results via Banach's FPT

Theorem 3.1. *Let (As1) valid. If*

$$(k_1+k_2+k_3)\,(l_1+l_2 l_3\,(b-\mathfrak{a})) < 1, \qquad (14)$$

then, (2) has a unique mild solution on J, where k_1, k_2, k_3 are given by (13).

Proof. We switch the problem (25) into a FP problem, we consider the operator $\tilde{\mathcal{S}} : \mathcal{C} \to \mathcal{C}$ as

$$\begin{aligned}
\left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}) &= \frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})} \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i\; I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=\delta_i} \right. \\
&\quad \left. - I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=b} \right) \\
&\quad + I_{\mathfrak{a}+}^{r_1;\xi} f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\, d\varphi\right).
\end{aligned}$$

Clearly, the solution of (2) is as a FP of the operator $\tilde{\mathcal{S}}$.

By (As1), for any $\varkappa, y \in \mathcal{C}$ and $\mathfrak{t} \in J$, we get

$$\begin{aligned}
&\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}) - \left(\tilde{\mathcal{S}}\mathfrak{y}\right)(\mathfrak{t})\right| \\
&\leq \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})}\left(\sum_{\mathrm{i}=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(r_1+\eta_i)} \int_{\mathfrak{a}}^{\delta_i} \xi'(\varsigma)(\xi(\delta_i) - \xi(\varsigma))^{r_1+\eta_i-1}\right. \\
&\times \left| f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right) - f\left(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \mathfrak{y}(\varphi))\, d\varphi\right)\right| d\varsigma \\
&+ \frac{1}{\Gamma(2+r_1-\mathfrak{v})} \int_{\mathfrak{a}}^{b} \xi'(\varsigma)(\xi(b) - \xi(\varsigma))^{1+r_1-\mathfrak{v}} \\
&\left.\times \left| f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right) - f\left(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \mathfrak{y}(\varphi))\, d\varphi\right)\right| d\varsigma\right) \\
&+ \frac{1}{\Gamma(r_1)} \int_{\mathfrak{a}}^{\tau} \xi'(\varsigma)(\xi(\tau) - \xi(\varsigma))^{r_1-1} \\
&\times \left| f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right) - f\left(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \mathfrak{y}(\varphi))\, d\varphi\right)\right| d\varsigma \\
&\leq \sum_{\mathrm{i}=1}^{\mathfrak{m}} |\theta_i| \frac{(\xi(b) - \xi(\mathfrak{a}))^{r_1+\eta_i+\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})\Gamma(r_1+\eta_i+1)} (l_1 + l_2 l_3 (b-\mathfrak{a})) \|x - y\| \\
&+ \frac{(\xi(b) - \xi(\mathfrak{a}))^{1+r_1}}{|\varpi|\Gamma(\mathfrak{v})\Gamma(3+r_1-\gamma)} (l_1 + l_2 l_3 (b-\mathfrak{a})) \|x - y\| \\
&+ \frac{(\xi(b) - \xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)} (l_1 + l_2 l_3 (b-\mathfrak{a})) \|x - y\| \\
&\leq (k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b-\mathfrak{a})) \|x - y\|.
\end{aligned}$$

Thus

$$\left\|\left(\tilde{\mathcal{S}}\varkappa\right) - \left(\tilde{\mathcal{S}}\mathfrak{y}\right)\right\| \leq (k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b-\mathfrak{a})) \|\varkappa - \mathfrak{y}\|.$$

From (14), $\tilde{\mathcal{S}}$ is a contraction. As an outcome of Banach's FPT, $\tilde{\mathcal{S}}$ has a unique FP which is a unique mild solution of (2) on J. □

3.2. Existence results via Schauder's FPT

Theorem 3.2. *Suppose that the hypotheses (As1)-(As2) are satisfied. Then, (2) has at least one mild solution on J.*

Proof. Let $\Omega = \{\varkappa \in \mathcal{C} : \|\varkappa\| \leq M_0\}$ be a non-empty closed bounded convex subset of $\mathcal{C}$, and M_0 is chosen such

$$M_0 \geq w^* (k_1 + k_2 + k_3),$$

where k_1, k_2, k_3 are given by (13). It is a known that continuity of the functions f and h implies that the operator $\tilde{\mathcal{S}}$ is continuous. It remain to demonstrate that the operator $\tilde{\mathcal{S}}$ is compact and will be given in the following steps.

Step 1. We show that $\tilde{\mathcal{S}}(\Omega) \subset \Omega$.

Let $w^* = \sup\{w(\mathfrak{t}) : \mathfrak{t} \in J\}$. For $\varkappa \in \Omega$, we have

$$\begin{aligned}
\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t})\right| &\leq \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})}\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i\, I_{\mathfrak{a}+}^{r_1+\eta_i;\xi}\left|f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right)\right|\Bigg|_{\mathfrak{t}=\delta_i}\right. \\
&\left. + I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi}\left|f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right)\right|\Bigg|_{\mathfrak{t}=b}\right) \\
&+ I_{\mathfrak{a}+}^{r_1;\xi}\left|f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right)\right| \\
&\leq \sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i| \frac{w^*(\xi(b) - \xi(\mathfrak{a}))^{r_1+\eta_i+\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})\Gamma(r_1+\eta_i+1)} + \frac{w^*(\xi(b) - \xi(\mathfrak{a}))^{1+r_1}}{|\varpi|\Gamma(\mathfrak{v})\Gamma(3+r_1-\mathfrak{v})} \\
&+ \frac{w^*(\xi(b) - \xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)} \\
&\leq w^*(k_1 + k_2 + k_3),
\end{aligned}$$

and consequently

$$\left\|\tilde{\mathcal{S}}\varkappa\right\| \leq M_0,$$

Hence, $\tilde{\mathcal{S}}(\Omega) \subset \Omega$ and the set $\tilde{\mathcal{S}}(\Omega)$ is uniformly bounded.

Step 2. $\tilde{\mathcal{S}}$ sends bounded sets of $\mathcal{C}$ into equicontinuous sets.

For $\mathfrak{t}_1, \mathfrak{t}_2 \in J$, $\mathfrak{t}_1 < \mathfrak{t}_2$ and for $\varkappa \in \Omega$, we have

$$\begin{aligned}
&\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}_2) - \left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}_1)\right| \\
&\leq \frac{(\xi(\mathfrak{t}2) - \xi(\mathfrak{a}))^{\mathfrak{v}-1} - (\xi(\mathfrak{t}_1) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})} \\
&\times \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(r_1+\eta_i)} \int_{\mathfrak{a}}^{\delta_i} \xi'(\varsigma)(\xi(\delta_i) - \xi(\varsigma))^{r_1+\eta_i-1}\left|f\left(\varsigma, \varkappa(\varsigma), \int_{\mathfrak{a}}^{\varsigma} h(\varsigma, \varphi, \varkappa(\varphi))\, d\varphi\right)\right| d\varsigma\right. \\
&\left. + \frac{1}{\Gamma(2+r_1-\mathfrak{v})} \int_{\mathfrak{a}}^{b} \xi'(\varsigma)(\xi(b) - \xi(\varsigma))^{1+r_1-\mathfrak{v}}\left|f\left(\varsigma, \varkappa(\varsigma), \int_{\mathfrak{a}}^{\varsigma} h(\varsigma, \varphi, \varkappa(\varphi))\, d\varphi\right)\right| d\varsigma\right) \\
&+ \frac{1}{\Gamma(r_1)} \int_{a}^{\mathfrak{t}_1} \xi'(\varsigma)\left((\xi(\mathfrak{t}_2) - \xi(\varsigma))^{r_1-1} - (\xi(\mathfrak{t}_1) - \xi(\varsigma))^{r_1-1}\right) \\
&\times \left|f\left(\varsigma, \varkappa(\varsigma), \int_{\mathfrak{a}}^{\varsigma} h(\varsigma, \varphi, \varkappa(\varphi))\, d\varphi\right)\right| d\varsigma + \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \xi'(\varsigma)(\xi(\tau_2) - \xi(\varsigma))^{r_1-1} \\
&\times \left|f\left(\varsigma, \varkappa(\varsigma), \int_{\mathfrak{a}}^{\varsigma} h(\varsigma, \varphi, \varkappa(\varphi))\, d\varphi\right)\right| d\varsigma \\
&\leq \frac{\left((\xi(\mathfrak{t}_2) - \xi(\mathfrak{a}))^{\mathfrak{v}-1} - (\xi(\mathfrak{t}_1) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}\right) w^*}{|\varpi|\Gamma(\mathfrak{v})} \\
&\times \left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}} \frac{|\theta_i|}{\Gamma(r_1+\eta_i)} \int_{\mathfrak{a}}^{\delta_i} \xi'(\varsigma)(\xi(\delta_i) - \xi(\varsigma))^{r_1+\eta_i-1}\, d\varsigma\right. \\
&\left. + \frac{1}{\Gamma(2+r_1-\mathfrak{v})} \int_{\mathfrak{a}}^{b} \xi'(\varsigma)(\xi(b) - \xi(\varsigma))^{1+r_1-\mathfrak{v}}\, d\varsigma\right) \\
&+ \frac{w^*}{\Gamma(r_1+1)}\left((\xi(\mathfrak{t}_2) - \xi(\mathfrak{a}))^{r_1} - (\xi(\mathfrak{t}_1) - \xi(\mathfrak{a}))^{r_1}\right).
\end{aligned}$$

As $\mathfrak{t}_1 \to \mathfrak{t}_2$, we obtain

$$\left|\left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}_2) - \left(\tilde{\mathcal{S}}\varkappa\right)(\mathfrak{t}_1)\right| \to 0.$$

Hence $\tilde{\mathcal{S}}(B_r)$ is equicontinuous. The Arzela-Ascoli theorem implies that $\tilde{\mathcal{S}}$ is compact. Thus by Schauder FPT, we prove that $\tilde{\mathcal{S}}$ has at least one FP $\varkappa \in \Omega$ that is in fact a mild solution of (2) on J. □

4. Ulam stability results

In this portion, we discuss the various types of Ulam stability for the ξ-Hilfer problem (2).

Theorem 4.1. *Suppose that the hypothesis (As1) and condition (14) are satisfied. Then, the first Eq of (2) is UH stable.*

Proof. Let $\epsilon > 0$. Let $\tilde{\varkappa} \in \mathcal{C}$ be any solution of the inequality

$$\left| {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\tilde{\varkappa}(\mathfrak{t}) - f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right)\right| \leq \epsilon,\ \ \mathfrak{t} \in J.$$

Then, there exists $v \in \mathcal{C}$ such that

$$ {}^{H}D_{\mathfrak{a}+}^{r_1,r_2;\xi}\tilde{\varkappa}(\mathfrak{t}) = f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right) + v(\mathfrak{t}),\ \ \mathfrak{t} \in J, \tag{15}$$

and $|v(\mathfrak{t})| \leq \epsilon$, $\mathfrak{t} \in J$. In view of Lemma 2.15, we get

$$\tilde{\varkappa}(\mathfrak{t}) = \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})} A_{\tilde{\varkappa}} + I_{\mathfrak{a}+}^{r_1;\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right) + I_{\mathfrak{a}+}^{r_1;\xi} v(\mathfrak{t}), \tag{16}$$

is mild solution of Eq (15), where

$$\begin{aligned} A_{\tilde{\varkappa}} = \Bigg(& \sum_{\mathfrak{i}=1}^{\mathfrak{m}} |\theta_i|\, I_{\mathfrak{a}+}^{r_1+\eta_i;\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=\delta_i} \\ & - I_{\mathfrak{a}+}^{2+r_1-\mathfrak{v};\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right)\Bigg|_{\mathfrak{t}=b}\Bigg). \end{aligned} \tag{17}$$

From Eq (16), we have

$$\begin{aligned} & \left|\tilde{\varkappa}(\mathfrak{t}) - \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})} A_{\tilde{\varkappa}} - I_{\mathfrak{a}+}^{r_1;\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right)\right| \\ & \leq I_{\mathfrak{a}+}^{r_1;\xi} |v(\mathfrak{t})| \leq \epsilon \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}. \end{aligned} \tag{18}$$

Let $\tilde{\varkappa} \in \mathcal{C}$ be mild solution of the problem

$$\begin{cases} {}^{H}D_{a+}^{r_1,r_2;\xi}\tilde{\varkappa}(\mathfrak{t}) = f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right), \\ \varkappa(\mathfrak{a}) = \tilde{\varkappa}(\mathfrak{a}),\ I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\varkappa(b) = I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\tilde{\varkappa}(b), \end{cases} \tag{19}$$

where $I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\varkappa(b) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi}\varkappa(\delta_i)$ and $I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\tilde{\varkappa}(b) = \sum_{\mathfrak{i}=1}^{\mathfrak{m}} \theta_i I_{\mathfrak{a}+}^{\eta_i;\xi}\tilde{\varkappa}(\delta_i)$. By Lemma 2.15, the equivalent FIE of (19) is

$$\tilde{\varkappa}(\mathfrak{t}) = \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi\Gamma(\mathfrak{v})} A_{\tilde{\varkappa}} + I_{\mathfrak{a}+}^{r_1;\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\, d\varphi\right),$$

where $A_{\tilde{\varkappa}}$ is given by (17).

Now, by using the assumption (As1), we obtain

$$
\begin{aligned}
&|A_{\varkappa}-A_{\tilde{\varkappa}}| \\
&\leq \frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})}\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\frac{|\theta_i|}{\Gamma(r_1+\eta_i)}\int_{\mathfrak{a}}^{\delta_i}\xi'(\varsigma)(\xi(\delta_i)-\xi(\varsigma))^{r_1+\eta_i-1}\right. \\
&\times\left|f\left(\varsigma,\varkappa(\varsigma),\int_{\mathfrak{a}}^{\varsigma}h(\varsigma,\varphi,\varkappa(\varphi))\,d\varphi\right)-f\left(\varsigma,\tilde{\varkappa}(\varsigma),\int_{\mathfrak{a}}^{\varsigma}h(\varsigma,\varphi,\tilde{\varkappa}(\varphi))\,d\varphi\right)\right|d\varsigma \\
&+\frac{1}{\Gamma(2+r_1-\mathfrak{v})}\int_{\mathfrak{a}}^{b}\xi'(\varsigma)(\xi(b)-\xi(\varsigma))^{1+r_1-\mathfrak{v}} \\
&\times\left.\left|f\left(\varsigma,\varkappa(\varsigma),\int_{\mathfrak{a}}^{\varsigma}h(\varsigma,\varphi,\varkappa(\varphi))\,d\varphi\right)-f\left(\varsigma,\tilde{\varkappa}(\varsigma),\int_{\mathfrak{a}}^{\varsigma}h(\varsigma,\varphi,\tilde{\varkappa}(\varphi))\,d\varphi\right)\right|d\varsigma\right) \\
&\leq \frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}}{|\varpi|\Gamma(\mathfrak{v})}\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}\frac{|\theta_i|(l_1+l_2l_3)}{\Gamma(r_1+\eta_i)}\int_{\mathfrak{a}}^{\delta_i}\xi'(\varsigma)(\xi(\delta_i)-\xi(\varsigma))^{r_1+\eta_i-1}\right. \\
&\times|\varkappa(\varsigma)-\tilde{\varkappa}(\varsigma)|\,d\varsigma+\frac{l_1+l_2l_3}{\Gamma(2+r_1-\mathfrak{v})}\int_{\mathfrak{a}}^{b}\xi'(\varsigma)(\xi(b)-\xi(\varsigma))^{1+r_1-\mathfrak{v}}|\varkappa(\varsigma)-\tilde{\varkappa}(\varsigma)|\,d\varsigma \\
&\leq \frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}(l_1+l_2l_3)}{|\varpi|\Gamma(\mathfrak{v})} \\
&\times\left(\sum_{\mathfrak{i}=1}^{\mathfrak{m}}|\theta_i|\,I_{\mathfrak{a}+}^{r_1+\eta_i;\xi}|x(\delta_i)-\tilde{x}(\delta_i)|+I_{\mathfrak{a}+}^{2+r_1-\gamma;\xi}|x(b)-\tilde{x}(b)|\right|\Bigg).
\end{aligned} \tag{20}
$$

Because $\varkappa(b)=\tilde{\varkappa}(b)$, we must have $\varkappa(\delta_i)=\tilde{\varkappa}(\delta_i)$, $i=1,2,...,m$. Therefore, from inequality (20), we obtain $A_{\varkappa}=A_{\tilde{\varkappa}}$. From (18) and (As1), we get

$$
\begin{aligned}
&|\tilde{\varkappa}(\mathfrak{t})-\varkappa(\mathfrak{t})| \\
&=\left|\tilde{\varkappa}(\mathfrak{t})-\frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\tilde{\mathfrak{f}}\Gamma(\mathfrak{v})}A_x-I_{\mathfrak{a}+}^{r_1;\xi}f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}}h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right)\right| \\
&\leq\left|\tilde{\varkappa}(\mathfrak{t})-\frac{(\xi(\mathfrak{t})-\xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\tilde{\mathfrak{f}}\Gamma(\mathfrak{v})}A_{\tilde{x}}-I_{\mathfrak{a}+}^{r_1;\xi}f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}}h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right)\right| \\
&+\left|I_{\mathfrak{a}+}^{r_1;\xi}f\left(\mathfrak{t},\tilde{\varkappa}(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}}h(\mathfrak{t},\varphi,\tilde{\varkappa}(\varphi))\,d\varphi\right)-I_{\mathfrak{a}+}^{r_1;\xi}f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}}h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right)\right| \\
&\leq\epsilon\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}+\frac{l_1+l_2l_3}{\Gamma(r_1)}\int_{\mathfrak{a}}^{\tau}\xi'(\varsigma)(\xi(\mathfrak{t})-\xi(\varsigma))^{r_1-1}|\tilde{x}(\varsigma)-x(\varsigma)|\,d\varsigma.
\end{aligned}
$$

Applying Lemma 2.11 with $u(t)=|\tilde{x}(\mathfrak{t})-x(\mathfrak{t})|$, $v(\mathfrak{t})=\epsilon\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}$ and $z(\mathfrak{t})=\frac{l_1+l_2l_3}{\Gamma(r_1)}$, we obtain

$$
\begin{aligned}
&|\tilde{\varkappa}(\mathfrak{t})-\varkappa(\mathfrak{t})| \\
&\leq\epsilon\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}\left[1+\int_{\mathfrak{a}}^{\mathfrak{t}}\sum_{k=1}^{\infty}\frac{[l_1+l_2l_3]^k}{\Gamma(kr_1)}\xi'(\varsigma)(\xi(t)-\xi(\varsigma))^{k\alpha-1}\,d\varsigma\right] \\
&\leq\epsilon\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}\left[1+\sum_{k=1}^{\infty}\frac{[(l_1+l_2l_3)(\xi(b)-\xi(\mathfrak{a}))^{r_1}]^k}{\Gamma(kr_1+1)}\right] \\
&=\epsilon\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}E_{r_1}\left((l_1+l_2l_3)(\xi(b)-\xi(\mathfrak{a}))^{r_1}\right)
\end{aligned}
$$

By setting

$$
k_f=\frac{(\xi(b)-\xi(\mathfrak{a}))^{r_1}}{\Gamma(r_1+1)}E_{r_1}\left((l_1+l_2l_3)(\xi(b)-\xi(\mathfrak{a}))^{r_1}\right).
$$

we obtain

$$|\tilde{\varkappa}(\mathfrak{t}) - \varkappa(\mathfrak{t})| \leq k_f \epsilon. \tag{21}$$

Therefore, the first Eq of (2) is UH stable. □

Remark 4.2. *Define $\phi_f : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi_f(\epsilon) = k_f \epsilon$. Then, $\phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\phi_f(0) = 0$. Then inequality (21) can be written as*

$$|\tilde{\varkappa}(\mathfrak{t}) - \varkappa(\mathfrak{t})| \leq \phi_f(\epsilon).$$

Thus, the first Eq of (2) is GUH stable.

In the next, we introduce the following function

(As3) the function $\phi \in C([\mathfrak{a}, b], \mathbb{R}^+)$ is increasing and there is a constant $\lambda_\phi > 0$ such that

$$I_{\mathfrak{a}+}^{r_1;\xi} \phi(t) \leq \lambda_\phi \phi(t), \ \forall \mathfrak{t} \in J.$$

Theorem 4.3. *Assume that the hypotheses (As1), (As3) and condition (14) are satisfied. Then, the first Eq of (2) is UHR stable.*

Proof. Let any $\epsilon > 0$. Let $\tilde{\varkappa} \in \mathcal{C}$ be any solution of the inequality

$$\left| {}^H D_{\mathfrak{a}+}^{r_1, r_2; \xi} \tilde{\varkappa}(\mathfrak{t}) - f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \tilde{\varkappa}(\varphi)) \, d\varphi\right)\right| \leq \epsilon \phi(\mathfrak{t}), \ \mathfrak{t} \in J.$$

Then, proceeding as in the proof of Theorem 4.1. From Remark 2.10, for some continuous function v such that $|v(\mathfrak{t})| < \epsilon \phi(\mathfrak{t})$, we get

$$\begin{aligned} &\left| \tilde{\varkappa}(t) - \frac{(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{\mathfrak{v}-1}}{\varpi \Gamma(\mathfrak{v})} A_{\tilde{\varkappa}} - I_{\mathfrak{a}+}^{r_1;\xi} f\left(\mathfrak{t}, \tilde{\varkappa}(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t}, \varphi, \tilde{\varkappa}(\varphi)) \, d\varphi\right)\right| \\ &\leq I_{\mathfrak{a}+}^{r_1;\xi} |v(\mathfrak{t})| \leq \epsilon I_{\mathfrak{a}+}^{r_1;\xi} |\phi(\mathfrak{t})| \leq \epsilon \lambda_\phi \phi(\mathfrak{t}), \ \mathfrak{t} \in J. \end{aligned}$$

Taking $\tilde{\varkappa} \in \mathcal{C}$ as any solution of (19), and following same steps as in the proof of Theorem 4.1,we get

$$\begin{aligned} &|\tilde{\varkappa}(\mathfrak{t}) - \varkappa(\mathfrak{t})| \\ &\leq \epsilon \lambda_\phi \phi(\mathfrak{t}) + \frac{l_1 + l_2 l_3}{\Gamma(r_1)} \int_{\mathfrak{a}}^{\mathfrak{t}} \xi'(\varsigma) (\xi(\mathfrak{t}) - \xi(\varsigma))^{r_1 - 1} |\tilde{\varkappa}(\varsigma) - \varkappa(\varsigma)| \, d\varsigma, \ \mathfrak{t} \in J. \end{aligned}$$

By applying Corollary 2.12, we obtain

$$\begin{aligned} |\tilde{\varkappa}(\mathfrak{t}) - \varkappa(\mathfrak{t})| &\leq \epsilon \lambda_\phi \phi(\mathfrak{t}) E_{r_1}\left((l_1 + l_2 l_3)(\xi(\mathfrak{t}) - \xi(\mathfrak{a}))^{r_1}\right) \\ &\leq \epsilon \lambda_\phi \phi(\mathfrak{t}) E_{r_1}\left((l_1 + l_2 l_3)(\xi(b) - \xi(\mathfrak{a}))^{r_1}\right). \end{aligned}$$

By taking a constant

$$k_{\phi,\, f} = \lambda_\phi \phi(\mathfrak{t}) E_{r_1}\left((l_1 + l_2 l_3)(\xi(b) - \xi(\mathfrak{a}))^{r_1}\right).$$

we obtain

$$|\tilde{\varkappa}(\mathfrak{t}) - \varkappa(\mathfrak{t})| \leq k_{\phi, f} \epsilon \phi(\mathfrak{t}). \tag{22}$$

Therefore, the first Eq (2) is UHR stable. □

Remark 4.4. *By putting $\epsilon = 1$ in the inequality (22), we deduce that first Eq of (2) is GUHR stable.*

5. Examples

In this section, we consider some particular cases of the nonlinear FIDE to apply our results in the study of existence and Ulam stabilities, specifically, UH and UHR.

Consider the nonlinear FIDE of the form

$$\begin{cases} {}^{H}D_{a+}^{\alpha,\beta;\xi}\varkappa(\mathfrak{t}) = f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}} h(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right), \ \mathfrak{t}\in(\mathfrak{a},b), \\ \varkappa(\mathfrak{a})=0,\ I_{\mathfrak{a}+}^{2-\mathfrak{v};\xi}\varkappa(b)=\sum_{i=1}^{\mathfrak{m}}\theta_i I_{\mathfrak{a}+}^{\eta_i;\xi}\varkappa(\delta_i), \end{cases} \tag{23}$$

The following examples are particular cases of the FIDE given by (23).

Example 5.1. *Consider the FIDE given by (23). Taking* $\xi(\mathfrak{t})=\log\mathfrak{t}$, $r_2\to 0$, $\mathfrak{a}=1$, $b=e$, $r_1=\frac{3}{2}$, $\theta_1=\frac{1}{2}$, $\theta_2=\frac{1}{10}$, $\eta_1=\frac{1}{4}$, $\eta_2=\frac{5}{2}$, $\delta_1=\frac{3}{2}$, $\delta_2=2$ *and* f, h *are continuous functions defined by*

$$f\left(\mathfrak{t},\varkappa(\mathfrak{t}),\int_{\mathfrak{a}}^{\mathfrak{t}}\mathrm{h}(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right)=\frac{1}{4}\varkappa(\mathfrak{t})+\frac{1}{10}\int_1^{\mathfrak{t}}\frac{1}{\varphi\exp(\mathfrak{t}^2-1)+4}\varkappa(\varphi)\,d\varphi,$$
$$\mathrm{h}(\mathfrak{t},\varphi,\varkappa(\varphi))=\frac{1}{\varphi\exp(\mathfrak{t}^2-1)+4}\varkappa(\varphi).$$

Then, the problem (23) reduce to the following

$$\begin{cases} {}^{Ha}D_{1+}^{\frac{3}{2},0;\log\mathfrak{t}}\varkappa(\mathfrak{t})=\frac{1}{4}\varkappa(\mathfrak{t})+\frac{1}{10}\int_1^{\mathfrak{t}}\frac{1}{\varphi\exp(\mathfrak{t}^2-1)+4}\varkappa(\varphi)\,d\varphi,\ \mathfrak{t}\in(1,e), \\ \varkappa(1)=0,\ I_{1+}^{\frac{1}{2};\log\mathfrak{t}}\varkappa(e)=\frac{1}{2}I_{1+}^{\frac{1}{4};\log\mathfrak{t}}\varkappa\left(\frac{3}{2}\right)+\frac{1}{10}I_{1+}^{\frac{5}{2};\log\mathfrak{t}}\varkappa(2), \end{cases} \tag{24}$$

which is nonlinear FIDE involving Hadamard FD. In this case $\mathfrak{v}=\frac{3}{2}$. *Set*

$$f(\mathfrak{t},\varkappa,\mathfrak{y})=\frac{1}{4}\varkappa+\frac{1}{10}\mathfrak{y},\ \forall\varkappa,\mathfrak{y}\in\mathbb{R}.$$

For $\varkappa_i,\mathfrak{y}_i\in\mathbb{R}$, $i=1,2$ *and* $\mathfrak{t}\in[1,e]$, *using the hypothesis (As1), we get*

$$|f(\mathfrak{t},\varkappa_1,\mathfrak{y}_1)-f(\mathfrak{t},\varkappa_2,\mathfrak{y}_2)|\le\frac{1}{4}|\varkappa_1-\varkappa_2|+\frac{1}{10}|\mathfrak{y}_1-\mathfrak{y}_2|,$$

and

$$\begin{aligned}|\mathrm{h}(\mathfrak{t},\varphi,\varkappa_1)-\mathrm{h}(\mathfrak{t},\varphi,\varkappa_2)| &\le \frac{1}{\varphi\exp(\mathfrak{t}^2-1)+4}|\varkappa_1-\varkappa_2| \\ &\le \frac{1}{5}|\varkappa_1-\varkappa_2|\end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1=\frac{1}{4}$, $l_2=\frac{1}{10}$ *and* $l_3=\frac{1}{5}$ *We will check that condition (14) is satisfied. Indeed*

$$\begin{aligned}&(k_1+k_2+k_3)(l_1+l_2l_3(b-\mathfrak{a})) \\ &\simeq(0.5+0.79+0.75)\left(\frac{1}{4}+\frac{1}{50}\right) \\ &\simeq 0.55<1.\end{aligned}$$

Then by Theorem 3.1, (24) has a unique mild solution on $[1,e]$. *Further, by Theorem 4.1 we conclude that the first Eq of (24) is UH stable with*

$$k_f=\frac{1}{\Gamma\left(\frac{5}{2}\right)}E_{\frac{3}{2}}\left(\frac{27}{100}\right).$$

Define

$$\phi(\mathfrak{t}) = \log(\mathfrak{t})^{\frac{3}{2}}, \ \mathfrak{t} \in [1, e].$$

Then, ϕ is continuous increasing function such that

$$\begin{aligned} I_{1+}^{\frac{3}{2};\log \mathfrak{t}} \phi(\mathfrak{t}) &= \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_1^{\mathfrak{t}} \left(\log \frac{\mathfrak{t}}{\varsigma}\right)^{\frac{1}{2}} \log(\mathfrak{t})^{\frac{3}{2}} \frac{d\varsigma}{\varsigma} \\ &\leq \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_1^{\mathfrak{t}} \left(\log \frac{\mathfrak{t}}{\varsigma}\right)^{\frac{1}{2}} \frac{d\varsigma}{\varsigma} \\ &\leq \frac{1}{\Gamma\left(\frac{5}{2}\right)} \log(\mathfrak{t})^{\frac{3}{2}}. \end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{5}{2}\right)}$ and $\phi(\mathfrak{t}) = \log(\mathfrak{t})^{\frac{3}{2}}$, hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (24) is UHR stable.

Example 5.2. *Consider the FIDE given by (23). Taking $\xi(\mathfrak{t}) = \mathfrak{t}$, $r_2 \to 0$, $\mathfrak{a} = 0$, $b = 1$, $r_1 = \frac{5}{4}$, $\theta_1 = 3$, $\theta_2 = 5$, $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\delta_2 = \frac{1}{2}$and f, h are continuous functions defined by*

$$f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} \mathrm{h}(\mathfrak{t}, \varphi, \varkappa(\varphi))\, d\varphi\right) = \frac{1}{8}\varkappa(\mathfrak{t}) + \frac{1}{6}\int_0^1 \frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2) + 9} \frac{|x(\varphi)|}{|x(\varphi)| + 1} d\varphi,$$

$$\mathrm{h}(\mathfrak{t}, \varphi, \varkappa(\varphi)) = \frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2) + 9} \frac{|x(\varphi)|}{|x(\varphi)| + 1}.$$

Then, the problem (23) reduce to the following

$$\begin{cases} {}^{RL}D_{1+}^{\frac{5}{4},0;\mathfrak{t}} \varkappa(\mathfrak{t}) = \frac{1}{8}\varkappa(\mathfrak{t}) + \frac{1}{6}\int_0^1 \frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2)+9} \frac{|x(\varphi)|}{|x(\varphi)|+1} d\varphi, \ \mathfrak{t} \in (0, 1), \\ \varkappa(0) = 0, \ I_{0+}^{\frac{3}{4};\mathfrak{t}} \varkappa(1) = 3 I_{0+}^{\frac{1}{4};\mathfrak{t}} \varkappa\left(\frac{1}{4}\right) + 5 I_{0+}^{\frac{1}{2};\mathfrak{t}} \varkappa\left(\frac{1}{2}\right), \end{cases} \tag{25}$$

which is nonlinear FIDE involving R-L FD. In this case $\mathfrak{v} = \frac{5}{4}$. Set

$$f(\mathfrak{t}, \varkappa, \mathfrak{y}) = \frac{1}{8}\varkappa + \frac{1}{6}\mathfrak{y}, \ \forall \varkappa, \mathfrak{y} \in \mathbb{R}.$$

For $\varkappa_i, \mathfrak{y}_i \in \mathbb{R}$, $i = 1, 2$ and $\mathfrak{t} \in [0, 1]$, using the hypothesis (As1), we get

$$|f(\mathfrak{t}, \varkappa_1, \mathfrak{y}_1) - f(\mathfrak{t}, \varkappa_2, \mathfrak{y}_2)| \leq \frac{1}{8}|\varkappa_1 - \varkappa_2| + \frac{1}{6}|\mathfrak{y}_1 - \mathfrak{y}_2|,$$

and

$$\begin{aligned} |\mathrm{h}(\mathfrak{t}, \varphi, \varkappa_1) - \mathrm{h}(\mathfrak{t}, \varphi, \varkappa_2)| &= \left| \frac{\sin(\mathfrak{t})}{\exp(\mathfrak{t}^2) + 9} \left(\frac{|\varkappa_1|}{|\varkappa_1| + 1} - \frac{|\varkappa_2|}{|\varkappa_2| + 1} \right) \right| \\ &\leq \frac{1}{\exp(\mathfrak{t}^2) + 9} \left(\frac{|\varkappa_1 - \varkappa_2|}{(1 + |\varkappa_1|)(1 + |\varkappa_2|)} \right) \\ &\leq \frac{1}{10}(|\varkappa_1 - \varkappa_2|), \end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{8}$, $l_2 = \frac{1}{6}$ and $l_3 = \frac{1}{10}$ We will check that condition (14) is satisfied. Indeed

$$\begin{aligned} &(k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b - \mathfrak{a})) \\ &\simeq (1.51 + 0.14 + 0.88)\left(\frac{1}{8} + \frac{1}{60}\right) \\ &\simeq 0.36 < 1. \end{aligned}$$

Then by Theorem 3.1, the (25) has a unique mild solution on $[0,1]$*. Moreover, by Theorem 4.1 we conclude that the first Eq of (25) is UH stable with*

$$k_f = \frac{1}{\Gamma\left(\frac{9}{4}\right)} E_{\frac{5}{4}}\left(\frac{17}{120}\right).$$

Define

$$\phi(\mathfrak{t}) = \mathfrak{t}^{\frac{5}{4}}, \ \mathfrak{t} \in [0,1].$$

Then, ϕ *is continuous increasing function such that*

$$\begin{aligned} I_{0+}^{\frac{5}{4};\mathfrak{t}}\phi(\mathfrak{t}) &= \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\frac{1}{4}}\,\mathfrak{t}^{\frac{5}{4}} d\varsigma \\ &\leq \frac{1}{\Gamma\left(\frac{5}{4}\right)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\varsigma)^{\frac{1}{4}}\, d\varsigma \\ &\leq \frac{1}{\Gamma\left(\frac{9}{4}\right)}\mathfrak{t}^{\frac{5}{4}}. \end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{9}{4}\right)}$ *and* $\phi(\mathfrak{t}) = \mathfrak{t}^{\frac{5}{4}}$*, hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (25) is UHR stable.*

Example 5.3. *Consider the FIDE given by (23). Taking* $\xi(\mathfrak{t}) = \mathfrak{t}$*,* $r_2 \to \frac{1}{2}$*,* $\mathfrak{a} = 0$*,* $b = 1$*,* $r_1 = \frac{7}{4}$*,* $\theta_1 = 3$*,* $\theta_2 = 5$*,* $\eta_1 = \frac{1}{4}$*,* $\eta_2 = \frac{1}{2}$*,* $\delta_1 = \frac{1}{4}$*,* $\delta_2 = \frac{1}{2}$ *and* f*,* h *are continuous functions defined by*

$$f\left(\mathfrak{t}, \varkappa(\mathfrak{t}), \int_{\mathfrak{a}}^{\mathfrak{t}} \mathrm{h}(\mathfrak{t},\varphi,\varkappa(\varphi))\,d\varphi\right) = \frac{1}{9}\varkappa(\mathfrak{t}) + \frac{1}{30}\int_0^{\mathfrak{t}} \frac{\cos(\mathfrak{t})}{\exp(\mathfrak{t})+5}\frac{|\varkappa(\varphi)|}{|\varkappa(\varphi)|+1}d\varphi,$$
$$\mathrm{h}(\mathfrak{t},\varphi,x(\varphi)) = \frac{\cos(\mathfrak{t})}{\exp(\mathfrak{t})+5}\frac{|\varkappa(\varphi)|}{|\varkappa(\varphi)|+1}.$$

Then, the problem (23) reduce to the following

$$\begin{cases} {}^{H}D_{0+}^{\frac{7}{4},\frac{1}{2};\mathfrak{t}}\varkappa(\mathfrak{t}) = \frac{1}{9}\varkappa(\mathfrak{t}) + \frac{1}{30}\int_0^{\mathfrak{t}} \frac{\cos(\mathfrak{t})}{\exp(\mathfrak{t})+5}\frac{|\varkappa(\varphi)|}{|\varkappa(\varphi)|+1}d\varphi, \ \mathfrak{t} \in (0,1), \\ \varkappa(0) = 0, \ I_{0+}^{\frac{1}{8};\mathfrak{t}}\varkappa(1) = 3I_{0+}^{\frac{1}{4};\mathfrak{t}}\varkappa\left(\frac{1}{4}\right) + 5I_{0+}^{\frac{1}{2};\mathfrak{t}}\varkappa\left(\frac{1}{2}\right), \end{cases} \tag{26}$$

which is nonlinear FIDE involving Hilfer FD. In this case $\mathfrak{v} = \frac{15}{8}$*. Set*

$$f(\mathfrak{t},\varkappa,\mathfrak{y}) = \frac{1}{9}\varkappa + \frac{1}{30}\mathfrak{y}, \ \forall \varkappa, \mathfrak{y} \in \mathbb{R}.$$

For $\varkappa_i, \mathfrak{y}_i \in \mathbb{R}$*,* $i = 1,2$ *and* $\mathfrak{t} \in [0,1]$*, using the hypothesis (As1), we get*

$$|f(\mathfrak{t},\varkappa_1,\mathfrak{y}_1) - f(\mathfrak{t},\varkappa_2,\mathfrak{y}_2)| \leq \frac{1}{9}|\varkappa_1 - \varkappa_2| + \frac{1}{30}|\mathfrak{y}_1 - \mathfrak{y}_2|,$$

and

$$\begin{aligned} |\mathrm{h}(\mathfrak{t},\varphi,\varkappa_1) - \mathrm{h}(\mathfrak{t},\varphi,\varkappa_2)| &= \left|\frac{\cos(\mathfrak{t})}{\exp(\mathfrak{t})+5}\left(\frac{|\varkappa_1|}{|\varkappa_1|+1} - \frac{|\varkappa_2|}{|\varkappa_2|+1}\right)\right| \\ &\leq \frac{1}{\exp(\mathfrak{t})+5}\left(\frac{|\varkappa_1 - \varkappa_2|}{(1+|\varkappa_1|)(1+|\varkappa_2|)}\right) \\ &\leq \frac{1}{6}(|\varkappa_1 - \varkappa_2|), \end{aligned}$$

thus, the assumption (As1) is satisfied with $l_1 = \frac{1}{9}$, $l_2 = \frac{1}{30}$ *and* $l_3 = \frac{1}{6}$ *We will check that condition (14) is satisfied. Indeed*

$$\begin{aligned}
&(k_1 + k_2 + k_3)(l_1 + l_2 l_3 (b - \mathfrak{a})) \\
&\simeq (3.07 + 0.5 + 0.62)\left(\frac{1}{9} + \frac{1}{180}\right) \\
&\simeq 0.49 < 1.
\end{aligned}$$

Then by Theorem 3.1, the (26) has a unique mild solution on $[0, 1]$. *Further, by Theorem 4.1 we conclude that the first Eq of (26) is UH stable with*

$$k_f = \frac{1}{\Gamma\left(\frac{11}{4}\right)} E_{\frac{7}{4}}\left(\frac{7}{60}\right).$$

Define

$$\phi(\mathfrak{t}) = \mathfrak{t}^{\frac{7}{4}}, \ \mathfrak{t} \in [0, 1].$$

Then, ϕ *is continuous increasing function such that*

$$\begin{aligned}
I_{0^+}^{\frac{7}{4};\mathfrak{t}} \phi(\mathfrak{t}) &= \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \varsigma)^{\frac{1}{5}} \mathfrak{t}^{\frac{7}{4}} d\varsigma \\
&\leq \frac{1}{\Gamma\left(\frac{7}{4}\right)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \varsigma)^{\frac{3}{4}} d\varsigma \\
&\leq \frac{1}{\Gamma\left(\frac{11}{5}\right)} \mathfrak{t}^{\frac{7}{4}}.
\end{aligned}$$

Therefore, for $\lambda_\phi = \frac{1}{\Gamma\left(\frac{11}{5}\right)}$ *and* $\phi(\mathfrak{t}) = \mathfrak{t}^{\frac{7}{4}}$, *hypothesis (As3) is satisfied. Hence, by Theorem 4.3 the first Eq of (26) is UHR stable.*

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