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# RECURRENT SYSTEMS OF TENSORS AND GAUGE FIELDS IN SPACE-TIME (*)

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ABSTRACT

The concept of a recurrent system of tensors in Riemannian geometry is defined and it is shown that the recurrence vectors of the system behave like potentials of a gauge filed. This allows a generalization of Maxwell's equations to all gauge phenomena including gravitation.

## 1. INTRODUCTION.

The present article is concerned with a generalization of the classical definition of a recurrent tensor and with a physical interpretation of recurrent systems in space-time.

As is wellknown, a nonzero tensor $\lambda$ in a Riemannian space is called recurrent if

$$\nabla_i \lambda = A_i \lambda \tag{1.1}$$

with $\nabla$ denoting the operator of covariant differentiation relative to the Riemannian connection.

Let Greek indices range from 1 to r with the usual summation convention. Consider a set of r linearly independent tensors $\lambda_\alpha$ of order s such that

$$\nabla_i \lambda_\alpha = A^\beta{}_{\alpha i} \lambda_\beta \tag{1.2}$$

for some vectors $A^\beta{}_{\alpha i}$ . If (1.2) holds then we will say that the $\lambda_\alpha$ define a recurrent system of the type (r, s) with the recurrence vectors $A^\beta{}_{\alpha i}$.

Here are two remarkable examples of tensors defining a recurrent system:

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* Dedicated to the memory of Dr. Saffet Nezihî ŞENER (1929–1956).

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i) The vectors of a basis for a field of parallel planes (cf. Willmore 1959).

ii) The generating bivectors of the infinitesimal holonomy group (cf. Hlavaty 1960).

## 2. PROPERTIES OF A RECURRENT SYSTEM.

Let us first note that the form of (1.2) remains, as can be easily verified, unaltered under the transformation

$$\lambda_\alpha \to G^\beta{}_\alpha \, \lambda_\beta \tag{2.1}$$

with the $G^\beta{}_\alpha$ forming a regular matrix G. From (1.2) it then follows that the matrix $A_i = (A^\beta{}_\alpha)_i$ transforms according to

$$A_i \to G^{-1} A_i G + G^{-1} \partial_i G. \tag{2.2}$$

In the study of recurrent systems it is convenient to introduce a new type of covariant differentiation D which operates with both Latin and Greek indices by means of coefficients of the Riemannian connection and $A^\beta{}_{\alpha i}$, respectively. With this convention (1.2) takes on the following simpler form:

$$D_i \, \lambda_\alpha = 0. \tag{2.3}$$

Now, by the Ricci identity, (2.3) yields*

$$\Sigma \, \lambda_{\alpha.a} \, R^a{}_{bij} + \lambda_\beta \, F^\beta{}_{\alpha ij} = 0, \tag{2.4}$$

where

$$F^\beta{}_{\alpha ij} = \nabla_i \, A^\beta{}_{\alpha j} - \nabla_j \, A^\beta{}_{\alpha i} + A^\beta{}_{\gamma i} \, A^\gamma{}_{\beta j} - A^\beta{}_{\gamma j} \, A^\gamma{}_{\alpha i} \tag{2.5}$$

or in matrix notation

$$F_{ij} = \nabla_i \, A_j - \nabla_j \, A_i + A_i \, A_j - A_j \, A_i. \tag{2.6}$$

It can also be easily verified that (2.1) induces on this matrix a similarity transformation:

$$F_{ij} \to G^{-1} F_{ij} \, G. \tag{2.7}$$

From (2.4) we have, in view of (2.3),

$$\Sigma \, \lambda_{\alpha.a} \, \nabla_k \, R^a{}_{bij} + \lambda_\beta \, D_k \, F^\beta{}_{\alpha ij} = 0. \tag{2.8}$$

As the $\lambda_\alpha$ are linearly independent, the Bianchi identity in (2.8) gives, if we rename tensor indices,

* Explicitly, (2.4) reads $\sum_u \lambda_{\alpha b_1 .. a .. b_s} \, R^a{}_{b_u ij} + \lambda_{\beta b_1 ... b_s} \, F^\beta{}_{\alpha ij} = 0$

$$D_{[c} F_{ab]} = O. \tag{2.9}$$

For reasons that will be apparent later on we introduce here the "divergence" of $F^{ab}$

$$D_a F^{ab} = J^b \tag{2.10}$$

and show that it is "conserved":

$$D_b J^b = O. \tag{2.11}$$

This is a direct consequence of the identity

$$(D_a D_b - D_b D_a) F^{\beta}{}_{\alpha}{}^{ij} = R^i{}_{cab} F^{\beta}{}_{\alpha}{}^{cj} + R^j{}_{cab} F^{\beta}{}_{\alpha}{}^{ic} + F^{\beta}{}_{\gamma ab} F^{\gamma}{}_{\alpha}{}^{ij} - F^{\gamma}{}_{\alpha ab} F^{\beta}{}_{\gamma}{}^{ij}. \tag{2.12}$$

But it can also be proved as follows. From (2.3) and (2.8) we first have

$$\Sigma \lambda_{\alpha.a} \nabla^h \nabla^k R^a{}_{bij} + \lambda_\beta D_h D_k F^{\beta}{}_{\alpha ij} = O. \tag{2.13}$$

Now if we contract (2.13) and make use of the identity

$$\nabla^i \nabla^j R^{ij}{}_{ab} = O \tag{2.14}$$

we see that (2.11) holds.

## 3. A PHYSICAL INTERPRETATION OF RECURRENT SYSTEMS IN SPACE-TIME.

The results of the preceding section, particularly (2.2), (2.6) and (2.7) suggest the idea of interpreting the $A^{\beta}{}_{\alpha i}$ as potential vectors of a gauge field $F_{ij}$. In accordence with this view (2.9) and (2.10) may be regarded as the generalized Maxwell's equations for the gauge field.

This analogy with electromagnetic theory will play a leading role in our subsequent considerations, as will be seen presently.

For the sake of generality, we consider two tensors $H_{ab}$ and $K_{ab}$ which may also possess other (Latin or Greek) indices except for being antisymmetric in the indices *ab*. If both tensors obey (2.9) one can show that (see Appendix) the (symmetric) tensor

$$T_{ab} = \tfrac{1}{2} (H_a{}^i K_{bi} + H_b{}^i K_{ai} - \tfrac{1}{2} g_{ab} H^{ij} K_{ij}) = \tfrac{1}{2} (H_a{}^i K_{bi} + H_a{}^{*i} K^*_{bi}) \tag{3.1}$$

satisfies the divergence condition

$$D_b T^{ab} = \tfrac{1}{2} (H^{ab} K_b + K^{ab} H_b) \tag{3.2}$$

with

$$D_a H^{ab}, = H^b, \; D_a K^{ab} = K^b. \tag{3.3}$$

If we set*

$$H_{ab} = F^{\mu}{}_{\nu ab}, \; K_{ab} = F^{\rho}{}_{\sigma ab} \tag{3.4}$$

the relations (3.1) become

$$T_{ab}{}^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma} = \tfrac{1}{2} (F^{\mu}{}_{\nu a}{}^{i} \, F^{\rho}{}_{\sigma bi} + F^{\mu}{}_{\nu b}{}^{i} \, F^{\rho}{}_{\sigma ai} - \tfrac{1}{2} g_{ab} F^{\mu}{}_{\nu}{}^{ij} F^{\rho}{}_{\sigma ij})$$
$$= \tfrac{1}{2} (F^{\mu}{}_{\nu a}{}^{i} \, F^{\rho}{}_{\sigma bi} + F^{\mu}{}_{\nu}{}^{*i}_{a} \, F^{\rho}{}_{\sigma}{}^{*}_{bi}) \tag{3.5}$$

and

$$D_b T^{ab}{}^{\mu}{}_{\nu}{}^{\rho}{}_{\sigma} = \tfrac{1}{2} (F^{\mu}{}_{\nu}{}^{ab} J^{\rho}{}_{\sigma b} + F^{\rho}{}_{\sigma}{}^{ab} J^{\mu}{}_{\nu b}), \tag{3.6}$$

where

$$D_a F^{\mu}{}_{\nu}{}^{ab} = J^{\mu}{}_{\nu}{}^{b}. \tag{3.7}$$

Contracting $\mu$ with $\sigma$, $\nu$ with $\rho$ in (3.5), (3.6):

$$T_{ab} = \mathrm{tr}\,(F_a{}^i F_{bi} - \tfrac{1}{4} g_{ab} F^{ij} F_{ij}) \tag{3.8}$$
$$= \mathrm{tr}\,\tfrac{1}{2} (F_a{}^i F_{bi} + F^{*i}_{a} F^{*}_{bi}),$$

$$\nabla_b T^{ab} = \mathrm{tr}\,(F^{ab} J_b), \tag{3.9}$$

where $T_{ab} = T_{ab}{}^{\mu}{}_{\nu}{}^{\nu}{}_{\mu}$ and tr denotes the trace of a matrix.

The tensor $T_{ab}$ defined by (3.8) is the energy tensor of the gauge field. In accordence with this interpretation (3.9) constitutes a generalization of the formula relative to the divergence of the Maxwell tensor in electromagnetic theory. Here, therefore, $J_b$ stands for the gauge current and tr $(F^{ab} J_b)$ for the generalized Lorentz force.

Gauge invariant expressions, like the energy tensor, are of some physical interest. In particular

$$f_{ab} = \mathrm{tr}\, F_{ab}, \; j^b = \mathrm{tr}\, J^b \tag{3.10}$$

display that property. If we take traces in (2.9), (2.10) we have

$$\nabla_{[c} f_{ab]} = 0, \tag{3.11}$$

$$\nabla_a f^{ab} = j^b. \tag{3.12}$$

It is worth remarking that (3.11), (3.12) are formally identical with Maxwell's equations for an electromagnetic field.

Note that the results obtained so far in this section rest upon (2.9), (2.10) alone which can also be thought of regardless of any recurrent system. But if we assume the existence of a recurrent system in space-time we are led to other conclusions as well.

* Note that this method applies to any pair of tensors of the types $F^{\mu}{}_{\nu ab}$, $R^m{}_{nab}$

First, consider a flat space-time. As the curvature tensor equals zero (2.4) implies

$$F_{ij} = O. \tag{3.13}$$

In other words, space-time cannot be flat if a gauge field is present.

Secondly, consider a space-time with the curvature condition

$$\nabla_a R^a{}_{bij} = O. \tag{3.14}$$

Contracting (2.8) then gives

$$D_i F^{ij} = O. \tag{3.15}$$

Noting that (3.14) holds in particular if Einstein's tensor is zero (absence of energy distribution) we conclude that every (nonzero) gauge current possesses an energy distribution in space-time.

## 4. GRAVITATION: A GAUGE PHENOMENON.

The aim of this section is to demonstrate that gravitation itself consists in a gauge phenomenon, and to indicate certain formulae pertaining to the gravitational field.

If the $\lambda_{\alpha a}$ denote a basis of the tangent space to space-time one can write

$$\nabla_i \lambda_{\alpha a} = A^\beta{}_{\alpha i} \lambda_{\beta a} \tag{4.1}$$

so that the $\lambda_{\alpha a}$ define a recurrent system in the sense of Sect. 1. Evidently, (2.1) in this case reduces to

$$\lambda_{\alpha a} R^a{}_{bij} + \lambda_{\beta b} F^\beta{}_{\alpha ij} = O. \tag{4.2}$$

The matrix $(\lambda_{\alpha a})$ being necessarily invertible, the gauge field $F_{ij}$ in (4.2) can be regarded as an equivalent representative of the gravitational field.

For a unified treatement of the gauge fields it is useful to introduce the differential forms

$$A = A_i \, dx^i \tag{4.3}$$

$$F = \tfrac{1}{2} F_{ij} \, dx^i \wedge dx^j \tag{4.4}$$

where $\wedge$ indicates the exterior multiplication. In this notation (2.2), (2.6) become respectively

$$A \to G^{-1} A G + G^{-1} d G, \tag{4.5}$$

$$F = d A + A \wedge A. \tag{4.6}$$

In (4.5), (4.6) one sees that A is a connection 1-form and that F is the curvavature 2– form of this connection, In terms of differential forms (2.7) may now be written as

$$F \to G^{-1} F\, G, \tag{4.7}$$

which is the usual rule for the transformation of the curvature 2- form. As to (2.9), this is merely the Bianchi identity:

$$d\, F = F \wedge A - A \wedge F. \tag{4.8}$$

Returning now to gravitation we denote (cf. Lichnérowicz 1955) respectively by $\omega = (\omega^a{}_b)$ and $\Omega = (\Omega^a{}_b)$ the connection and curvature forms of space-time relative to a coframe $\theta = (\theta^a)$. Then in addition to the condition

$$d\, \theta + \omega \wedge \theta = o, \tag{4.9}$$

which means that the torsion of space-time is zero, we have

$$\Omega = d\, \omega + \omega \wedge \omega. \tag{4.10}$$

It is wellknown that under a change of coframe $\theta \to G^{-1}\, \theta$, det $G \neq O$ the connection and curvature forms transform according to

$$\omega \to G^{-1}\, \omega G + G^{-1}\, d\, G \tag{4.11}$$

$$\Omega \to G^{-1}\, \Omega\, G \tag{4.12}$$

We see that the formalism of the preceding section applies to gravitation if we note that A relative to (4.1) is the connection form of space-time.

In the light of the above analysis gravitation appears to be a particular type of gauge phenomenon which derives from the potential 1-form $\omega$.

Following further the parallelism between gravitation and other gauge phennomena one is led, according to (2.9) and (2.10), to consider

$$\nabla_{[c}\, R_{ab]ij} = O \tag{4.13}$$

and

$$\nabla_a\, R^a{}_{bij} = S_{bij} \tag{4.14}$$

as the field equations of gravitation. If we take $H_{ab} = R_{abmn}$, $K_{ab} = R_{abrs}$ in (3.1) and (3.2) we obtain at once the gravitational counterparts of (3.5), (3.6):

$$U_{abmnrs} = \tfrac{1}{2}\, (R_a{}^i{}_{mn}\, R_{birs} + R_b{}^i{}_{mn}\, R_{airs} - \tfrac{1}{2}\, g_{ab}\, R^{ij}{}_{mn}\, R_{ijrs})$$

$$= \tfrac{1}{2} (R_a{}^i{}_{mn} R_{birs} + R_a{}^{*i}{}_{mn} R_b{}^*{}_{irs}), \qquad (4.15)$$

$$\nabla_b U^{ab}{}_{mnrs} = \tfrac{1}{2} (R^{ab}{}_{mn} S_{brs} + R^{ab}{}_{rs} S_{bmn}). \qquad (4.16)$$

A detailed study of the tensor $U_{abmnrs}$ may be found elsewhere (Öktem, 1968 a, b) and we shall not here consider this tensor further. However, let us only note that its contracted form

$$\begin{aligned} U_{ab} &= R_a{}^{ikj} R_{bijk} - \tfrac{1}{4} g_{ab} R^{ijlk} R_{ijkl} \\ &= \tfrac{1}{2} (R_a{}^{ikj} R_{bijk} + R_a{}^{*ikj} R^*_{bijk}) \end{aligned} \qquad (4.17)$$

is nothing but the gravitational analog of (3.8), and could be interpreted as the energy tensor for the gravitational field itself. Now in view of (4.17) the contraction of (4.16) yields

$$\nabla_b U^{ab} = R^{abji} S_{bij}. \qquad (4.18)$$

In the right-hand side of this equality occurs the expression for the gravitational Lorentz force with the tensor $S_{bij}$ representing the gravitational current. Intuitively, this latter tensor seems somehow connected with the concept of "matter"; for the condition $S_{bij} = 0$ implies (if the physical interpretation presented above is valid) the vanishing of all gauge currents in space-time. By (4.18) $U_{ab}$ is conserved if $S_{bij} = 0$. Furthermore, one can prove that $U_{ab} = 0$ for an Einstein space-time. The converse, however, need not be true.

## 5. CONCLUDING REMARKS.

In this work we introduce the concept of a recurrent system of tensors in Riemannian geometry and examine its relevance to gauge theories. We observe that the recurrence vectors associated with the system exhibit certain remarkable similarities with the potentials of a gauge field and we try to elucidate the physical implications of such an interpretation. From this study we conclude that every gauge field derives from a connection 1-form and consists in the curvature 2-form of the connection. This result, which applies to all gauge phenomena including gravitation, seems to constitute a common trait of all gauge fields. Finally, let us note that the infinitesimal holonomy group* (generating bivectors of which define a recurrent system) could play an essential role in a group-theoretical approach to the subject of gauge theories in space-time.

* In gauge theories the holonomy group was first considered by Loos (1967).

## APPENDIX.

Let

$$D_a H_{bc} + D_b H_{ca} + D_c H_{ab} = O. \qquad (A.1)$$

If we contract (A.1) with $K^{bc}$ we obtain, after some play with indices,

$$D_b (H_{ai} K^{bi}) - \tfrac{1}{2} K^{ij} D_a H_{ij} = H_{ab} K^b. \qquad (A.2)$$

Interchanging $H_{ab}$, $K_{ab}$ in this expression we also have

$$D_b (K_{ai} H^{bi}) - \tfrac{1}{2} H^{ij} D_a K_{ij} = K_{ab} H^b . \qquad (A.3)$$

The forumla (3.2) then follows at once from (A.2) and (A.3). The second equality in (3.1) is easily verified on account of the identity

$$H_b{}^i K_{ai} - H_a{}^{*i} K_b{}^*{}_i = \tfrac{1}{2} g_{ab} H^{ij} K_{ij}, \qquad (A.4)$$

where the asterisk relative to a pair of indices denotes the dual operation:

$$H_a{}^*{}_b = \tfrac{1}{2} \eta_{abij} H^{ij}, \ (\eta_{abij} = \eta_{[abij]}, \ \eta_{1234} = |g|^{1/2}) \qquad (A.5)$$


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