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On boundary value problems for the Boussinesq-type equation with dynamic and non-dynamic boundary conditions

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Abstract

The work studies boundary value problems with non-dynamic and dynamic boundary conditions for one- and two-dimensional Boussinesq-type equations in domains representing a trapezoid, triangle, "curvilinear" trapezoid, "curvilinear" triangle, truncated cone, cone, truncated "curvilinear" cone, and "curvilinear" cone. Combining the methods of the theory of monotone operators and a priori estimates, in Sobolev classes, we have established theorems on the unique weak solvability of the boundary value problems under study.

Keywords: Boussinesq equation, dynamical boundary condition, non-cylindrical domains, non-dynamical boundary condition .
2010 MSC: 35K10, 35K20, 35K55.

1. Introduction

The theory of Boussinesq equations and its modifications always attracts the attention of mathematicians and applied scientists. The Boussinesq equation, as well as its modifications, take an important place in the description of the motion of liquids and gases, including in the theory of non-stationary filtration in porous media [1]–[11]. The works [12]–[17] can also be noted here. In recent years, boundary value problems for these equations have been actively studied, since they simulate processes in porous media. Processes occurring

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in porous media acquire special importance for deep understanding and comprehension in the problems of exploration and efficient development of oil and gas fields.

The papers [18]–[20] previously studied the solvability of the boundary value problems for one-dimensional Boussinesq-type and Burgers equations with Dirichlet boundary conditions in a domain, which is represented by a trapezoid or a triangle, respectively.

In this paper, we study the questions of the correctness of the formulation of boundary value problems for one- and two-dimensional Boussinesq-type equations in a domain on the moving part of the boundary of which dynamic nonlinear conditions are set. Domains are represented by a trapezoid, triangle, truncated cone, cone, truncated "curvilinear" cone, and "curvilinear" cone. We establish theorems on the unique weak solvability of the considered boundary problems.

2. Statements of the problems and main results

Problem 1. Let $\Omega_t = \{0 < x < t\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$. In the domain $Q_{xt} = \Omega_t \times (t_0, T)$, which is a trapezoid, we consider an initial boundary value problem for Boussinesq-type equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f, \;\; (x,t) \in Q_{xt}, \tag{1}$$

with boundary conditions

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t) \;\text{ at } \{x = t, \, t \in (t_0, T)\}, \;\; u = 0 \;\text{ at } \{x = 0, \, t \in (t_0, T)\}, \tag{2}$$

where $\frac{du(t,t)}{dt} = [\partial_t u(x,t) + \partial_x u(x,t)]_{|x=t}$, and with initial condition

$$u(x, t_0) = u_0(x), \;\; x \in \Omega_{t_0} = (0, t_0); \;\; u(t_0, t_0) = u_{00}, \tag{3}$$

where $f(x,t)$, $g(t)$, $u_0(x)$ are given functions, u_{00} is a given number.

Problem 2. Let $\Omega_t = \{0 < x < t\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $T < \infty$. In the domain $Q_{xt} = \Omega_t \times (0, T)$, representing a triangle, we consider a boundary value problem for the Boussinesq-type equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f, \;\; (x,t) \in Q_{xt}, \tag{4}$$

with boundary conditions

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t) \;\text{ at } \{x = t, \, t \in (0, T)\}, \;\; u = 0 \;\text{ at } \{x = 0, \, t \in (0, T)\}, \tag{5}$$

where $\frac{du(t,t)}{dt} = [\partial_t u(x,t) + \partial_x u(x,t)]_{|x=t}$, and the functions $f(x,t)$, $g(t)$ are given.

Problem 3. Let $\Omega_t = \{0 < x < \varphi(t)\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$, $\varphi(t) \in C^1([t_0, T])$, $0 \leq \varphi'(t) \leq C_1 = \text{const}$, $\varphi(t_0) > 0$. In the domain $Q_{xt} = \Omega_t \times (t_0, T)$, representing a "curvilinear" trapezoid, we consider the initial-boundary problem for the Boussinesq-type equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f, \;\; (x,t) \in Q_{xt}, \tag{6}$$

with boundary conditions

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t) \;\text{ at } \{x = \varphi(t), \, t \in (t_0, T)\}, \;\; u = 0 \;\text{ at } \{x = 0, \, t \in (t_0, T)\}, \tag{7}$$

where $\frac{du(t,t)}{dt} = [\partial_t u(x,t) + \varphi'(t)\partial_x u(x,t)]_{|x=\varphi(t)}$, and with initial condition

$$u(x, t_0) = u_0(x), \;\; x \in \Omega_{t_0} = (0, t_0); \;\; u(\varphi(t_0), t_0) = u_{00}, \tag{8}$$

where $f(x,t)$, $g(t)$, $u_0(x)$ are given functions, u_{00} is a given number.

Problem 4. Let $\Omega_t = \{0 < x < \varphi(t)\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $T < \infty$, $\varphi(t) \in C^1([0,T])$, $0 \leq \varphi'(t) \leq C_1 = \text{const}$, $\varphi(0) = 0$, $\exists\,\varepsilon : 0 < \varepsilon \ll T, \varphi'(t) = 1\ \forall t \in [0,\varepsilon]$. In the domain $Q_{xt} = \Omega_t \times (0,T)$, which is a "curvilinear" triangle, we consider a boundary value problem for Boussinesq-type equation

$$\partial_t u - \partial_x\left(|u|\partial_x u\right) = f,\ \ (x,t) \in Q_{xt}, \tag{9}$$

with boundary conditions

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t)\ \text{ at } \{x = \varphi(t),\, t \in (0,T)\},\ \ u = 0\ \text{ at } \{x = 0,\, t \in (0,T)\}, \tag{10}$$

where $\frac{du(t,t)}{dt} = \left[\partial_t u(x,t) + \varphi'(t)\partial_x u(x,t)\right]_{|x=\varphi(t)}$, and the functions $f(x,t)$, $g(t)$ are given.

Remark 2.1. *We consider problems with a dynamic boundary condition only on the moving part of the boundary. The latter in no way detracts from the generality; this is done only for the sake of simplicity of presentation. It would be possible to put a dynamic condition on the fixed part of the boundary as well.*

Problem 5. Let $x = (x_1, x_2)$, $\Omega_t = \{|x| < t\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$. In the domain $Q_{xt} = \Omega_t \times (t_0, T)$, which is a truncated cone, we consider an initial-boundary problem for a two-dimensional Boussinesq-type equation

$$\partial_t u - \sum_{i=1}^{2} \partial_{x_i}\left(|u|\partial_{x_i} u\right) = f,\ \ (x,t) \in Q_{xt}, \tag{11}$$

with boundary conditions

$$D_t u + \frac{1}{2}|u|u = g(x,t)\ \text{ at } (x,t) \in \Sigma_{xt} \equiv \partial\Omega_t \times (t_0, T), \tag{12}$$

where $D_t u(x,t)\big|_{|x|=t} \triangleq \left[\partial_t u(x,t) + \partial_{\vec{n}} u(x,t)\right]\big|_{|x|=t}$, $\vec{n}$ is a unit outward normal to the circle $|x| = t$, and with initial condition

$$u(x,t_0) = u_0(x),\ \ x \in \Omega_{t_0},\ \ u(x,t_0) = u_{00}(x),\ \ x \in \partial\Omega_{t_0}, \tag{13}$$

where $f(x,t)$, $g(x,t)$, $u_0(x)$, $u_{00}(x)$ are given functions.

Problem 6. Let $x = (x_1, x_2)$, $\Omega_t = \{|x| < t\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $T < \infty$. In the domain $Q_{xt} = \Omega_t \times (0,T)$, which is a cone, we consider a boundary value problem for a two-dimensional Boussinesq-type equation

$$\partial_t u - \sum_{i=1}^{2} \partial_{x_i}\left(|u|\partial_{x_i} u\right) = f,\ \ (x,t) \in Q_{xt}, \tag{14}$$

with boundary conditions

$$D_t u + \frac{1}{2}|u|u = g(x,t)\ \text{ at } (x,t) \in \Sigma_{xt} \equiv \partial\Omega_t \times (0, T), \tag{15}$$

where $D_t u(x,t)\big|_{|x|=t} \triangleq \left[\partial_t u(x,t) + \partial_{\vec{n}} u(x,t)\right]\big|_{|x|=t}$, $\vec{n}$ is a unit outward normal to the circle $|x| = t$, and the functions $f(x,t)$, $g(x,t)$ are given.

Problem 7. Let $x = (x_1, x_2)$, $\Omega_t = \{|x| < \varphi(t)\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $0 < t_0 < T < \infty$, $\varphi(t) \in C^1([t_0,T])$, $0 \leq \varphi'(t) \leq C_1 = \text{const}$, $\varphi(t_0) > 0$. In the domain $Q_{xt} = \Omega_t \times (t_0, T)$, representing a truncated cone (with a curvilinear generatrix determined by the function $\varphi(t)$), we consider an initial boundary value problem for a two-dimensional Boussinesq-type equation

$$\partial_t u - \sum_{i=1}^{2} \partial_{x_i} (|u| \partial_{x_i} u) = f, \ \ (x,t) \in Q_{xt}, \tag{16}$$

with boundary conditions

$$D_t u + \frac{1}{2}|u|u = g(x,t) \ \text{ at } (x,t) \in \Sigma_{xt} \equiv \partial\Omega_t \times (t_0, T), \tag{17}$$

where $D_t u(x,t)\big|_{|x|=\varphi(t)} \triangleq [\partial_t u(x,t) + \varphi'(t)\partial_{\vec{n}} u(x,t)]\big|_{|x|=\varphi(t)}$, $\vec{n}$ is a unit outward normal to the circle $|x| = \varphi(t)$, and with initial condition

$$u(x,t_0) = u_0(x), \ \ x \in \Omega_{t_0}, \ \ u(x,t_0) = u_{00}(x), \ \ x \in \partial\Omega_{t_0}, \tag{18}$$

where $f(x,t)$, $g(x,t)$, $u_0(x)$, $u_{00}(x)$ are given functions.

Problem 8. Let $x = (x_1, x_2)$, $\Omega_t = \{|x| < \varphi(t)\}$ and $\partial\Omega_t$ be the boundary of Ω_t, $T < \infty$, $\varphi(t) \in C^1([0,T])$, $0 \le \varphi'(t) \le C_1 = \text{const}$, $\varphi(0) = 0$, $\exists\, \varepsilon : 0 < \varepsilon \ll T, \varphi'(t) = 1 > 0 \ \forall\, t \in [0, \varepsilon]$. In the domain $Q_{xt} = \Omega_t \times (0,T)$, representing a cone (with a curvilinear generatrix determined by the function $\varphi(t)$), we consider a boundary value problem for a two-dimensional Boussinesq-type equation

$$\partial_t u - \sum_{i=1}^{2} \partial_{x_i} (|u| \partial_{x_i} u) = f, \ \ (x,t) \in Q_{xt}, \tag{19}$$

with boundary conditions

$$D_t u + \frac{1}{2}|u|u = g(x,t) \ \text{ at } (x,t) \in \Sigma_{xt} \equiv \partial\Omega_t \times (0, T), \tag{20}$$

where $D_t u(x,t)\big|_{|x|=\varphi(t)} \triangleq [\partial_t u(x,t) + \varphi'(t)\partial_{\vec{n}} u(x,t)]\big|_{|x|=\varphi(t)}$, $\vec{n}$ is a unit outward normal to the circle $|x| = \varphi(t)$, and the functions $f(x,t)$, $g(x,t)$ are assumed to be given.

3. Main results

Using and developing the results of [18]–[19], we have established the validity of the following theorems.

Theorem 3.1 (Trapezoid). *Let*

$$\begin{aligned} &f \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)), && u_0 \in H^{-1}(\Omega_{t_0}), \\ &g \in L_{3/2}((t_0,T)), && u_{00} \text{ is a given number.} \end{aligned} \tag{21}$$

Then the initial boundary value problem (1)–(3) *has a unique solution*

$$\begin{aligned} &u \in L_3((t_0,T); L_3(\Omega_t)) \cap L_\infty((t_0,T); H^{-1}(\Omega_t)), \\ &\partial_t u \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)), \\ &v(s) \in L_3(0, \sqrt{2}\,(T-t_0)), \\ &v'(s) \in L_{3/2}(0, \sqrt{2}\,(T-t_0)), \end{aligned} \tag{22}$$

where $t \in (t_0,T)$, $s \in (0, \sqrt{2}\,(T-t_0))$, $v(s)\big|_{s=\sqrt{2}\,(t-t_0)} = u(t,t)$, $s = s(t) = \sqrt{2}\,(t-t_0)$.

Theorem 3.2 (Triangle)**.** *Let*

$$f \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)), \;\; g \in L_{3/2}((0,T)). \tag{23}$$

Then the boundary value problem (4)–(5) *has a unique solution*

$$\begin{aligned} & u \in L_3((0,T); L_3(\Omega_t)) \cap L_\infty((0,T); H^{-1}(\Omega_t)), \\ & \partial_t u \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)), \\ & v(s) \in L_3(0, \sqrt{2}\,T) \cap L_\infty(0, \sqrt{2}\,T), \\ & v'(s) \in L_{3/2}(0, \sqrt{2}\,T), \end{aligned} \tag{24}$$

where $t \in (0,T),\; s \in (0, \sqrt{2}\,T),\; v(s)\big|_{s=\sqrt{2}\,t} = u(t,t),\; s = s(t) = \sqrt{2}\,t$*; and for* $\{x \to 0+,\; x \to t-0,\; t \to 0+\}$ *we have*

$$\begin{cases} u(x,t) = \mathcal{O}\left(x^{-\alpha_0}(t-x)^{-\alpha+\alpha_0}t^{-\beta}\right), \\ \alpha_0 \le \alpha < \frac{1}{3}, \;\; \alpha + \beta < \frac{2}{3}. \end{cases} \tag{25}$$

Theorem 3.3 ("Curvilinear" trapezoid)**.** *Let*

$$\begin{aligned} & f \in L_{3/2}((t_0,T); W_{3/2}^{-1}(\Omega_t)), \quad && u_0 \in H^{-1}(\Omega_{t_0}), \\ & g \in L_{3/2}((t_0,T)), && u_{00} \text{ is a given number.} \end{aligned} \tag{26}$$

Then the initial boundary value problem (6)–(8) *has a unique solution*

$$\begin{aligned} & u \in L_3((t_0,T); L_3(\Omega_t)) \cap L_\infty((t_0,T); H^{-1}(\Omega_t)), \\ & \partial_t u \in L_{3/2}((t_0,T); W_{3/2}^{-1}(\Omega_t)), \\ & v(s) \in L_3(0, s(T)), \\ & v'(s) \in L_{3/2}(0, s(T)), \end{aligned} \tag{27}$$

where $v(s)\big|_{s=s(t)} = u(\varphi(t),t),\;\; s = s(t) = \int\limits_{t_0}^{t} \sqrt{1+[\varphi'(\tau)]^2}d\tau,\; s \in (0, s(T)),\; t \in (t_0, T).$

Theorem 3.4 ("Curvilinear" triangle)**.** *Let*

$$f \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)), \;\; g \in L_{3/2}((0,T)). \tag{28}$$

Then the boundary value problem (9)–(10) *has a unique solution*

$$\begin{aligned} & u \in L_3((0,T); L_3(\Omega_t)) \cap L_\infty((0,T); H^{-1}(\Omega_t)), \\ & \partial_t u \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)), \\ & v(s) \in L_3(0, s(T)) \cap L_\infty(0, s(T)), \\ & v'(s) \in L_{3/2}(0, s(T)), \end{aligned} \tag{29}$$

where $v(s)\big|_{s=s(t)} = u(\varphi(t),t),\;\; s = s(t) = \int\limits_{t_0}^{t} \sqrt{1+[\varphi'(\tau)]^2}d\tau,\; s \in (0, s(T)),\; t \in (0,T)$*; and for* $\{x \to 0+,\; x \to t-0,\; t \to 0+\}$ *we have*

$$\begin{cases} u(x,t) = \mathcal{O}\left(x^{-\alpha_0}(t-x)^{-\alpha+\alpha_0}t^{-\beta}\right), \\ \alpha_0 \le \alpha < \frac{1}{3}, \;\; \alpha + \beta < \frac{2}{3}. \end{cases} \tag{30}$$

Theorem 3.5 (Truncated cone). *Let*

$$\begin{aligned} &f \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)), \qquad u_0 \in H^{-1}(\Omega_{t_0}),\\ &g \in L_{3/2}((t_0,T); W^{-2/3}_{3/2}(\partial\Omega_t)), \quad u_{00} \in L_2(\Omega_{t_0}) \text{ are given functions.} \end{aligned} \tag{31}$$

Then the initial boundary value problem (11)–(13) *has a unique solution*

$$\begin{aligned} &u \in L_3((t_0,T); L_3(\Omega_t)) \cap L_\infty((t_0,T); H^{-1}(\Omega_t)),\\ &\partial_t u \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)),\\ &v(x,s) \in L_3((0,\sqrt{2}\,(T-t_0)); W^{2/3}_3(\partial\Omega_t)) \cap L_\infty((0,\sqrt{2}\,(T-t_0)); L_2(\partial\Omega_t)),\\ &\partial_s v(x,s) \in L_{3/2}((0,\sqrt{2}\,(T-t_0)); W^{-2/3}_{3/2}(\partial\Omega_t)), \end{aligned} \tag{32}$$

where $t \in (t_0,T),\; s = \sqrt{2}\,(t-t_0) \in (0, \sqrt{2}\,(T-t_0))$,

$$v(x,s)\big|_{s=\sqrt{2}\,(t-t_0),\,|x|=t} = u(x,t)\big|_{|x|=t},$$

$$\partial_s v(x,s)\big|_{s=\sqrt{2}\,(t-t_0),\,|x|=t} = D_t u(x,t)\big|_{|x|=t} \triangleq [\partial_t u(x,t) + \partial_{\vec{n}} u(x,t)]\,\big|_{|x|=t}.$$

Theorem 3.6 (Cone). *Let*

$$f \in L_{3/2}((0,T); W^{-1}_{3/2}(\Omega_t)), \;\; g \in L_{3/2}((0,T); W^{-2/3}_{3/2}(\partial\Omega_t)). \tag{33}$$

Then the boundary value problem (14)–(15) *has a unique solution*

$$\begin{aligned} &u \in L_3((0,T); L_3(\Omega_t)) \cap L_\infty((0,T); H^{-1}(\Omega_t)),\\ &\partial_t u \in L_{3/2}((0,T); W^{-1}_{3/2}(\Omega_t)),\\ &v(x,s) \in L_3((0,\sqrt{2}\,T); W^{2/3}_3(\partial\Omega_t)) \cap L_\infty((0,\sqrt{2}\,T); L_2(\partial\Omega_t)),\\ &\partial_s v(x,s) \in L_{3/2}((0,\sqrt{2}\,T); W^{-2/3}_{3/2}(\partial\Omega_t)), \end{aligned} \tag{34}$$

where $t \in (0,T),\; s = \sqrt{2}\,t \in (0, \sqrt{2}\,T)$,

$$v(x,s)\big|_{s=\sqrt{2}\,t,\,|x|=t} = u(x,t)\big|_{|x|=t},$$

$$\partial_s v(x,s)\big|_{s=\sqrt{2}\,t,\,|x|=t} = D_t u(x,t)\big|_{|x|=t} \triangleq [\partial_t u(x,t) + \partial_{\vec{n}} u(x,t)]\,\big|_{|x|=t};$$

and for $\{|x| \to 0+,\; |x| \to t-0,\; t \to 0+\}$ *we have*

$$\begin{cases} u(x,t) = \mathcal{O}\left(|x|^{-\alpha_0}(t-|x|)^{-\alpha+\alpha_0} t^{-\beta}\right),\\ \alpha_0 \le \alpha < \frac{1}{3}, \;\; \alpha+\beta < \frac{2}{3}. \end{cases} \tag{35}$$

Theorem 3.7 (Truncated "curvilinear" cone). *Let*

$$\begin{aligned} &f \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)), \qquad u_0 \in H^{-1}(\Omega_{t_0}),\\ &g \in L_{3/2}((t_0,T); W^{-2/3}_{3/2}(\partial\Omega_t)), \quad u_{00} \in L_2(\Omega_{t_0}) \text{ are given functions.} \end{aligned} \tag{36}$$

Then the initial boundary value problem (16)–(18) *has a unique solution*

$$\begin{aligned} &u \in L_3((t_0,T); L_3(\Omega_t)) \cap L_\infty((t_0,T); H^{-1}(\Omega_t)),\\ &\partial_t u \in L_{3/2}((t_0,T); W^{-1}_{3/2}(\Omega_t)),\\ &v(x,s) \in L_3((0,s(T)); W^{2/3}_3(\partial\Omega_t)) \cap L_\infty((0,s(T)); L_2(\partial\Omega_t)),\\ &\partial_s v(x,s) \in L_{3/2}((0,s(T)); W^{-2/3}_{3/2}(\partial\Omega_t)), \end{aligned} \tag{37}$$

where

$$t \in (t_0, T),\ \ s \in (0, s(T)),\ \ s = s(t) = \int\limits_{t_0}^{t} \sqrt{1 + [\varphi'(\tau)]^2} d\tau,$$

$$v(x,s)\big|_{s=s(t),\,|x|=\varphi(t)} = u(x,t)\big|_{|x|=\varphi(t)},$$

$$\partial_s v(x,s)\big|_{s=s(t),\,|x|=\varphi(t)} = D_t u(x,t)\big|_{|x|=\varphi(t)} \triangleq \left[\partial_t u(x,t) + \varphi'(t)\partial_{\vec{n}} u(x,t)\right]\big|_{|x|=\varphi(t)}.$$

Theorem 3.8 ("Curvilinear" cone)**.** *Let*

$$f \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)),\ \ g \in L_{3/2}((0,T); W_{3/2}^{-2/3}(\partial\Omega_t)). \tag{38}$$

Then the boundary value problem (19)–(20) *has a unique solution*

$$\begin{aligned} & u \in L_3((0,T); L_3(\Omega_t)) \cap L_\infty((0,T); H^{-1}(\Omega_t)), \\ & \partial_t u \in L_{3/2}((0,T); W_{3/2}^{-1}(\Omega_t)), \\ & v(x,s) \in L_3((0,s(T)); W_3^{2/3}(\partial\Omega_t)) \cap L_\infty((0,s(T)); L_2(\partial\Omega_t)), \\ & \partial_s v(x,s) \in L_{3/2}((0,s(T)); W_{3/2}^{-2/3}(\partial\Omega_t)), \end{aligned} \tag{39}$$

where

$$t \in (0, T),\ \ s \in (0, s(T)),\ \ s = s(t) = \int\limits_{0}^{t} \sqrt{1 + [\varphi'(\tau)]^2} d\tau,$$

$$v(x,s)\big|_{s=s(t),\,|x|=\varphi(t)} = u(x,t)\big|_{|x|=\varphi(t)},$$

$$\partial_s v(x,s)\big|_{s=s(t),\,|x|=\varphi(t)} = D_t u(x,t)\big|_{|x|=\varphi(t)} \triangleq \left[\partial_t u(x,t) + \varphi'(t)\partial_{\vec{n}} u(x,t)\right]\big|_{|x|=\varphi(t)};$$

and for $\{|x| \to 0+,\ |x| \to t-0,\ t \to 0+\}$ *we have*

$$\begin{cases} u(x,t) = \mathcal{O}\left(|x|^{-\alpha_0}(t-|x|)^{-\alpha+\alpha_0} t^{-\beta}\right), \\ \alpha_0 \le \alpha < \frac{1}{3},\ \ \alpha + \beta < \frac{2}{3}. \end{cases} \tag{40}$$

4. Schemes of proofs of theorems 3.1–3.8

Let us give a scheme of the proof using Theorems 3.1–3.4. For example, Problem 1 is divided into two subproblems:

Problem 1.1. Find a solution to the following Cauchy problem for an (ordinary) differential equation

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t) \ \text{ at } \{x = t,\, t \in (t_0, T)\}, \tag{41}$$

where $\frac{du(t,t)}{dt} = [\partial_t u(x,t) + \partial_x u(x,t)]_{|x=t}$, with initial condition

$$u(t_0, t_0) = u_{00}, \tag{42}$$

where $g(t)$ is a given function, u_{00} is a given number.

Under the conditions of Theorem 3.1 in the Cauchy problem (41)–(42) the operator $\frac{1}{2}|u|u$ has the monotonicity condition. This allows us to establish the validity of the assertion that Problem 1.1 has a unique solution $\{v(s),\ s \in (0, \sqrt{2}\,T)\}$, moreover $v(s) \in L_3(0, \sqrt{2}\,(T-t_0)),\ v'(s) \in L_{3/2}(0, \sqrt{2}\,(T-t_0))$, which allows us to obtain the Dirichlet boundary condition $u(t,t) = v(\sqrt{2}\,(t-t_0))$, on the moving boundary $x = t$ of the domain Q_{xt} where $u(t_0, t_0) = v(0) = u_{00}$.

Thus, we get the following initial boundary value problem

Problem 1.2. Find a solution to the initial boundary value problem for the Boussinesq equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f, \ \ (x,t) \in Q_{xt}, \tag{43}$$

with boundary conditions

$$u(x,t) = h(t) \ \text{ at } \{x = t,\, t \in (t_0, T)\}, \ \ u(x,t) = 0 \ \text{ at } \{x = 0,\, t \in (t_0, T)\}, \tag{44}$$

and with initial condition

$$u(x, t_0) = u_0(x), \ \ x \in \Omega_{t_0} = (0, t_0), \tag{45}$$

where $f(x,t)$, $h(t) = v(\sqrt{2}\,(t - t_0))$, $u_0(x)$ are given functions.

The solvability of the problem (43)–(45) was previously established by us in [18].

Thus, the solvability of Problems 1.1 and 1.2 allows us to obtain the assertion of Theorem 3.1. This is a brief outline of the proof of this theorem.

Now about the proof of Theorem 3.2. First of all, let us formulate an analog of Problem 1.1.

Problem 2.1. Find a solution to the Cauchy problem for an (ordinary) differential equation

$$\frac{du}{dt} + \frac{1}{2}|u|u = g(t) \ \text{ at } \{x = t,\, t \in (0, T)\}, \tag{46}$$

where $\frac{du(t,t)}{dt} = [\partial_t u(x,t) + \partial_x u(x,t)]_{|x=t}$, with initial condition

$$u(0, 0) = 0, \tag{47}$$

where $g(t)$ is a given function.

Under the conditions of Theorem 3.2 in the Cauchy problem (46)–(47) the operator $\frac{1}{2}|u|u$ has the monotonicity condition. This allows us to establish the validity of the assertion that Problem 2.1 has a unique solution $\{v(s),\ s \in (0, \sqrt{2}T)\}$, moreover $v(s) \in L_3(0, \sqrt{2}T)$, $v'(s) \in L_{3/2}(0, \sqrt{2}T)$, which allows us to obtain the Dirichlet boundary condition $u(t,t) = v(\sqrt{2}\,t)$, where $u(0,0) = v(0) = 0$.

Thus, we get the following boundary value problem

Problem 2.2. Find a solution to the boundary value problem for the Boussinesq equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f, \ \ (x,t) \in Q_{xt}, \tag{48}$$

with boundary conditions

$$u(x,t) = h(t) \ \text{ at } \{x = t,\, t \in (0, T)\}, \ \ u(x,t) = 0 \ \text{ at } \{x = 0,\, t \in (0, T)\}, \tag{49}$$

where $f(x,t)$, $h(t) = v(\sqrt{2}\,t)$ are given functions.

The solvability of the problem (48)–(49) was established by us earlier in [19].

Remark 4.1. *Let us show that the solution $u(x,t)$ of the boundary value problem (23)–(24) having a singularity of the order specified in (25) will belong to the space $L_3(Q_{xt}^{t_0})$, where $Q_{xt}^{t_0} = \{x, t |\, 0 < x < t,\ 0 < t < t_0 \ll T\}$. For this purpose, it suffices to show that the following integral is bounded for $t_0 \to 0+$:*

$$\int_{Q_{xt}^{t_0}} x^{-3\alpha_0}(t - x)^{-3\alpha+3\alpha_0} t^{-3\beta}\, dx\, dt. \tag{50}$$

We have

$$\int_0^{t_0} t^{-3\beta} \int_0^t x^{-3\alpha_0}(t - x)^{-3\alpha+3\alpha_0}\, dx\, dt = \left\| \begin{array}{l} x = t\sin^2\theta \\ 0 < \theta < \pi/2 \\ dx = 2\sin\theta\cos\theta\, d\theta \end{array} \right\| =$$

$$= 2\int_0^{t_0} t^{1-3\alpha-3\beta} \int_0^{\pi/2} \sin^{1-6\alpha_0}\theta \cos^{1-6\alpha+6\alpha_0}\theta \, d\theta \, dt.$$

It is easy to check that under the conditions of Theorem 3.2, in the last expression the inner integral takes a finite value. Calculating the outer integral, we have

$$\int_0^{t_0} t^{1-3\alpha-3\beta} \, dt = \frac{1}{2-3(\alpha+\beta)} t_0^{2-3(\alpha+\beta)},$$

which, under the conditions of Theorem ("1"), is also bounded from above.

Note that if the order of the singularity of the solution $u(x,t)$ is higher than in (25), then this function is no longer an element of the space $L_3(Q_{xt}^{t_0})$.

Thus, the solvability of Problems 2.1 and 2.2 allows us to obtain the assertion of Theorem 2.2. This is a brief outline of the proof of this theorem.

Theorems 3.3 and 3.4 are proved similarly to Theorems 3.1 and 3.2.

Let us proceed to the proof of Theorems 3.5–3.8. Let us give a proof scheme using the theorems 3.5–3.6 as an example. For example, Problem 5 is divided into two subproblems:

Problem 5.1. Find a solution to the following Cauchy problem for a differential equation

$$D_t u + \frac{1}{2}|u|u = g(x,t) \text{ at } \{|x| = t, \, t \in (t_0, T)\}, \tag{51}$$

where $D_t u(x,t) \triangleq [\partial_t u(x,t) + \partial_{\vec{n}} u(x,t)]\big|_{|x|=t}$, $\vec{n}$ is a unit outward normal to the circle $|x| = t$, with initial condition

$$u(x,t_0) = u_{00}(x), \; x \in \{|x| = t\}, \tag{52}$$

where $g(x,t)$, $u_{00}(x)$ are given functions.

Under the conditions of Theorem 3.5 in the Cauchy problem (51)–(52) the operator $\frac{1}{2}|u|u$ has the monotonicity condition. This allows us to establish the validity of the assertion that Problem 5.1 has a unique solution $\{v(x,s), \; s \in (0, \sqrt{2}T)\}$, moreover $v(x,s) \in L_3((0, \sqrt{2}(T - t_0)); W_3^{2/3}(\partial\Omega_t)) \cap L_\infty((0, \sqrt{2}(T - t_0)); L_2(\partial\Omega_t))$, $\partial_s v(x,s) \in L_{3/2}((0, \sqrt{2}(T-t_0)); W_{3/2}^{-2/3}(\partial\Omega_t))$, which allows us to obtain the Dirichlet boundary condition on the moving boundary $|x| = t$ of Q_{xt} $u(x,t)\big|_{|x|=t} = v(x, \sqrt{2}(t-t_0))$, where $u(x,t_0) = v(x,0) = u_{00}(x), \; x \in \{|x| = t\}$.

Thus, we get the following initial-boundary problem

Problem 5.2. Find a solution to the initial boundary value problem for the Boussinesq equation

$$\partial_t u - \partial_x(|u|\partial_x u) = f, \;\; (x,t) \in Q_{xt}, \tag{53}$$

with boundary conditions

$$u(x,t) = h(x,t) \text{ at } \{|x| = t, \, t \in (t_0, T)\}, \tag{54}$$

and with initial condition

$$u(x,t_0) = u_0(x), \;\; x \in \Omega_{t_0} = (|x| < t_0), \tag{55}$$

where $f(x,t)$, $h(x,t) = v(x, \sqrt{2}(t-t_0))$ are given functions.

The solvability of the problem (53)–(55) is established in the same way as in [18], following [21]–[24].

Thus, the solvability of Problems 5.1 and 5.2 allows us to obtain the assertion of Theorem 3.5. This is a brief outline of the proof of this theorem.

Now about the proof of Theorem 3.6. First of all, we formulate an analog of Problem 5.1.

Problem 6.1. Find a solution to the Cauchy problem for a differential equation

$$D_t u + \frac{1}{2}|u|u = g(x,t) \text{ at } \{|x| = t, \, t \in (0,T)\}, \tag{56}$$

where $D_t u(x,t) \triangleq [\partial_t u(x,t) + \partial_x u(x,t)]\big|_{|x|=t}$, with initial condition

$$u(x,0) = u_{00}(x),\ x \in \{|x| = 1\}, \tag{57}$$

where $g(x,t)$ is a given function.

Under the conditions of Theorem 3.6 in the Cauchy problem (56)–(57) the operator $\frac{1}{2}|u|u$ has the monotonicity condition. This allows us to establish the validity of the assertion that Problem 6.1 has a unique solution $\{v(x,s),\ s \in (0,\sqrt{2}\,T)\}$, moreover $v(x,s) \in L_3((0,\sqrt{2}\,T); W_3^{2/3}(\partial\Omega_t)) \cap L_\infty((0,\sqrt{2}\,T); L_2(\partial\Omega_t))$, $\partial_s v(x,s) \in L_{3/2}((0,\sqrt{2}\,T); W_{3/2}^{-2/3}(\partial\Omega_t))$, which allows us to obtain the Dirichlet boundary condition on the moving boundary $|x| = t$ of Q_{xt} $u(x,t)\big|_{|x|=t} = v(x,\sqrt{2}\,t)$, where $u(x,0) = v(x,0) = u_{00}(x)$.

Thus, we get the following boundary value problem

Problem 6.2. Find a solution to the boundary value problem for the Boussinesq equation

$$\partial_t u - \partial_x (|u|\partial_x u) = f,\ \ (x,t) \in Q_{xt}, \tag{58}$$

with boundary conditions

$$u(x,t) = h(x,t) \text{ at } \{x = t,\ t \in (0,T)\},\ \ u(x,t) = 0 \text{ at } \{x = 0,\ t \in (0,T)\}, \tag{59}$$

where $f(x,t)$, $h(x,t) = v(x,\sqrt{2}\,t)$ are given functions.

The solvability of the problem (58)–(59) is set in the same way as in [19], following [21]–[24].

Thus, the solvability of Problems 6.1 and 6.2 allows us to obtain the assertion of Theorem 2.6. This is a brief outline of the proof of this theorem.

Theorems 3.7 and 3.8 are proved similarly to Theorems 3.5 and 3.6.

Conclusion

In the work boundary value problems for one- and two-dimensional Boussinesq-type equations in domains representing a trapezoid, a triangle, a "curvilinear" trapezoid, a "curvilinear" triangle, a truncated cone, a cone, a truncated "curvilinear" cone, and " curvilinear" cone are studied. Using the methods of the theory of monotone operators and a priori estimates, we prove theorems on their unique weak solvability in Sobolev classes.

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