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PAGES: 218-224

ORIGINAL PDF URL: https://dergipark.org.tr/tr/download/article-file/1130511

Some Characterizations of h-Regular Γ-Hemiring in terms of Cubic h-Ideals

Debabrata Mandal[1]*

Abstract
The aim of this paper is to study h-hemiregular and h-intra-hemiregular Γ-hemiring using the combined concept of cubic set and h-ideals. We have defined two types of compositions of cubic sets and used these to obtain some characterizations of h-hemiregular and h-intra-hemiregular Γ-hemiring.

Keywords: Γ-hemiring, Cubic (h-, h-bi-, h-quasi-) ideal, h-intra-hemiregular, h-hemiregular.
2010 AMS: Primary 08A72, Secondary 16Y99

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1. Introduction

Semirings [3], a common generalization of rings and distributive lattices, arise naturally in graph theory, automata theory, mathematical modelling, functional analysis etc.. We know that ideals of ring are a very important tool to describe the structure theory and it is useful in various purposes. If we try to obtain similar results in case of semiring, we find that there are some limitations. To make the connection between the ring ideals and semiring ideals Henriksen [4] defined a special kind of ideals, called k-ideal. Iizuka [5] extended the results to a similar kind of ideals called h-ideals. As a continuation of this, La Torre [10] studied h-ideals and k-ideals in hemirings and tried to investigate the gap between the ring ideals and semiring ideals.

In 1965, Zadeh [16] proposed the theory of fuzzy sets. After that we have seen that it is a very useful mathematical tool for describing the vague or complex or illdefined systems. Rosenfeld [15] used the concept to study of fuzzy algebraic structure. Since then many researchers have developed these ideas. Jun et al [6] applied the concept to initiate the study of fuzzy h-ideals in hemiring. Sardar et al [13, 14], Ma et al [11] extended some of these results in more general setting of hemiring i.e. Γ-hemiring. Jun et al [7, 8] initiated the study of cubic subgroups and cubic sets. Khan et al [9] applied this in case of cubic h-ideals of hemirings. Chinnadurai [1, 2] used this notion to study cubic bi-ideals and cubic lateral ideals in near-ring and ternary near-ring respectively.

As a continuation of this, the main aim of this paper is to study h-hemiregularity and h-intra-hemiregularity criterion of Γ-hemiring using cubic h-ideals, cubic h-bi-ideals and cubic h-quasi-ideals.

2. Primary Ideas

We know that a hemiring is a nonempty set S on which operations addition and multiplication have been defined such that $(S,+)$ is a commutative monoid with identity 0, $(S,\cdot)$ is a semigroup and multiplication distributes over addition from either side. In addition to that, $0 \cdot s = 0 = s \cdot 0$ for all $s \in S$. As an extension of this Γ-hemiring can be defined as follows:

For two additive commutative semigroups with zero, S and Γ, there exists a mapping $S \times \Gamma \times S \to S$ ($(a,\alpha,b) \mapsto a\alpha b$) which satisfy the following conditions:

i) $(a+b)\alpha c = a\alpha c + b\alpha c$,

ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,

iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,

iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$,

v) $0_S\alpha a = 0_S = a\alpha 0_S$,

vi) $a0_\Gamma b = 0_S = b0_\Gamma a$

for all $a,b,c \in S$ and for all $\alpha, \beta \in \Gamma$.

We now summon up the definitions of several types of ideals.
A subset I of a Γ-hemiring S is called a left(resp. right) ideal of S if I is closed under addition and $S\Gamma I \subseteq I$ (resp.$I\Gamma S \subseteq I$).
A subset Q of a Γ-hemiring S is called a quasi-ideal of S if Q is closed under addition and $S\Gamma Q \cap Q\Gamma S \subseteq Q$.
A subset B of a Γ-hemiring S is called a bi-ideal if B is closed under addition and $B\Gamma S\Gamma B \subseteq B$.
A left ideal H of S is called a left *h*-ideal if $x,z \in S$, $a,b \in H$ and $x+a+z = b+z$ implies $x \in H$. A right *h*-ideal is defined analoguesly.

We now remind the definition of cubic set and characteristic cubic set. For a non-empty set X, a cubic set C in X is a structure $C = < \widetilde{\mu}, f >$ where $\widetilde{\mu} = [\mu^-, \mu^+]$ is an interval valued fuzzy set and f is a fuzzy set in X. For any non-empty set G of a set X, the characteristic cubic set of G is defined to be the structure $\chi_G(x) = < x, \widetilde{\zeta}_{\chi_G}(x), \eta_{\chi_G}(x) : x \in X >$ where

$$\widetilde{\zeta}_{\chi_G}(x) = \begin{cases} [1,1] \approx \widetilde{1} \text{ if } x \in G \\ [0,0] \approx \widetilde{0} \text{ otherwise.} \end{cases}$$

and

$$\eta_{\chi_G}(x) = \begin{cases} 0 \text{ if } x \in G \\ 1 \text{ otherwise.} \end{cases}$$

3. Cubic *h*-Ideals

In this section, we recall some definitions and results from [12] which will be used to develop the main portion of the paper.

Definition 3.1. *Let* $< \widetilde{\mu}, f >$ *be a non empty cubic subset of a Γ-hemiring S. Then* $< \widetilde{\mu}, f >$ *is called a cubic left ideal [respectively, cubic right ideal] of S if*

(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x), f(y)\}$ *and*

(ii) $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y)$, $f(x\gamma y) \leq f(y)$ *[respectively,* $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(x), f(x\gamma y) \leq f(x)$*].*

for all $x,y \in S$, $\gamma \in \Gamma$.
Note: For cubic left or right ideal $< \widetilde{\mu}, f >$ *of a Γ-hemiring S,* $\widetilde{\mu}(0) \supseteq \widetilde{\mu}(x)$ *and* $f(0) \leq f(x)$ *for all* $x \in S$.

Definition 3.2. *A cubic left ideal* $< \widetilde{\mu}, f >$ *of a Γ-hemiring S is called a cubic left h-ideal if for all* $a,b,x,z \in S$, $x+a+z = b+z$ *then* $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ *and* $f(x) \leq \max\{f(a), f(b)\}$.

Definition 3.3. *Let* $A = < \widetilde{\mu}, f >$ *and* $B = < \widetilde{\theta}, g >$ *be two cubic sets of a Γ−hemiring S. Define intersection of A and B by*

$$A \cap B = < \widetilde{\mu}, f > \cap < \widetilde{\theta}, g > = < \widetilde{\mu} \cap \widetilde{\theta}, f \cup g >.$$

Definition 3.4. *Let* $A = < \widetilde{\mu}, f >$ *and* $B = < \widetilde{\theta}, g >$ *be two cubic sets of a Γ−hemiring S. Define composition of A and B by*

$$A\Gamma_{ch}B = < \widetilde{\mu}, f > \Gamma_{ch} < \widetilde{\theta}, g > = < \widetilde{\mu}\Gamma_{ch}\widetilde{\theta}, f\Gamma_{ch}g >$$

where

$$\widetilde{\mu}\Gamma_{ch}\widetilde{\theta}(x) = \cup[\cap\{\widetilde{\mu}(a_1), \widetilde{\mu}(a_2), \widetilde{\theta}(b_1), \widetilde{\theta}(b_2)\}]$$
$$\quad\quad\quad {}_{x+a_1\gamma b_1+z=a_2\delta b_2+z}$$
$$= \widetilde{0}, \text{if } x \text{ cannot be expressed as } x+a_1\gamma b_1+z = a_2\delta b_2+z.$$

and

$$f\Gamma_{ch}g(x) \quad = \inf_{x+a_1\gamma b_1+z=a_2\delta b_2+z}\{\max\{f(a_1),f(a_2),g(b_1),g(b_2)\}\}$$
$$= 1, \text{if } x \text{ cannot be expressed as above}$$

for $x,z,a_1,a_2,b_1,b_2 \in S$ *and* $\gamma,\delta \in \Gamma$.

Definition 3.5. *Let* $A =< \widetilde{\mu},f >$ *and* $B =< \widetilde{\theta},g >$ *be two cubic sets of a* $\Gamma-$*hemiring S. Define generalized composition of A and B by*

$$Ao_{ch}B =< \widetilde{\mu},f > o_{ch} < \widetilde{\theta},g >=< \widetilde{\mu}o_{ch}\widetilde{\theta},fo_{ch}g >$$

where

$$\widetilde{\mu}o_{ch}\widetilde{\theta}(x) \quad = \quad \cup[\cap_i\{\cap\{\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\theta}(b_i),\ \widetilde{\theta}(d_i)\}\}]$$
$$x+\sum_{i=1}^{n}a_i\gamma_ib_i+z = \sum_{i=1}^{n}c_i\delta_id_i+z$$
$$= \widetilde{0}, \text{if } x \text{ cannot be expressed as above}$$

and

$$fo_{ch}g(x) \quad = \inf[\max_i\{\max\{f(a_i),f(c_i),g(b_i),g(d_i)\}\}]$$
$$x+\sum_{i=1}^{n}a_i\gamma_ib_i+z = \sum_{i=1}^{n}c_i\delta_id_i+z$$
$$= 1, \text{if } x \text{ cannot be expressed as above}$$

where $x,z,a_i,b_i,c_i,d_i \in S$ *and* $\gamma_i,\delta_i \in \Gamma$, *for* $i \in \{1,...,n\}$.

Lemma 3.6. *Let* $A =< \widetilde{\mu}_1,f >$, $B =< \widetilde{\mu}_2,g >$ *be two cubic h-ideal of a* $\Gamma-$*hemiring S. Then* $A\Gamma_{ch}B \subseteq Ao_{ch}B \subseteq A\cap B \subseteq A($ *and* $B)$.

Definition 3.7. *A cubic subset* $< \widetilde{\mu},f >$ *of a* Γ*-hemiring S is called cubic h-bi-ideal if for all* $x,y,z,a,b \in S$ *and* $\alpha,\beta \in \Gamma$ *we have*

   i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x),f(y)\}$

   ii) $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x\alpha y) \leq \max\{f(x),f(y)\}$

   iii) $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(z)\}$, $f(x\alpha y\beta z) \leq \max\{f(x),f(z)\}$

   iv) *If* $x+a+z = b+z$ *then* $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a),\widetilde{\mu}(b)\}$, $f(x) \leq \max\{f(a),f(b)\}$

Definition 3.8. *A cubic subset* $< \widetilde{\mu},f >$ *of a* Γ*-hemiring S is called cubic h-quasi-ideal if for all* $x,y,z,a,b \in S$ *we have*

   i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x),\widetilde{\mu}(y)\}$, $f(x+y) \leq \max\{f(x),f(y)\}$

   ii) $(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})\cap(\widetilde{\zeta}_{\chi_S}o_{ch}\widetilde{\mu}) \subseteq \widetilde{\mu}$, $(fo_{ch}\eta_{\chi_S})\cup(\eta_{\chi_S}o_{ch}f) \supseteq f$,

   iii) *If* $x+a+z = b+z$ *then* $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a),\widetilde{\mu}(b)\}$, $f(x) \leq \max\{f(a),f(b)\}$

Lemma 3.9. *Any cubic h-quasi-ideal of S is a cubic h-bi-ideal of S.*

4. Cubic *h*-Hemiregularity and Cubic *h*-Intra-Hemiregularity

In this section, we study the concept of *h*-hemiregularity and *h*-intra-hemiregularity in Γ-hemiring by using cubic *h*-ideal, cubic *h*-bi-ideal and cubic *h*-quasi-ideal.

Definition 4.1. *[11] A* Γ*-hemiring S is said to be h-hemiregular if for each* $x \in S$, *there exist* $a,b \in S$ *and* $\alpha,\beta,\gamma,\delta \in \Gamma$ *such that* $x+x\alpha a\beta x+z = x\gamma b\delta x+z$.

We now try to find some characterizations of *h*-hemiregular Γ-hemiring in terms of cubic *h*-ideals.

Theorem 4.2. *Let S be an h-hemiregular Γ-hemiring. Then for any cubic right h-ideal $A =< \widetilde{\mu}, f >$ and any cubic left h-ideal $B =< \widetilde{v}, g >$ of S, we have $A\Gamma_{ch}B = A \cap B$.*

Proof. Let S be an *h*-hemiregular Γ-hemiring. By Lemma 3.6, we have $A\Gamma_{ch}B \subseteq A \cap B$.

Since S is *h*-hemiregular, for any $a \in S$, there exist $z, x_1, x_2 \in S$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \Gamma$ such that $a + a\alpha_1 x_1 \beta_1 a + z = a\alpha_2 x_2 \beta_2 a + z$.

Now for any $e, b, c, d \in S$ and $\gamma, \delta \in \Gamma$, the general expression of a as $a + e\gamma b + z = c\delta d + z$ implies that

$$(\widetilde{\mu}\Gamma_{ch}\widetilde{v})(a) = \bigcup_{a+e\gamma b+z=c\delta d+z}\{\cap\{\widetilde{\mu}(e),\ \widetilde{\mu}(c), \widetilde{v}(b), \widetilde{v}(d)\}\}$$

$$\supseteq \bigcap_{a+a\alpha_1 x_1 \beta_1 a+z=a\alpha_2 x_2 \beta_2 a+z}\{\widetilde{\mu}(a\alpha_1 x_1), \widetilde{\mu}(a\alpha_2 x_2), \widetilde{v}(a)\}$$

$$\supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(a), \widetilde{v}(a)\}$$

$$= \cap\{\widetilde{\mu}(a), \widetilde{v}(a)\} = (\widetilde{\mu} \cap \widetilde{v})(a).$$

$$(f\Gamma_{ch}g)(a) = \inf_{a+e\gamma b+z=c\delta d+z}\{\max\{f(e), f(c), g(b), g(d)\}\}$$

$$\leq \max_{a+a\alpha_1 x_1 \beta_1 a+z=a\alpha_2 x_2 \beta_2 a+z}\{f(a\alpha_1 x_1), f(a\alpha_2 x_2), g(a)\}$$

$$\leq \max\{f(a), f(a), g(a)\}$$

$$= \max\{f(a), g(a)\} = (f \cup g)(a).$$

Therefore $(A \cap B) \subseteq (A\Gamma_{ch}B)$.

Hence $A\Gamma_{ch}B = A \cap B$. □

Corollary 4.3. *If S be a h-hemiregular Γ-hemiring, then for any cubic right h-ideal $A =< \widetilde{\mu}, f >$ and any cubic left h-ideal $B =< \widetilde{v}, g >$ of S we have $Ao_{ch}B = A \cap B$.*

Theorem 4.4. *Let S be a h-hemiregular Γ-hemiring. Then*
(i) $A \subseteq Ao_{ch}\chi_S o_{ch}A$ for every cubic h-bi-ideal $A =< \widetilde{\mu}, f >$ of S.
(ii) $A \subseteq Ao_{ch}\chi_S o_{ch}A$ for every cubic h-quasi-ideal $A =< \widetilde{\mu}, f >$ of S.

Proof. Suppose that $A =< \widetilde{\mu}, f >$ be any cubic h-bi-ideal of S and x be any element of S. Since S is *h*-hemiregular there exist $a, b, z \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

Now for any general expression of x as $x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S}o_{ch}\widetilde{\mu})(x)$$
$$= \bigcup_{x+\sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z}(\cap\{(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(a_i), (\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(c_i), \widetilde{\mu}(b_i), \widetilde{\mu}(d_i)\})$$

$$\supseteq \bigcap_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(x\alpha a), (\widetilde{\mu}o_{ch}\widetilde{\zeta}_{\chi_S})(x\gamma b), \widetilde{\mu}(x)\}$$

$$= \cap\{\bigcup_{x\alpha a+\sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z}(\cap\{(\widetilde{\mu}(a_i), \widetilde{\mu}(c_i))\}),\quad \bigcup_{x\gamma b+\sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z}(\cap\{(\widetilde{\mu}(a_i), \widetilde{\mu}(c_i))\}),$$

$$\widetilde{\mu}(x)\}$$
$$\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(x), \widetilde{\mu}(x)\}(\text{since } x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a \text{ and } x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b).$$
$$= \widetilde{\mu}(x).$$

$$(fo_{ch}\eta_{\chi_S}o_{ch}f)(x)$$
$$= \inf(\max\{(fo_{ch}\eta_{\chi_S})(a_i), (fo_{ch}\eta_{\chi_S})(c_i), f(b_i), f(d_i)\})$$
$${}_{x+\sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z}$$

$$\leq \max_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(fo_{ch}\eta_{\chi_S})(x\alpha a),(fo_{ch}\eta_{\chi_S})(x\gamma b),f(x)\}$$

$$= \max_{x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z \ \ x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z}\{\quad \inf(\max\{(f(a_i),f(c_i))\}) \quad , \quad \inf(\max\{(f(a_i),f(c_i))\}) \quad ,f(x)\}$$

$$\leq \max\{f(x),f(x),f(x)\}(\text{since } x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a \text{ and } x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b).$$
$$= f(x).$$

This implies that $A \subseteq Ao_{ch}\chi_{S}o_{ch}A$.

(i)⇒(ii) By Lemma 3.9 "Any cubic *h*-quasi-ideal of *S* is a cubic *h*-bi-ideal of *S*". Thus, if the proof of (ii) is made, it is straightforward to see that (i) is true by Lemma 3.9. □

Theorem 4.5. *Let S be a h-hemiregular Γ-hemiring. Then*
(i) $A\cap B \subseteq Ao_{ch}Bo_{ch}A$ for every cubic h-bi-ideal $A =< \widetilde{\mu}, f >$ and every cubic h-ideal $B =< \widetilde{\nu}, g >$ of S.
(ii) $A\cap B \subseteq Ao_{ch}Bo_{ch}A$ for every cubic h-quasi-ideal $A =< \widetilde{\mu}, f >$ and every cubic h-ideal $B =< \widetilde{\nu}, g >$ of S.

Proof. Suppose *S* is a *h*-hemiregular Γ-hemiring and $A =< \widetilde{\mu}, f >$, $B =< \widetilde{\nu}, g >$ be any cubic *h*-bi-ideal and cubic*h*-ideal of *S*, respectively and *x* be any element of *S*. Since *S* is *h*-hemiregular, there exist $a,b,z \in S$ and $\alpha,\beta,\gamma,\delta \in \Gamma$ such that $x+x\alpha a\beta x+z=x\gamma b\delta x+z$.

Now for any general expression of *x* as $x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$, where $a_i,b_i,c_i,d_i \in S$ and $\gamma_i,\delta_i \in \Gamma$, we have

$$(\widetilde{\mu}o_{ch}\widetilde{\nu}o_{ch}\widetilde{\mu})(x)$$
$$= \cup(\cap\{(\widetilde{\mu}o_{ch}\widetilde{\nu})(a_i),(\widetilde{\mu}o_{ch}\widetilde{\nu})(c_i),\widetilde{\mu}(b_i),\widetilde{\mu}(d_i)\})$$
$$\qquad x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

$$\supseteq \cap\{(\widetilde{\mu}o_{ch}\widetilde{\nu})(x\alpha a),(\widetilde{\mu}o_{ch}\widetilde{\nu})(x\gamma b),\widetilde{\mu}(x)\}$$
$$\quad x+x\alpha a\beta x+z=x\gamma b\delta x+z$$

$$= \cap\{\cup(\cap\{(\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\nu}(b_i),\widetilde{\nu}(d_i))\}),\cup(\cap\{(\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\nu}(b_i),\widetilde{\nu}(d_i))\}),$$
$$\qquad x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z \quad x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

$$\widetilde{\mu}(x)\}$$
$$\supseteq \cap\{\cap\{\widetilde{\mu}(x),\widetilde{\nu}(a\beta x\alpha a),\widetilde{\nu}(b\delta x\alpha a)\},\cap\{\widetilde{\mu}(x),\widetilde{\nu}(a\beta x\gamma b),\widetilde{\nu}(b\delta x\gamma b)\},\widetilde{\mu}(x)\}$$
$$(\text{since } x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a \text{ and } x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b)$$
$$\supseteq \cap\{\widetilde{\mu}(x),\widetilde{\nu}(x)\} = (\widetilde{\mu}\cap\widetilde{\nu})(x).$$

$$(fo_{ch}go_{ch}f)(x)$$
$$= \inf(\max\{(fo_{ch}g)(a_i),(fo_{ch}g)(c_i),f(b_i),g(d_i)\})$$
$$\qquad x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

$$\leq \max_{x+x\alpha a\beta x+z=x\gamma b\delta x+z}\{(fo_{ch}g)(x\alpha a),\ (fo_{ch}g)(x\gamma b),f(x)\}$$

$$= \max\{\inf(\max\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),\inf(\max\{(f(a_i),f(c_i),g(b_i),g(d_i))\}),$$
$$\qquad x\alpha a+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z \quad x\gamma b+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

$$f(x)\}$$
$$\leq \max\{\max\{f(x),g(a\beta x\alpha a),g(b\delta x\alpha a)\},\cap\{f(x),g(a\beta x\gamma b),g(b\delta x\gamma b)\},f(x)\}$$
$$(\text{since } x\alpha a+x\alpha a\beta x\alpha a+z\alpha a=x\gamma b\delta x\alpha a+z\alpha a \text{ and } x\gamma b+x\alpha a\beta x\gamma b+z\gamma b=x\gamma b\delta x\gamma b+z\gamma b)$$
$$\supseteq \max\{f(x),g(x)\} = (f\cup g)(x).$$

(i)⇒(ii) This is straightforward using the Lemma 3.9. □

Definition 4.6. *[13]A Γ-hemiring S is said to be h-intra-hemiregular if for each $x \in S$, there exist $z,a_i,a_i^{'},b_i,b_i^{'} \in S$, and $\alpha_i,\beta_i,\gamma_i,\delta_i,\eta \in \Gamma$, $i\in \mathbf{N}$, such that $x+\sum_{i=1}^{n}a_i\alpha_i x\eta x\beta_i a_i^{'}+z=\sum_{i=1}^{n}b_i\gamma_i x\eta x\delta b_i^{'}+z$.*

We now try to find a characterization of *h*-intrahemiregular Γ-hemiring in terms of cubic *h*-ideals.

Theorem 4.7. *Let S be a h-intra-regular Γ-hemiring. Then $A \cap B \subseteq A o_{ch} B$ for every cubic left h-ideal $A = < \widetilde{\mu}, f >$ and every cubic right h-ideal $A = < \widetilde{v}, g >$ of S.*

Proof. Suppose S is h-intra-hemiregular. Let $A = < \widetilde{\mu}, f >$ and $A = < \widetilde{v}, g >$ be any cubic left h-ideal and cubic right h-ideal of S respectively. Now let $x \in S$. Then by hypothesis there exist $z, a_i, a_i', b_i, b_i' \in S$, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, $i \in \mathbf{N}$, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$. Now for any general expression of x as $x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$(\widetilde{\mu} o_{ch} \widetilde{v})(x) = \cup [\underset{i}{\cap} \{ \cap \{ \widetilde{\mu}(a_i), \widetilde{\mu}(c_i), \widetilde{v}(b_i), \ \widetilde{v}(d_i) \} \}]$$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\supseteq \underset{i}{\cap} [\cap \{ \widetilde{\mu}(a_i \alpha_i x), \widetilde{\mu}(b_i \gamma_i x), \widetilde{v}(x \beta_i a_i'), \widetilde{v}(x \delta_i b_i') \}]$$

$$x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$$

$$\supseteq \cap \{ \widetilde{\mu}(x), \widetilde{v}(x) \} = (\widetilde{\mu} \cap \widetilde{v})(x).$$

$$(f o_{ch} g)(x) = \inf [\underset{i}{\max} \{ \max \{ f(a_i), f(c_i), g(b_i), g(d_i) \} \}]$$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\leq \underset{i}{\max} [\max \{ f(a_i \alpha_i x), f(b_i \gamma_i x), g(x \beta_i a_i'), g(x \delta_i b_i') \}]$$

$$x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$$

$$\leq \max \{ f(x), g(x) \} = (f \cup g)(x).$$

Hence the proof is completed. □

We now combine the concepts of *h*-hemiregularity and *h*-intra-hemiregularity of a Γ-hemiring and obtain a characterization.

Theorem 4.8. *Let S be both h-hemiregular and h-intra-hemiregular Γ-hemiring. Then*
(i) $A = A o_{ch} A$ for every cubic h-bi-ideal $A = < \widetilde{\mu}, f >$ of S.
(ii) $A = A o_{ch} A$ for every cubic h-quasi-ideal $A = < \widetilde{\mu}, f >$ of S.

Proof. Suppose S be both h-hemiregular and h-intra-hemiregular Γ-hemiring. Let $x \in S$ and $A = < \widetilde{\mu}, f >$ be any cubic h-bi-ideal of S. Since S is both h-hemiregular and h-intra-hemiregular there exist $z, a_i, b_i, c_i, d_i \in S$ and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i', \eta \in \Gamma$, $i \in \mathbf{N}$ such that $x + \sum_{i=1}^{n} x \alpha_i a_i \alpha_i' x \eta x \beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x \gamma_i c_i \gamma_i' x \eta x \delta_i' d_i \delta_i x + z$.

Now for any general expression of x as $x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$, where $a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, we have

$$(\widetilde{\mu} o_{ch} \widetilde{\mu})(x)$$
$$= \cup [\underset{i}{\cap} \{ \cap \{ \widetilde{\mu}(a_i), \widetilde{\mu}(c_i), \widetilde{\mu}(b_i), \ \widetilde{\mu}(d_i) \} \}]$$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\supseteq \underset{i}{\cap} [\cap \{ \widetilde{\mu}(x \alpha_i a_i \alpha_i' x), \widetilde{\mu}(x \beta_i' b_i \beta_i x), \widetilde{\mu}(x \gamma_i c_i \gamma_i' x), \widetilde{\mu}(x \delta_i' d_i \delta_i x) \}]$$

$$x + \sum_{i=1}^{n} x \alpha_i a_i \alpha_i' x \eta x \beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x \gamma_i c_i \gamma_i' x \eta x \delta_i' d_i \delta_i x + z$$

$$\supseteq \widetilde{\mu}(x).$$

$(f o_{ch} f)(x)$

$= \inf_i [\max\{\max\{f(a_i), f(c_i), f(b_i), f(d_i)\}\}]$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$\leq \max_i [\max\{f(x\alpha_i a_i \alpha_i' x), f(x\beta_i' b_i \beta_i x), f(x\gamma_i c_i \gamma_i' x), f(x\delta_i' d_i \delta_i x)\}]$

$$x + \sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x \eta x \beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x \eta x \delta_i' d_i \delta_i x + z$$

$\leq f(x)$.

Now $A o_{ch} A \subseteq A o_{ch} \chi_S \subseteq A$. Hence $A o_{ch} A = A$ for every cubic *h*-bi-ideal A of S.

(i)⇒(ii) This is straightforward using the Lemma 3.9. □

5. Conclusion

In this paper, I have studied some properties *h*-hemiregular and *h*-intra-hemiregular Γ-hemiring using the concept of cubic *h*-ideal, cubic *h*-bi-ideal and cubic *h*-quasi-ideal. At the end section, I also acquire some characterizations of *h*-hemiregular and *h*-intra-hemiregular Γ-hemiring. Interested reader may find some other feature of these types of Γ-hemiring and extend the obtained result using the concept of neutrosophic set and neutrosophic ideal.

Acknowledgement

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