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Picard and Picard-Krasnoselskii iteration methods for generalized proportional Hadamard fractional integral equations

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Abstract

In the current paper, some existence and uniqueness results for a generalized proportional Hadamard fractional integral equation are established via Picard and Picard-Krasnoselskii iteration methods together with the Banach contraction principle. A simulative example was provided to verify the applicability of the theoretical findings.

Keywords: Picard method Picard-Krasnoselskii method Proportional Hadamard fractional integral.
2010 MSC: 26A33, 39B12, 45H05.

1. Introduction & Preliminaries

We deal with the following generalized proportional Hadamard fractional integral equation (GPHFIE):

$$\begin{aligned}\psi(t) &= \varphi(t) + \frac{\lambda}{\delta^{\gamma}\Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi(s)) \frac{ds}{s}, \quad t \in [a, b], \\ &= \varphi(t) + \lambda {}_a^H\mathcal{I}^{\delta,\gamma} \mathcal{G}(t, \psi(t)),\end{aligned} \tag{1}$$

where the proportionality index $\delta \in (0, 1]$, $\lambda \in \mathbb{R}$, ${}_a^H\mathcal{I}^{\delta,\gamma}$ denotes the left-sided generalized proportional Hadamard fractional integral of order γ $(0 < \gamma < 1)$, and the functions $\varphi : [a, b] \to \mathbb{R}$ and $\mathcal{G} : [a, b] \times \mathbb{R} \to \mathbb{R}$

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are continuous.

It is worthwhile to remark that the aforesaid problem (1) has the advantage of generalizing many earlier papers, for example, when $\delta \to 1$, the GPHFIE (1) is reduced to the Hadamard fractional integral equation

$$\psi(t) = \varphi(t) + \frac{\lambda}{\Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi(s)) \, \frac{ds}{s}, \quad t \in [a, b],$$

which studied in several forms in recent papers, see [9, 10, 12, 20].

Newly, Jarad et al. [8] suggested a new definition concerning the generalized fractional proportional fractional integral operator as:

Definition 1.1. *Take $\delta \in (0,1]$ and $\gamma > 0$. The left-sided generalized proportional fractional integral of the function $\omega \in L^1[a,b]$ of order γ is defined by*

$$({}_a\mathcal{I}^{\gamma,\delta}\omega)(t) = \frac{1}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(t-s)} (t-s)^{\gamma-1} \omega(s) \, ds, \qquad t \in [a, b]. \tag{2}$$

Inspired by the above definition, Rahman et al. [17] gave the definition of generalized proportional Hadamard fractional integral as follows:

Definition 1.2. *Take $\delta \in (0,1]$ and $\gamma > 0$. The left-sided generalized proportional Hadamard fractional integral of the function ω of order γ is defined by*

$$\left({}_a^H\mathcal{I}^{\gamma,\delta}\omega\right)(t) = \frac{1}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \omega(s) \, \frac{ds}{s}, \quad a < t. \tag{3}$$

The concept of these new operators was disseminated in many prominent areas previously with notable contributions, see [1, 2, 3, 5, 7, 23].

On the other hand, the utilizing of iterative methods for finding solutions to nonlinear equations has attracted the heed of many researchers. For instance, El-Sayed et al. [6] used Picard and Adomian decomposition methods for the following fractional quadratic integral equation:

$$x(t) = a(t) + g(t, x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \, ds, \quad t \in [0,1], \ \alpha \in \mathbb{R}^+,$$

where $a : [0,1] \to \mathbb{R}^+$ and the functions $f, g : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous.

In [13], Marasi et al. proved an extension of the Picard-Lindelöf existence and uniqueness theorem for the fractional differential equations with Caputo-Fabrizio fractional derivatives:

$${}_a^{CF}\mathcal{D}^\alpha u(t) = g(t, u(t)) \, ds, \quad t \in [0,1], \ u(0) = u_0,$$

where ${}_a^{CF}\mathcal{D}^\alpha$ is the fractional Caputo-Fabrizio derivative of order $\alpha \in (0,1)$.

In [14], Micula used Picard iteration method for approximating solutions of fractional integral equations of the second kind of the form:

$$x(t) = \frac{a(t)}{\Gamma(\alpha)} \int_0^t b(s)(t-s)^{\alpha-1} x(s) \, ds + f(t), \quad t \in [0,T], \ \alpha \in (0,1),$$

where the functions $a, b, f : [0,T] \to \mathbb{R}$ are continuous. In [15], Okeke and Abbas introduced the Picard-Krasnoselskii hybrid iterative process which is a hybrid of Picard and Krasnoselskii iterative processes. They concluded that their newly introduced iterative process converges faster than all of Picard and Krasnoselskii iterative processes. Furthermore, Abdeljawad et al. [4] proved strong and weak convergence results for a class of mappings which is much more general than that of Suzuki non-expansive mappings on Banach space through the Picard-Krasnoselskii hybrid iteration process. Using a numerical example, they proved that the Picard-Krasnoselskii hybrid iteration process converges faster than both of the Picard and Krasnoselskii iteration processes. It should be noted that the Picard iteration is the simple iteration for approximate solution of fixed point equation for non-linear contraction mapping.

Some recent contributions on investigating fractional differential and integral equations via Picard iterative method can be located in the papers [18, 21, 22].

Motivated by the consequences in the above-mentioned articles, our target in the current work is to give more general results in a framework of the generalized proportional Hadamard fractional integral operator.

The main recency of the existing work could be listed as:

i) We propose a new category of fractional integral equations in terms of the generalized proportional Hadamard fractional integral.

ii) From the choice of the proportionality index $\delta = 1$, we get the well-known Hadamard fractional integral equations.

iii) With the aid of the Picard and Picard-Krasnoselskii iteration methods together with the Banach contraction principle, the mains results are established.

The following result will have a pivotal role in proving one of the main findings in the current paper.

Lemma 1.3. *(Lemma 2, [19]) Let $\{z_n\}$ be a non-negative sequence with the property*

$$z_{n+1} \leq (1 - \eta_n) z_n.$$

If $\{\eta_n\} \subset (0, 1)$ and $\sum_{n=1}^{\infty} \eta_n = \infty$. Then $\lim_{n\to\infty} z_n = 0$.

Going forward, we will introduce these assumptions.

(**A1**) The function $\varphi : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$.

(**A2**) The function $\mathcal{G} : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded with

$$M = \max_{(t,\psi)\in[a,b]\times\mathbb{R}} |\mathcal{G}(t, \psi)|.$$

(**A3**) There exists a constant $L > 0$ such that

$$|\mathcal{G}(t, \psi_1) - \mathcal{G}(t, \psi_2)| \leq L|\psi_1 - \psi_2|, \quad \forall t \in [a, b], \psi_i, \psi_2 \in \mathbb{R}.$$

2. Uniqueness result via Banach contraction principle

Let $\mathbf{C} = C([a, b], \mathbb{R})$ be the Banach space of all continuous functions from $[a, b]$ into $\mathbb{R}$ with the norm $\|x\| = \max_{t\in[a,b]} |x(t)|$.

Theorem 2.1. *If the assumptions $(\mathbf{A1}) - (\mathbf{A3})$ are fulfilled, provided that*

$$\frac{L|\lambda|(\ln b - \ln a)^{\gamma}}{\delta^{\gamma}\Gamma(\gamma + 1)} < 1. \tag{4}$$

Then the GPHFIE (1) has a unique solution in $\mathbf{C}$.

Proof. Define the set

$$\Omega_k := \{x \in \mathbf{C} : |x - \varphi(t)| \leq k\},$$

where $k = \frac{M|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)}$.

Clearly, the set Ω_k is non-empty, convex and compact in the Banach space $(\mathbf{C}, \|\cdot\|)$.

Define the integral operator $\mathcal{H} : \mathbf{C} \to \mathbf{C}$ by

$$(\mathcal{H}\psi)(t) = \varphi(t) + \frac{\lambda}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi(s)) \frac{ds}{s}, \quad t \in [a,b],\ \psi \in \mathbf{C}. \tag{5}$$

It obvious that the solution $\psi \in \mathbf{C}$ of the GPHFIE (1) is equivalent to the fixed point of the operator $\mathcal{H}$, i.e. $\mathcal{H}\psi = \psi$.

First, we show that $\mathcal{H}$ maps Ω_k into Ω_k.
For $\psi \in \Omega_k$ and since $\left|e^{\frac{\delta-1}{\delta}(\ln t - \ln s)}\right| \leq 1, \quad \forall t > s,\ \delta \in (0,1]$, using (**A2**), one has

$$\begin{aligned}
|\psi(t) - \varphi(t)| &\leq \frac{|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t \left|e^{\frac{\delta-1}{\delta}(\ln t - \ln s)}\right| (\ln t - \ln s)^{\gamma-1} |\mathcal{G}(s, \psi(s))| \frac{ds}{s} \\
&\leq \frac{M|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} \frac{ds}{s} \\
&\leq \frac{M|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)} = k,
\end{aligned}$$

Which implies that $\mathcal{H} : \Omega_k \to \Omega_k$.

Next, we show that $\mathcal{H}$ is a contraction. For $\psi_1, \psi_2 \in \Omega_k$, using (**A3**), we get

$$\begin{aligned}
|(\mathcal{H}\psi_1)(t) - (\mathcal{H}\psi_2)(t)| &\leq \frac{|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t \left|e^{\frac{\delta-1}{\delta}(\ln t - \ln s)}\right| (\ln t - \ln s)^{\gamma-1} |\mathcal{G}(s, \psi_1(s)) - \mathcal{G}(s, \psi_2(s))| \frac{ds}{s} \\
&\leq \frac{|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} L|\psi_1(s) - \psi_2(s)| \frac{ds}{s}.
\end{aligned}$$

Taking the maximum of the above inequality implies to

$$\|\mathcal{H}\psi_1 - \mathcal{H}\psi_2\| \leq \frac{L|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)} \|\psi_1 - \psi_2\|.$$

In the light of the condition (4), we infer that $\mathcal{H}$ is a contraction and consequently, $\mathcal{H}$ has a unique fixed point which is the unique solution of GPHFIE (1). □

3. Picard iterative process

The Picard iterative process [16] defined by the sequence $\{\psi_n\}_{n=0}^\infty$ as follows:

$$\begin{cases} \psi_0 &= \varphi \in \mathbf{C}, \\ \psi_n &= \mathcal{H}\psi_{n-1}, \quad n \geq 1, \end{cases} \tag{6}$$

is used to examine the existence and uniqueness of solutions of the GPHFIE (1). We begin with

$$\begin{aligned}\psi_0(t) &= \varphi(t) \in \mathbf{C},\\ \psi_n(t) &= \psi_0(t) + \frac{\lambda}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi_{n-1}(s)) \frac{ds}{s}, \quad n \geq 1.\end{aligned} \tag{7}$$

For $n = 1$, (7) reads

$$\psi_1(t) - \psi_0(t) = \frac{\lambda}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi_0(s)) \frac{ds}{s},$$

which implies that

$$|\psi_1(t) - \psi_0(t)| \leq \frac{M|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)}. \tag{8}$$

For $n \geq 2$, we get

$$\begin{aligned}|\psi_n(t) - \psi_{n-1}(t)| &\leq \frac{|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t \left| e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} \right| (\ln t - \ln s)^{\gamma-1} |\mathcal{G}(s, \psi_{n-1}(s)) - \mathcal{G}(s, \psi_{n-2}(s))| \frac{ds}{s}\\ &\leq \frac{L|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} |\psi_{n-1}(s) - \psi_{n-2}(s)| \frac{ds}{s}.\end{aligned}$$

Putting $n = 2$, then using (8), we obtain that

$$\begin{aligned}|\psi_2(t) - \psi_1(t)| &\leq \frac{L|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} |\psi_1(s) - \psi_0(s)| \frac{ds}{s}\\ &\leq \frac{ML|\lambda|^2(\ln b - \ln a)^{2\gamma}}{\delta^{2\gamma} \Gamma^2(\gamma+1)}.\end{aligned}$$

Similarly, for $n = 3$, we get

$$\begin{aligned}|\psi_3(t) - \psi_2(t)| &\leq \frac{L|\lambda|}{\delta^\gamma \Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} |\psi_2(s) - \psi_1(s)| \frac{ds}{s}\\ &\leq \frac{ML^2|\lambda|^3(\ln b - \ln a)^{3\gamma}}{\delta^{3\gamma} \Gamma^3(\gamma+1)}.\end{aligned}$$

Repeating this technique, we get

$$|\psi_n(t) - \psi_{n-1}(t)| \leq ML \left(\frac{L|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)} \right)^n.$$

Since $\frac{L|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma \Gamma(\gamma+1)} < 1$, then the infinite series $\sum_{n=1}^{\infty}(\psi_n(t) - \psi_{n-1}(t))$ is uniformly convergent.
On the other side, since $\psi_n(t) = \psi_0(t) + \sum_{i=1}^{n}(\psi_i(t) - \psi_{i-1}(t))$. Then the convergence of the sequence $\{\psi_n\}_{n=0}^{\infty}$ is equivalent to the convergence of the infinite series $\sum_{i=1}^{\infty}(\psi_i(t) - \psi_{i-1}(t))$ and the solution will be $\psi(t) = \lim_{n\to\infty} \psi_n(t)$.

Hence, in accordance with the aforesaid discussion, we conclude that the sequence $\{\psi_n\}_{n=0}^{\infty}$ is uniformly convergent and the assumption (**A1**) implies

$$\begin{aligned}\psi(t) &= \lim_{n\to\infty} \frac{\lambda}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi_{n-1}(s)) \frac{ds}{s}\\ &= \frac{\lambda}{\delta^\gamma \Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \mathcal{G}(s, \psi(s)) \frac{ds}{s}.\end{aligned}$$

This proves the existence of a solution.

Next, we show that the uniqueness of the solution. Let $\vartheta(t)$ be a solution of (1). Then

$$\vartheta(t)=\varphi(t)+\frac{\lambda}{\delta^{\gamma}\Gamma(\gamma)}\int_a^t e^{\frac{\delta-1}{\delta}(\ln t-\ln s)}\,(\ln t-\ln s)^{\gamma-1}\,\mathcal{G}(s,\vartheta(s))\,\frac{ds}{s},\quad t\in[a,b].$$

Thus, we get

$$\begin{aligned}
|\vartheta(t)-\psi_n(t)| &\le \frac{|\lambda|}{\delta^{\gamma}\Gamma(\gamma)}\int_a^t \left|e^{\frac{\delta-1}{\delta}(\ln t-\ln s)}\right|(\ln t-\ln s)^{\gamma-1}\left|\mathcal{G}(s,\vartheta(s))-\mathcal{G}(s,\psi_{n-1}(s))\right|\,\frac{ds}{s}\\
&\le \frac{L|\lambda|}{\delta^{\gamma}\Gamma(\gamma)}\int_a^t (\ln t-\ln s)^{\gamma-1}\,|\vartheta(s)-\psi_{n-1}(s)|\,\frac{ds}{s},\\
|\vartheta(t)-\psi_{n-1}(t)| &\le \frac{L|\lambda|}{\delta^{\gamma}\Gamma(\gamma)}\int_a^t (\ln t-\ln s)^{\gamma-1}\,|\vartheta(s)-\psi_{n-2}(s)|\,\frac{ds}{s},\\
|\vartheta(t)-\psi_{n-2}(t)| &\le \frac{L|\lambda|}{\delta^{\gamma}\Gamma(\gamma)}\int_a^t (\ln t-\ln s)^{\gamma-1}\,|\vartheta(s)-\psi_{n-3}(s)|\,\frac{ds}{s},\\
&\;\;\vdots\\
|\vartheta(t)-\varphi(t)| &\le \frac{M|\lambda|(\ln b-\ln a)^{\gamma}}{\delta^{\gamma}\Gamma(\gamma+1)}.
\end{aligned}$$

Hence, we get

$$|\vartheta(t)-\psi_n(t)|\le ML\left(\frac{L|\lambda|(\ln b-\ln a)^{\gamma}}{\delta^{\gamma}\Gamma(\gamma+1)}\right)^{n+1}.$$

Therefore,

$$\lim_{n\to\infty}\psi_n(t)=\vartheta(t)=\psi(t).$$

This completes the proof.

4. Picard-Krasnoselskii hybrid iterative process

The Krasnoselskii iterative process [11] is defined by the sequence $\{\theta_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases}\theta_1\in\mathbf{C},\\ \theta_{n+1}=(1-\mu)\theta_n+\mu\mathcal{H}\theta_n,\quad n\ge 1,\end{cases}\tag{9}$$

where $\mu\in(0,1)$.

Latterly, Okeke and Abbas [15] initiated the following Picard-Krasnoselskii hybrid iterative process by the sequence $\{w_n\}_{n=0}^{\infty}$:

$$\begin{cases}w_1=w\in\mathbf{C},\\ w_{n+1}=\mathcal{H}u_n,\\ u_n=(1-\mu)w_n+\mu\mathcal{H}w_n,\quad n\in\mathbb{N},\end{cases}\tag{10}$$

where $\mu\in(0,1)$.

Theorem 4.1. *If the assumptions* (**A1**) − (**A3**) *are fulfilled. Then the GPHFIE (1) has a unique solution* w^* *(say), and the Picard-Krasnoselskii hybrid iterative process (10) converges to* w^*.

Proof. Let $\{w_n\}_{n=0}^{\infty}$ be an iterative sequence created by the Picard-Krasnoselskii hybrid iterative process (10) for the operator $\mathcal{H}$ defined by (5) and w^* be a fixed point of the operator $\mathcal{H}$, i.e. $w^* = \mathcal{H}w^*$.
We show that $w_n \to w^*$ as $n \to \infty$. For $t \in [a,b]$, one has
$\|u_n - w^*\|$

$$\begin{aligned}
&= \|(1-\mu)w_n + \mu\mathcal{H}w_n - w^*\| \\
&\leq (1-\mu)\|w_n - w^*\| + \mu\|\mathcal{H}w_n - \mathcal{H}w^*\| \\
&= (1-\mu)\|w_n - w^*\| \\
&+ \mu \max_{t\in[a,b]} \left\{ \left| \frac{\lambda}{\delta^\gamma\Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \left(\mathcal{G}(s, w_n(s)) - \mathcal{G}(s, w^*(s))\right) \frac{ds}{s} \right| \right\} \\
&\leq (1-\mu)\|w_n - w^*\| + \mu \max_{t\in[a,b]} \left\{ \frac{|\lambda|}{\delta^\gamma\Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} \left|\mathcal{G}(s, w_n(s)) - \mathcal{G}(s, w^*(s))\right| \frac{ds}{s} \right\} \\
&\leq (1-\mu)\|w_n - w^*\| + \mu \max_{t\in[a,b]} \left\{ \frac{L|\lambda|}{\delta^\gamma\Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} |w_n(s) - w^*(s)| \frac{ds}{s} \right\} \\
&\leq (1-\mu)\|w_n - w^*\| + \mu \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)} \|w_n - w^*\| \\
&= \left(1 - \left(1 - \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)}\right)\mu\right) \|w_n - w^*\|. \qquad (11)
\end{aligned}$$

Using (10), we get

$$\begin{aligned}
\|w_{n+1} - w^*\| &= \|\mathcal{H}u_n - \mathcal{H}w^*\| \\
&= \max_{t\in[a,b]} \left\{ \left| \frac{\lambda}{\delta^\gamma\Gamma(\gamma)} \int_a^t e^{\frac{\delta-1}{\delta}(\ln t - \ln s)} (\ln t - \ln s)^{\gamma-1} \left(\mathcal{G}(s, u_n(s)) - \mathcal{G}(s, w^*(s))\right) \frac{ds}{s} \right| \right\} \\
&\leq \max_{t\in[a,b]} \left\{ \frac{|\lambda|}{\delta^\gamma\Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} \left|\mathcal{G}(s, u_n(s)) - \mathcal{G}(s, w^*(s))\right| \frac{ds}{s} \right\} \\
&\leq \max_{t\in[a,b]} \left\{ \frac{L|\lambda|}{\delta^\gamma\Gamma(\gamma)} \int_a^t (\ln t - \ln s)^{\gamma-1} |u_n(s) - w^*(s)| \frac{ds}{s} \right\} \\
&\leq \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)} \|u_n - w^*\|. \qquad (12)
\end{aligned}$$

It follows, from (11) and (12), that

$$\|w_{n+1} - w^*\| \leq \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)} \left(1 - \left(1 - \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)}\right)\mu\right) \|w_n - w^*\|. \qquad (13)$$

Since $\frac{L|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)} < 1$, then we obtain that

$$\|w_{n+1} - w^*\| \leq \left(1 - \left(1 - \frac{L|\lambda|\,(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)}\right)\mu\right) \|w_n - w^*\|. \qquad (14)$$

Note that $1 - \left(1 - \frac{L|\lambda|(\ln b - \ln a)^\gamma}{\delta^\gamma\Gamma(\gamma+1)}\right)\mu = \eta_n < 1$ and $\|w_n - w^*\| = z_n$. Thus, all the conditions of Lemma 1.3 are satisfied. Hence, $\lim_{n\to\infty} \|w_n - w^*\| = 0$. The proof is finished. □

5. Example

In this section, we give an example to examine the applicability of the theoretical findings.

Example 5.1. *Consider the following GPHFIE:*

$$\psi(t) = \frac{1}{2}t^3 + \frac{1}{5}\, {}^{H}_{1^+}\mathcal{I}^{\frac{1}{2},\frac{1}{4}} \left(\frac{t}{2} + \frac{\sin t}{5} \frac{|\psi(t)|}{1+|\psi(t)|} \right), \qquad t \in [1, e]. \tag{15}$$

Here, $\gamma = \frac{1}{2}$, $\delta = \frac{1}{4}$, $\lambda = \frac{1}{5}, \varphi(t) = \frac{1}{2}t^3$, and $\mathcal{G}(t,\psi) = \frac{t}{2} + \frac{\sin t}{5}\frac{|\psi|}{1+|\psi|}$. It is clear that φ and $\mathcal{G}$ are continuous functions on $[1, e]$ which imply that the assumptions (**A1**) and **(A2)** are satisfied and $M = \frac{e}{2} + \frac{\sin e}{5} \approx 1.368625949$.

Further, for each $t \in [1, e]$ and $\psi_1, \psi_2 \in \mathbb{R}$, we get

$$|\mathcal{G}(t,\psi_1) - \mathcal{G}(t,\psi_2)| \leq \frac{\sin e}{5}|\psi_1 - \psi_2|.$$

Thus, the assumption (**A3**) holds true with $L = \frac{\sin e}{5}$.

In view of the above parameters, the condition (4) can be obtained as:

$$\frac{L|\lambda|(\ln b - \ln a)^{\gamma}}{\delta^{\gamma}\Gamma(\gamma+1)} = \frac{2\sin e}{25\Gamma(3/2)} \approx 4.28108 \times 10^{-3} < 1.$$

Therefore, all the conditions of the Theorem 2.1 hold true. Hence, the GPHFIE (5.1) has a unique solution on $[1, e]$.

6. Conclusions

Based on Picard and Picard-Krasnoselskii iteration methods with the Banach contraction principle, novel existence and uniqueness results for a new category of generalized proportional Hadamard integral equations are established. In order to illustrate the obtained results, an example is proposed.

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