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ON THE REPRESENTATIONS AND CHARACTERS OF CAT^1-GROUPS AND CROSSED MODULES

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Abstract. Let G be a group and V a K-vector space. A K-linear representation of G with representation space V is a homomorphism $\phi : G \longrightarrow GL(V)$. The dimension of V is called the degree of ϕ. If ϕ is a representation of G, then the character ϕ is defined for $g \in G$ as $\psi_g(\phi) = Tr(\phi(g))$. In this paper we study the representations and characters of cat^1-groups and crossed modules. We show that for class functions ψ_1 and ψ_2 of crossed module $\chi = (G, M, \mu, \partial)$, the inner product is Hermitian. Also, if $\chi = (G, M, \mu, \partial)$ is a finite crossed module and ψ is an irreducible character of χ, then

$$\sum_{m\in M,\, g\in G} \psi(m, g)\psi(m^{-1}, g^{-1}) = |G||M|.$$

Moreover, we present some examples of the character tables of crossed modules.

1. Introduction

Cat^1-groups (or 1-cat groups) are the first in a series of models homotopy n-types introduced by Loday [9]. They are sometimes referred to simply as cat-groups [6] if the higher cat^n-groups are not also being considered; the term categorical group [13] is used for similar structures in which inverses for the group operations are only defined up to isomorphism[7].

The term crossed module was introduced by J.H.C. Whitehead in his work on combinatorial homotopy theory [12]. So many mathematician and many areas of mathematics have used crossed modules such as homotopy theory, homology and cohomology of groups, Algebra, K-theory etc.

In this paper we study the representations and characters of cat^1-groups and crossed modules.

2. Cat^1-groups

We recall some definitions and properties of cat^1-groups.

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Definition 1. A cat^1-group $\mathcal{C} = (G, P, i; s, t)$ consists of groups G and P, an embedding $i : P \longrightarrow G$ and epimorphisms $s, t : G \longrightarrow P$ satisfying:

(1) $si = ti = id_p$,
(2) $[ker\, s, ker\, t] = \{1_G\}$.

A morphism $\gamma : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ of cat^1-groups consists of a pair $\gamma_g : G_1 \longrightarrow G_2$ and $\gamma_p : P_1 \longrightarrow P_2$ that commute with the homomorphisms of C_1 and C_2.

With the obvious composition, there is a category $\mathbf{cat}^1$ of cat^1-groups and their morphisms.

Definition 2. If G and H are groups; with a left action of H on G, the semidirect product of G by H is the group $G \rtimes H = \{(g, h) \mid \; g \in G, \; h \in H\}$ with multiplication $(g, h)(g', h'^h g', hh')$. The inverse of (g, h) is $({}^{h^{-1}}g^{-1}, h^{-1})$.

Since, for any cat^1-group, $Ker\, s$ is normal in G and $iP \leqslant G$, it follows that there is an action of iP on $Ker\, s$ by conjugation. Hence, the semidirect product $Ker\, s \rtimes P$ is defined.

Lemma 1. [5] *For a cat^1-group (G, P, i, s, t), we have $G \cong Ker\, s \rtimes P$.*

3. Representations of cat^1-groups

Let K be a field and let C_0, C_1 be vector spaces over K. If $\delta : C_1 \longrightarrow C_0$ is a linear transformation, then

$$\mathcal{C} : \; C_1 \xrightarrow{\;\;\delta\;\;} C_0$$

is a length 1 chain complex of vector spaces. $\mathcal{C}$ can be considered as

$$\cdots \longrightarrow \circ \longrightarrow \; C_1 \xrightarrow{\;\;\delta\;\;} C_0 \; \longrightarrow \circ \longrightarrow \cdots$$

and so the composition trivially gives the zero map and δ is a differential [8]. Thus, every linear transformation can be considered as a chain complex. It will sometimes be convenient to blur the distinction between the linear transformation and its chain complex, and refer to δ itself as a chain complex. Suppose that in addition to $\mathcal{C}$ we have a chain complex $\mathcal{D} : \; D_1 \xrightarrow{\;\;\delta^{\mathcal{D}}\;\;} D_0$ i.e., write $\delta^{\mathcal{C}}$ for the differential in $\mathcal{C}$ to distinguish it. Then, a morphism between $\mathcal{C}$ and D is defined as follows.

Definition 3. A chain map $f : \mathcal{C} \longrightarrow \mathcal{D}$ consists of components $f_1 : C_1 \longrightarrow D_1$ and $f_0 : C_0 \longrightarrow D_0$ such that $\delta^D f_1 = f_0 \delta^D$, i.e., the following diagram commutes:

$$\begin{array}{ccc} C_1 & \xrightarrow{\;\;f_1\;\;} & D_1 \\ {\scriptstyle \delta^{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \delta^{\mathcal{D}}} \\ C_0 & \xrightarrow[\;\;f_0\;\;]{} & D_0 \end{array}$$

Suppose that $f : \mathcal{C} \longrightarrow \mathcal{D}$ and $g : \mathcal{D} \longrightarrow \varepsilon$ are chain maps. Then the composite $g\#_\circ f : \mathcal{C} \longrightarrow \varepsilon$ is defined by $(g\#_\circ f)_i := g_i f_i$, where $i = 0, 1$ and the composition on the right hand side is the usual one for linear maps.

Definition 4. Let K be a field. The category of length 1 chain complexes of K-vector spaces, and chain maps between them, is denoted by $Ch_K^{(1)}$.

This structure provides the foundation for a 2-groupoid. By restricting our attention to those chain maps that are invertible we obtain a subgroupoid of $Ch_K^{(1)}$, which we shall write as $inv\,Ch_K^{(1)}$. From Definition 4 before it is clear that a chain map $f : \mathcal{C} \longrightarrow \mathcal{D}$ is invertible precisely when both its components are invertible.

Definition 5. A chain isomorphism is an invertible chain map $f : \mathcal{C} \longrightarrow \mathcal{D}$.

Hence the morphisms of $invCh_K^{(1)}$ are precisely the chain isomorphisms of $Ch_K^{(1)}$. Let $\delta : C_1 \longrightarrow C_0$ be a linear transformation of vector spaces; this can and will be considered as an object in $Ch_K^{(1)}$ in the way explained earlier. The collection of all chain isomorphisms $\delta \longrightarrow \delta$ and homotopies between them is a 2-group and is also a cat^1-group. As isomorphisms from an object to itself are commonly known as automorphisms, we may call this structure an automorphism cat^1-group.

We recall that any element in an abstract vector space V of dimension n can be considered as an n-tuple in K^n. Linear transformations between vector spaces are equivalent to matrices over K; assuming standard bases, a linear transformation $\phi : K^n \longrightarrow K^m$ uniquely determines and is determined by an $m \times n$ matrix ϕ (or M_ϕ or $M(\phi)$) with coefficients in K. In particular, a linear isomorphism $K^n \longrightarrow K^n$ is equivalent to an element of $GL_n(K)$. In $Ch_K^{(1)}$, the objects are chain complexes of length 1. As we have seen, these are essentially the same as linear transformations. Hence a chain complex $\mathcal{C}$ with differential $\delta^{\mathcal{C}} : C_1 \longrightarrow C_0$ can be represented by an $n_0 \times n_1$ matrix $\Delta^{\mathcal{C}}$, where n_i is the dimension of C_i. Suppose that $\mathcal{D}$ is another chain complex, with differential $d^{\mathcal{D}} : D_1 \longrightarrow D_0$, where the dimension of D_i is m_i. A chain map $f : \mathcal{C} \longrightarrow \mathcal{D}$ is given by a pair of matrices $F_1(m_1 \times n_1)$ and $F_0(m_0 \times n_0)$. The commutativity of the chain map with the differentials is then expressed as

$$F_0 \Delta^{\mathcal{C}} = \Delta^{\mathcal{D}} F_1$$

which is an $m_0 \times n_1$ matrix as required. Any chain map $f : \mathcal{C} \longrightarrow \mathcal{D}$ in $inv\,Ch_K^{(1)}$ is invertible, so in this case D_i also has dimension n_i and the corresponding square matrices are non-singular, i.e., $F_1 \in GL_{n_1}(K)$ and $F_0 \in GL_{n_0}(K)$ and so before equation can then by rewritten as $\Delta^{\mathcal{C}} = F_0^{-1} \Delta^{\mathcal{D}} F_1$. A chain homotopy is a linear transformation, so corresponds to a matrix. So, we have another chain map $f' : \mathcal{C} \longrightarrow \mathcal{D}$ and a homotopy $h : f \simeq f'$, and a chain homotopy $h' : C_0 \longrightarrow D_1$ with a corresponding $n_1 \times n_0$ matrix H such that $H\Delta^{\mathcal{C}} = F_1^1 - F_1$, $\Delta^{\mathcal{D}} H = F_0^1 - F_0$. If $h : f \longrightarrow f'$ and $\hat{h} : f' \longrightarrow f''$, then the vertical composite $\hat{h} \neq_1 h$, which is given by the chain homotopy $(\hat{h}\#_1 h)' := \hat{h}' + h'$, corresponds to the matrix sum $\hat{H} + H$. On the other hand:

Definition 6. Let $h : f \simeq f'$ and $\hat{h} : f' \simeq f''$ such that $f, f', f'' : \mathcal{C} \longrightarrow \mathcal{D}$. Then the vertical composite $(\hat{h}\#_1 h) : f \simeq f''$ is the homotopy with chain homotopy component:

$$(\hat{h}\#_1 h)' := \hat{h}' + h'$$

Suppose that we have homotopies $h : f \simeq f' : \mathcal{C} \longrightarrow \mathcal{D}$ and $k : g \simeq g' : \mathcal{D} \longrightarrow \varepsilon$. Then the chain homotopy components of $g\#_0 h$, $k\#_0 f'$, and $k\#_0 h$ are represented by the matrices $G_1 H$, kF'_0 and $G_1 H + kF'_0$ respectively.

Definition 7. Let $\delta : C_1 \longrightarrow C_0$ be a linear transformation of K-vector spaces. The automorphism cat^1-group of δ, $Aut(\delta)$, consists of:

- the group $Aut(\delta)_1$ of all chain automorphisms $\delta \longrightarrow \delta$,
- the groups $Aut(\delta)_2$ of all homotopies on $Aut(\delta)_1$,
- morphisms $s, t : Aut(\delta)_2 \longrightarrow Aut(\delta)_1$, selecting the source and target of each homotopy,
- the morphism $i : Aut(\delta)_1 \longrightarrow Aut(\delta)_2$, which provides the identity homotopy on each chain automorphism.

4. Linear Representations

We recall that a cat^1-group [5] is the same thing as a 2-group. Therefore, we may look for representations of a cat^1-group $\mathcal{C}$ as 2-functors into a suitable 2-category, taking elements of P to 1-cells and elements $\mathcal{C} \rtimes P$ to 2-cells, so as to preserve the structures (all the 1-and 2-cells will have the same object, $\star$, as their 0-source and target, even if the target category has many objects). By analogy with groups and groupoids, the target 2-category of a linear representation should involve vector spaces or modules. We have seen in [5] that $Ch_K^{(1)}$ is a 2-category which generalises $Vect_K$, where $Vect_K$ is category of K-vector spaces and linear transformations, so this is suitable for our purpose.
Although its ramifications will be far-reaching, the actual definition of a representation is fairly obvious.

Definition 8. A linear representation of the cat^1-group $\mathcal{C}$ is a 2-functor $\phi : \mathcal{C} \longrightarrow Ch_K^{(1)}$.

Given $\mathcal{C}$, the first step towards defining ϕ is to find a chain complex (i.e., linear transformation) to act as the implicit target object, $\delta = \phi(\star)$. The group algebra functor [5] provides a canonical way of getting from a group homomorphism to a linear transformation, although it will sometimes be useful to make a different choice. Once δ is chosen, the elements of the cat^1-group must be mapped to elements of $Ch_k^{(1)}$, with elements of the base going to 1-cells and elements of the top group going to 2-cells . For ϕ to be a functor, this mapping must preserve identities and composition. Therefore, the image of $\mathcal{C}$ lies within $Aut(\delta)$. This δ is clearly analogous to the representation space of a group representation; since it is a chain complex rather than a vector space it will be called the representation complex of the representation. Recall that $Aut(\delta)$ is itself a cat^1-group [5], whose elements are linear transformations. Therefore, another way of considering the representation ϕ is to take it as a cat^1-group morphism

$$\phi : \mathcal{C} \longrightarrow Aut(\delta).$$

Definition 9. The right regular representation of a cat^1-group $\mathcal{C} = (G \rtimes P, P, s, t, i)$ is the 2-functor $\rho : \mathcal{C}^{op} \longrightarrow ch_K^{(1)}$ sending each $p \in P$ to the chain automorphism

$$\rho(p)(e_q) := e_{q\#_0 p}, \quad \rho(p)(\bar{v}_{c,q}) := \bar{v}_{c,qp}$$

and each $(c, p) \in G \rtimes P$ to the homotopy $\rho(c, p) : \rho(p) \longrightarrow \rho(\partial cp)$ with chain homotopy

$$\rho'(c, p)(e_q) := \bar{v}_{qc,qp},$$

where all chain automorphisms and homotopies reside in $Aut\,(\delta)$ for the linear transformation $\delta := \bar{\tau}|_{Ker\bar{\sigma}}$ obtained from the car^1- group algebra $\overline{K(\mathcal{C})}$ of $\mathcal{C}$.

Theorem 1. *(Cayley). For any cat^1-group $\mathcal{C}$, the right regular representation, as defined in Definition 9, exists.*

Definition 10. A 2-functor $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ is faithful, if for 2-cells $\alpha, \beta \in \mathcal{A}$,

$$\phi(\alpha) = \phi(\beta) \implies \alpha = \beta.$$

Definition 11. Let $\mathcal{C}$ be a cat^1-group. A representation $\phi : \mathcal{C} \longrightarrow ch_K^{(1)}$ is faithful if it is faithful as a 2-functor.

5. Crossed Modules

A crossed module is a 4-tuple $\chi = (G, M, \mu, \partial)$, where G and M are groups, μ is an action of G on M, and $\partial : M \longrightarrow G$ is a homomorphism, called the boundary map, that satisfies:

- $\partial(m^g) = g^{-1}(\partial m)g$, for all $m \in M$ and $g \in G$,
- $m^{\partial n} = n^{-1}mn$ for all $m, n \in M$.

A crossed module is finite if both G and M are finite groups.

Example 1. For a group G, we will denote by RG the crossed module $(G, 1, \mu, \partial)$, where 1 denote the trivial subgroup of G, and both the action μ and the boundary map ∂ are trivial.

Example 2. If G is a group, DG is the crossed module (G, G, μ, id), where μ is the conjugation action, $\mu(m, g) = g^{-1}mg$, and $id : g \longrightarrow g$ is the trivial map.

Definition 12. A morphism $\phi : \chi \longrightarrow \mathcal{Y}$ between the crossed modules $\chi = (G_1, M_1, \mu_\chi, \partial_\chi)$ and $\mathcal{Y} = (G_2, M_2, \mu_\mathcal{Y}, \partial_\mathcal{Y})$ is a pair (ϕ_1, ϕ_2), where $\phi_i : M_i \longrightarrow G_i$ and group homomorphisms for $i = 1, 2$, and the following relations hold:

- $\partial_\mathcal{Y} \circ \phi_2 = \phi_1 \circ \partial_\chi$,
- $\mu_\mathcal{Y} \circ (\phi_2 \times \phi_1) = \phi_2 \circ \mu_\chi$,

which simply express the commutativity of the diagrams:

$$\begin{array}{ccc} M_1 & \xrightarrow{\partial_\chi} & G_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ M_2 & \xrightarrow{\partial_\mathcal{Y}} & G_2 \end{array} \quad , \qquad \begin{array}{ccc} M_1 \times G_1 & \xrightarrow{\mu_\chi} & M_1 \\ {\scriptstyle \phi_2\times\phi_1}\downarrow & & \downarrow{\scriptstyle \phi_2} \\ M_2 \times G_2 & \longrightarrow & M_2 \end{array}$$

A consequence of the defining properties of a crossed module is that $K = Ker\,\partial$ is a central subgroup of M, $I = im\,\partial$ is a normal subgroup of G, and one has an exact sequence

$$1 \longrightarrow K \longrightarrow M \longrightarrow G \longrightarrow C \longrightarrow 1$$

where $C = G/I$ is the cokernel of ∂ [3]. In particular, $|M||C| = |K||G|$ for a finite crossed module.

From a crossed module $\chi = (G, M, \mu, \partial)$ we can construct a cat^1-group $\mathcal{C}(\chi)$: $(G \rtimes M, M, i, s, t)$. Here $s, t : M \rtimes G \longrightarrow M$, $i : M \longrightarrow G \times M$ are defined as $s(c, p) = p$, $t(c, p) = \partial(c)p$ and $i(p) = (1_G, p)$. Then $s|_M = t|_M = id_M$ and $[Ker\,s, Ker\,t] = 1_{G\rtimes M}$. Note that $(c, p) \in Ker\,s \Longleftrightarrow p = 1_M$, i.e., $Ker\,s = \{(c, 1_M)\} \cong G$; hence $t(c, 1_M) = \partial(c)1_M = \partial(c)$, so $\partial = t|_{Ker\,s}$ and we can recover χ from $\mathcal{C}(\chi)$. The same trick enables us to construct a crossed module $\chi(\mathcal{C})$ from any given cat^1-group $\mathcal{C}$. These constructions lead to the well-known equivalence between crossed modules and cat^1-groups [5].

Example 3. (a) Let $C_2 =, x|x^2 = 1 >$ and $I = \{1\}$. Then $C_2 \longrightarrow I$ is a crossed module and from $C_2 \longrightarrow I$ we get the cat^1-group $(C_2, I, i, 0, 0)$ where i is the inclusion $(1 \rightarrowtail 1)$ and s, t are both the zero map.

(b) Let $C_3 =< x|x^3 = 1 >$ and $C_2 =< y|y^2 = 1 >$. The zero homomorphism and the trivial action, 1, make $(C_3, C_2, 0, 1)$ a crossed module. The semidirect product corresponding to $(C_3, C_2, 0, 1)$ is $C_3 \rtimes C_2 \cong C_6$ and so (C_6, C_2, i, s, s), where the structural homomorphisms are identical and send odd powers of the generator of C_6 to the generator of C_2 and even powers to the identity.

(c) Let $C_4 =< x|x^4 = 1 >$ and $C_2 =< y|y^2 = 1 >$ and $y = x^2$. Action by conjugation fixes each element of C_4, and together with the boundary ∂ defined by $x \longrightarrow y$, gives a crossed module, (C_4, C_2, ∂). Since C_4 is abelian, action by conjugation fixes every element of C_4. (C_4, C_2, ∂) with action by conjugation yields $C_4 \rtimes C_2 \cong C_4 \times C_2$ and leads to the cat^1-group $(C_4 \times C_2, C_2, i, s, t)$ with s the projection onto C_2 and t the homomorphism sending both $(x, 1)$ and $(1, y)$ to y.

6. Character of Crossed Modules

To any finite crossed module $\chi = (G, M, \mu, \partial)$ we will associate a braided tensor category $\mathcal{M}(\chi)$. An object of $\mathcal{M}(\chi)$ is a 3-tuple (V, P, Q), where V is a complex linear space, while P and Q are maps $P : M \longrightarrow End(V)$ and $Q : G \longrightarrow GL(V)$ such that for all $g, h \in G$ and $m, n \in M$:

- $P(m)P(n) = \delta(m,n)P(m)$ where $\delta(m,n) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise,} \end{cases}$
- $\sum_{m\in M} P(m) = id_V$,
- $Q(g)Q(h) = Q(gh)$,
- $P(m)Q(g) = Q(g)P(m^g)$.

Definition 13. A morphism $\phi : (V_1, P_1, Q_1) \longrightarrow (V_2, P_2, Q_2)$ between two objects of $\mathcal{M}(\chi)$ is a linear map $\phi : V_1 \longrightarrow V_2$ such that $\phi \circ P_1(m) = P_2(m) \circ \phi$ for all $m \in M$ and $\phi \circ Q_1(g) = Q_2(g) \circ \phi$ for all $g \in G$.

EXAMPLE 4. The triple $\underline{\mathbf{1}} = (V, P, Q)$ with $V = \mathbb{C}$, $P(m) = \delta(m, 1)id_V$ and $Q(g) = id_V$, is a one dimensional object of $\mathcal{M}(\chi)$, that we will call the trivial object.

Definition 14. If (V, P, Q) is an object of $\mathcal{M}(\chi)$, a linear subspace $W < V$ is invariant if $P(m)W \subset W$ and $Q(g)W \subset W$ for all $m \in M$ and $g \in G$. An object (V, P, Q) is reducible if it has a nontrivial invariant subspace, otherwise it is irreducible.

Definition 15. The character of an object (V, P, Q) of $\mathcal{M}(\chi)$ is the complex valued function $\psi : M \times G \longrightarrow \mathcal{C}$ given by $\psi(m, g) = Tr_V(P(m)Q(g))$.

A function is called class function if it is constant on conjugate classes.

Proposition 1. *The character ψ of an object of $\mathcal{M}(\chi)$ is a class function of the crossed module.*

Proof. The proof is straightforward, because of, a complex valued function $\psi : M \times G \longrightarrow \mathcal{M}$ we have

- $\psi(m, g) = 0$ unless $m^g = m$, for $m \in M$ and $g \in G$;
- $\psi(m^h, h^{-1}gh) = \psi(m, g)$ for all $m \in M$ and $g, h \in G$.

□

Definition 16. Let $\chi = (G, M, \mu, \partial)$ be a crossed module and (m, g), (m', g') elements of $M \times G$. Then (m, g) and (m', g') are called the conjugate, if there is $h \in G$ such that $(m', g'^h, h^{-1}gh)$.

Definition 17. If $\chi = (G, M, \mu, \partial)$ is a crossed module and $(m, g) \in M \times G$, then conjugacy class (m, g) is

$$\{(m^h, h^{-1}gh) \mid \quad h \in G\}.$$

Proposition 2. *Let $\chi = (G, M, \mu, \partial)$ be a crossed module. Then (m, g), (m', g') elements of $M \times G$ are conjugate if and only if for all character ψ of χ, $\psi(m, g) = \psi(m', g')$.*

Proof. The proof is straightforward. □

Proposition 3. *Let $\chi = (G, M, \mu, \partial)$ be a crossed module. Then (m, g) is conjugate of (m^{-1}, g^{-1}), if and only if $\psi(m, g)$ is real for all $\psi \in Irr(\chi)$.*

Proof. We know that $\psi(m^{-1}, g^{-1}) = \overline{\psi(m, g)}$, additionally by Proposition 2 we have $\psi(m, g) = \psi(m^{-1}, g^{-1})$. Therefore, $\overline{\psi(m, g)} = \psi(m, g)$. So $\psi(m, g)$ is real for all $\psi \in Irr(\chi)$. □

Definition 18. Let $\chi = (G, M, \mu, \partial)$ be a crossed module and ψ be a character. Then $\psi(1, 1)$ is called the degree character of ψ.

Definition 19. Characters of degree 1 are called linear characters.

Proposition 4. *If ψ is the character of χ, $\psi \neq 0$ and ψ is a homomorphism, then ψ is a linear character.*

Proof. If ψ is a character (non zero) and a homomorphism, then

$$\psi(1, 1) = \psi\left((1, 1)^2\right) = \psi(1, 1)\psi(1, 1) = (\psi(1, 1))^2 .$$

So $\psi(1, 1) = 1$. Therefore, ψ is a linear character. □

The next step in the theory of characters is to put a Hermitian inner product structure on the space of class functions and prove that the irreducible characters form an orthonormal basis with respect to this inner product.

Irreducible characters, i.e., the characters of the irreducible objects of $\mathcal{M}(\chi)$, play a distinguished role, since any character may be written as a linear combination, of irreducible ones with non-negative integer coefficients. We refer to [2] for basic results about irreducible characters.

Theorem 2. *(Generalized Orthogonality Relation) If $\chi = (G, M, \mu, \partial)$ is a finite crossed module, then*

$$\frac{1}{|G|} \sum_{h \in G} \psi_p(m, h)\psi_q(m, h^{-1}g) = \frac{1}{d_p} \delta_{pq} \psi_p(m, g)$$

for $p, q \in Irr(\chi)$, where $d_p = \sum_{m \in M} \psi_p(m, 1)$ denotes the dimension of the irreducible p.

Theorem 3. *(Second Orthogonality Relation) If $\chi = (G, M, \mu, \partial)$ is a finite crossed module, then*

$$\sum_{p \in Irr(\chi)} \psi_p(m, g)\psi_p(n, h) = \sum_{z \in G} \delta(n, m^z)\delta(h^{-1}, g^z).$$

Example 5. Suppose that the triple $\underline{\mathbf{1}} = (V, P, Q)$, with $V = \mathbb{C}$, $P(m) = \delta(m, 1)id_V$ and $Q(g) = id_V$. Then, we have $\psi_{\underline{\mathbf{1}}}(m, g) = \delta(m, 1)$.

In a finite crossed module $\chi = (G, M, \mu, \partial)$, we put $I = im\partial$ and $K = ker\partial$.

Theorem 4. *To each irreducible $p \in Irr(\chi)$, associate the complex number*

$$w_p = \frac{1}{d_p} \sum_{m \in M} \psi_p(m, \partial m),$$

which turns out to be a root of unity, and $\psi_p(m, g\partial m) = w_p \psi_p(m, g)$, for all $m \in M$, $g \in G$. Additionally,

$$\sum_{p \in Irr(\chi)} d_p^2 w_p^{-1} = |G||K|.$$

According to [2] we have:

Theorem 5. *(Frobenius-Schur): Let $\chi = (G, M, , \mu, \partial)$ be a finite crossed module, then*

$$\nu_p = \frac{1}{|G|} \sum_{m \in M, g \in G} \delta(m^g, m^{-1}) \psi(m, g^2),$$

where ψ_p is a irreducible character, and additionally, ν_p may take only the values 0 and ± 1.

Definition 20. For class functions ψ_1, ψ_2 of a crossed module $\chi = (G, M, \mu, \partial)$ define

$$< \psi_1, \psi_2 >= \frac{1}{|G|} \sum_{m \in M,\, g \in G} \overline{\psi_1(m, g)} \psi_2(m, g)$$

where the bar denotes the complex conjugation.

Proposition 5. [2] *The set of class functions of a finite crossed module χ form a finite dimensional linear space $\mathcal{PL}(\chi)$, which carries the natural scalar product*

$$< \psi_1, \psi_2 >= \frac{1}{|G|} \sum_{m \in M,\, g \in G} \overline{\psi_1(m, g)} \psi_2(m, g)$$

where $\psi_1, \psi_2 \in \mathcal{PL}(\chi)$, and the bar denotes the complex conjugation.

Lemma 2. *For class functions ψ_1, ψ_2 of crossed module $\chi = (G, M, \mu, \partial)$, the inner product in Definition 20, is Hermitian.*

Proof. For all $\lambda \in \mathbb{C}$, $\psi_1, \psi_2, \psi_3 \in cf(\chi, \mathbb{C})$ we have

$$\begin{aligned} < \psi_1 + \lambda\psi_2, \psi_3 > &= \frac{1}{|G|} \sum_{m \in M\, g \in G} (\psi_1 + \lambda\psi_2)(m, g) \overline{\psi_3(m, g)} \\ &= \frac{1}{|G|} \sum \psi_1(m, g) \overline{\psi_3(m, g)} + \lambda \frac{1}{|G|} \sum \psi_2(m, g) \overline{\psi_3(m, g)} \\ &= < \psi_1, \psi_2 > + \lambda < \psi_2, \psi_3 > \end{aligned}$$

also

$$\begin{aligned} <\psi_1,\psi_2+\lambda\psi_3> &= \frac{1}{|G|}\sum\psi_1(m,g)\overline{(\psi_2+\lambda\psi_3)(m,g)} \\ &= \frac{1}{|G|}\sum\psi_1(m,g)\overline{\psi_2(m,g)}+\lambda\frac{1}{|G|}\sum\psi_1(m,g)\overline{\psi_3(m,g)} \\ &= <\psi_1,\psi_2>+\bar{\lambda}<\psi_1,\psi_3> \end{aligned}$$

and

$$\begin{aligned} <\psi_1,\psi_2> &= \frac{1}{|G|}\sum\psi_1(m,g)\overline{\psi_2(m,g)} \\ &= \frac{1}{|G|}\sum\overline{\overline{\psi_1(m,g)}\psi_2(m,g)} \\ &= \frac{1}{|G|}\sum\overline{\psi_2(m,g)\overline{\psi_1(m,g)}}=\overline{<\psi_2,\psi_1>} \end{aligned}$$

also

$$\begin{aligned} <\psi_1,\psi_1> &= \frac{1}{|G|}\sum\psi_1(m,g)\overline{\psi_1(m,g)} \\ &= \frac{1}{|G|}\sum|\psi_1(m,g)|^2\geq 0. \end{aligned}$$

□

Theorem 6. *(First Orthogonality Relation): Let $\chi=(G,M,\mu,\partial)$ be a finite crossed module and ψ_1, ψ_2 be the irreducible characters of χ on $\mathbb{C}$. Then with respect to the inner product $<,>$;*

$$\sum_{m\in M,g\in G}\psi_1(m,g)\psi_2(m^{-1},g^{-1})=0.$$

Proof. We have

$$\psi_1(m,g)=\sum_{m\in M,\,g\in G}a_{ii}(m,g),\qquad \psi_2(m,g)=\sum_{m\in M,\,g\in G}b_{jj}(m^{-1},g^{-1}).$$

Thus

$$\begin{aligned} \sum_{m\in M,\,g\in G}\psi_1(m,g)\psi_2(m^{-1},g^{-1}) &= \sum_{m\in M,\,g\in G}\sum_{i,j}a_{ii}(m,g)b_{jj}(m^{-1},g^{-1}) \\ &= \sum_{i,j}\sum_{m\in M,\,g\in G}a_{ii}(m,g)b_{jj}(m^{-1},g^{-1})=0. \end{aligned}$$

□

Theorem 7. [2] *(Generalization of Burnside's Classical Theorem) For a finite crossed module χ there are only finitely many isomorphism classes of irreducible* *objects in $M(\chi)$, and*

$$\sum_{p\in Irr(\chi)} d_p^2 = |G||M|,$$

where we denote by $Irr(\chi)$ the set of (isomorphism classes of) irreducible objects of $M(\chi)$, and d_p denotes the dimension of the irreducible $p \in Irr(\chi)$.

Theorem 8. *Let $\chi = (G, M, \mu, \partial)$ be a finite crossed module and ψ be an irreducible character of χ. Then*

$$\sum_{m\in M,\, g\in G} \psi(m, g)\psi(m^{-1}, g^{-1}) = |G||M|.$$

Proof. We have

$$\begin{aligned}\sum_{m\in M,\, g\in G} \psi(m, g)\psi(m^{-1}, g^{-1}) &= \sum_{m\in M,\, g\in G}\sum_{i,j} a_{ii}(m, g)b_{jj}(m^{-1}, g^{-1})\\ &= \sum_{i,j}\sum_{m\in M,\, g\in G} a_{ii}(m, g)b_{jj}(m^{-1}, g^{-1}) = |G||M|.\end{aligned}$$

□

Theorem 9. *If $\chi = (G, M, \mu, \partial)$ is a finite crossed module and ψ_1, ψ_2 are the irreducible and not equivalent characters of χ, then $\psi_1 \neq \psi_2$.*

Proof. If ψ_1, ψ_2 are not equivalent, by First Orthogonality Relation Theorem 6 we have,

$$\sum_{m\in M,\, g\in G} \psi_1(m, g)\psi_2(m^{-1}, g^{-1}) = 0.$$

But if $\psi_1 \neq \psi_2$ by Theorem 8 we have,

$$\sum_{m\in M,\, g\in G} \psi_1(m, g)\psi_2(m^{-1}, g^{-1}) = \sum_{m\in M,\, g\in G} \psi_1(m, g)\psi_1(m^{-1}, g^{-1}) = |M||G|$$

that is a contradiction. □

The notion of direct sum of objects of $M(\chi)$ is the obvious one:

$$(V_1, P_1, Q_1) \oplus (V_2, P_2, Q_2) = (V_1 \oplus V_2, P_1 \oplus P_2, Q_1 \oplus Q_2).$$

Theorem 10. [2] *(Maschke's Theorem) For a finite crossed module χ, any object of $M(\chi)$ decomposes uniquely (up to ordering) into a direct sum of irreducible objects.*

Lemma 3. *If $\chi = (G, M, \mu, \partial)$ is a finite crossed module, ψ is the character of χ and $g \in G$, $m \in M$, $(o(g), o(m)) = k$, then*

- *$\psi(g, m)$ is a sum of all kth roots of 1.*
- *$\psi(m^{-1}, g^{-1}) = \overline{\psi(m, g)}$.*
- *$\sum_{m\in M,\, g\in G} \psi(m, g)\psi(m^{-1}, g^{-1}) > 0$.*

Proof. The proof is straightforward. □

Definition 21. Let ψ be a character of crossed module $\chi = (G, M, \mu, \partial)$. Then

$$Ker\psi = \{(m, g) \mid \psi(m, g) = \psi(1, 1),\ m \in M,\ g \in G\}.$$

Definition 22. Let ψ be a character of crossed module $\chi = (G, M, \mu, \partial)$. Then

$$Z(\psi) = \{(m, g) \mid |\psi(m, g)| = \psi(1, 1),\ m \in M,\ g \in G\}.$$

Lemma 4. *Let ψ be a character of crossed module $\chi = (G, M, \mu, \partial)$ with $\psi = \sum n_i\psi_i$ for $\psi_i \in Irr(\chi)$. Then $Ker\psi = \bigcap\{Ker\psi_i \mid n_i > 0\}$. Also, we have $\bigcap\{Ker\psi_i \mid \psi_i \in Irr(\chi)\} = 1$.*

Proof. If $(m, g) \in Ker\psi$, then $\psi(m, g) = \psi(1, 1)$. But

$$\begin{aligned} \psi(1,1) &= |\psi(m,g)| = |\sum_{i=1}^{h} n_i\psi_i(m,g)| \\ &\leq \sum_{i=1}^{h} n_i|\psi_i(m,g)| \leq \sum_{i=1}^{h} n_i\psi_i(1,1) = \psi(1,1). \end{aligned}$$

So the above equality holds if and only if $\psi_i(m, g) = \psi(1, 1)$ for all $1 \leq i \leq h$. So $(m, g) \in Ker\psi_i$ for those of ψ_i that is positive for them. Thus $Ker\psi = \bigcap\{Ker\psi_i \mid n_i > 0\}$. Proof for second statement is outright. □

Lemma 5. *Let $\chi = (G, M, \mu, \partial)$ be a crossed module and $Y = (Y_1, Y_2, \mu, \partial)$ be a normal subcrossed module of them, ψ be a character of $\frac{\chi}{Y}$, then $\bar{\psi}$ with criterion $\bar{\psi}(m, g) = \psi(mY_2, gY_1)$ is a character of χ . For the converse, the characters of χ and Y that are in the kernel of them, are characters of $\frac{\chi}{Y}$.*

Proof. The proof is straightforward. □

Corollary 1. *Let $\chi = (G, M, \mu, \partial)$ be a crossed module and Y be a normal subcrossed module of them, then $Irr\frac{\chi}{Y} = \{\psi \in Irr(\chi) \mid Y \subseteq Ker\psi\}$.*

Proof. The proof is straightforward. □

7. Products of Characters

Let ψ and η be characters of crossed module $\chi = (G, M, \mu, \partial)$. The fact that $\psi + \eta$ is a character, is a triviality. We may define a new class function $\psi\eta$ of χ be setting $(\psi\eta)(m, g) = \psi(m, g)\eta(m, g)$. It is true but somewhat less trivial that $\psi\eta$ is a character.

Let V_1 and V_2 be $\mathbb{C}[G]$-modules. We shall construct a new $\mathbb{C}[G]$-module $V_1 \otimes V_2$ called the tensor product of V_1 and V_2. If $\{v_1, \ldots, v_n\}$ be a bases for V_1 and $\{w_1, \ldots, w_m\}$ for V_2, then $V_1 \otimes V_2$ is the $\mathbb{C}$-space spanned by the mn symbols $v_i \otimes w_j$. If $v \in V_1$ and $w \in V_2$, suppose $v = \sum a_iv_i$ and $w = \sum b_jw_j$, then $v \otimes w = \sum a_ib_j(v_i \otimes w_j) \in V_1 \otimes V_2$.

We define an action of G on $V_1 \otimes V_2$ by setting $(v_i \otimes w_j)g = v_ig \otimes w_jg$ and extending this by linearity to all of $V_1 \otimes V_2$.

Definition 23. The tensor product of the objects (V_1, P_1, Q_1) and (V_2, P_2, Q_2) is the triple $(V_1 \otimes V_2, P_{12}, Q_{12})$ where $P_{12} : m \longrightarrow \sum_{n \in M} P_1(n) \otimes P_2(n^{-1}m)$ and $Q_{12} : g \longrightarrow Q_1(g) \otimes Q_2(g)$.

The category $M(\chi)$ may be shown to be a monoidal tensor category, which in general fails of to be symmetric, but it is always braided, the braiding being provided by the map

$$\begin{aligned} R_{12} : V_1 \otimes V_2 &\longrightarrow V_2 \otimes V_1 \\ v_1 \otimes v_2 &\longrightarrow \sum_{m \in M} Q_2(\partial m) v_2 \otimes P_1(m) v_1. \end{aligned}$$

Characters behave well under the direct sums and tensor products. The character of a direct sum is just the sum of the characters of the summands, while the character of a tensor products is given by the formula

$$\psi_{A \otimes B}(m, g) = \sum_{n \in M} \psi_A(n, g) \psi_B(n^{-1}m, g),$$

if ψ_A and ψ_B are the characters of the factors.

Lemma 6. *If ψ_1, ψ_2 and ψ_3 are characters of a crossed module $\chi = (G, M, \mu, \partial)$, then $< \psi_1\psi_2, \psi_3 > = < \psi_1, \bar{\psi}_2\psi_3 >$.*

Proof.

$$\begin{aligned} < \psi_1\psi_2, \psi_3 > &= \frac{1}{|G|} \sum_{m \in M, g \in G} \psi_1(m, g)\psi_2(m, g)\overline{\psi_3(m, g)} \\ &= \frac{1}{|G|} \sum_{m \in M, g \in G} \psi_1(m, g)\overline{\overline{\psi_2(m, g)}\psi_3(m, g)} = < \psi_1, \bar{\psi}_2\psi_3 > . \end{aligned}$$

□

Proposition 6. *If ψ_1 and ψ_2 are the irreducible characters of crossed module $\chi = (G, M, \mu, \partial)$, then*

$$< \psi_1\psi_2, 1_\chi > = \begin{cases} 1 & \psi_1 = \bar{\psi}_2 \\ 0 & otherwise \end{cases}$$

Proof. By Lemma 6 we have $< \psi_1\psi_2, \psi_3 > = < \psi_1, \bar{\psi}_2\psi_3 >$, so $< \psi_1\psi_2, 1_\chi > = < \psi_1, \bar{\psi}_2 1_\chi > = < \psi_1, \bar{\psi}_2 >$. Also, if ψ_1 is a character of χ, then $\bar{\psi}_1$ is a character and ψ_1 is irreducible, then $\bar{\psi}_1$ is irreducible. Hence

$$< \psi_1\psi_2, 1_\chi > = < \psi_1, \bar{\psi}_2 > = \begin{cases} 1 & \psi_1 = \bar{\psi}_2 \\ 0 & \text{otherwise.} \end{cases}$$

□

In general, products of irreducible characters are not irreducible. For instance, if $\psi \in Irr(\chi)$, nevertheless, 1_χ is a constituent of $\psi\bar{\psi}$, since $< \psi\bar{\psi}, 1_\chi > = < \psi, \psi 1_\chi > = 1$.

Theorem 11. *If M and G are finite abelian groups and $\chi = (G, M, \mu, \partial)$ a crossed modules, then irreducible characters are linear characters.*

Proof. If G and M are finite abelian groups, then $\forall m \in M,\ g \in G$,

$$\text{class}(m, g) = \left\{(m', g') \mid (m', g'^h, h^{-1}gh),\ h \in G\right\} = \{(m, g)\}.$$

So $M \times G$ has $|G||M|$ conjugate class, and hence it has $|G||M|$ irreducible characters, for example $\psi_1, \psi_2, \ldots, \psi_{|G||M|}$. Therefore, $\sum_{i=1}^{|G||M|} \psi_i^2(1,1) = |G||M|$, hence $\psi(1,1) = 1$. □

In the end we present a few examples. But, first recall that:

Theorem 12. *If H, K and ϕ are as in the above definition, then $G = H \rtimes_\phi K$ is a group of order $|G| = |H||K|$.*

Theorem 13. *(The orthogonality relations).*

(1) *Let ψ be a character of a representation (V, P), then ψ is irreducible if and only if $< \psi, \psi >= 1$.*
(2) *If ψ_1, ψ_2 are characters of two non-isomorphic irreducible representations, then $< \psi_1, \psi_2 >= 0$.*

Theorem 14. *The numbers of irreducible characters are the same as the number of conjugate classes of G.*

Example 6. If $\chi = (\mathbb{Z}_p, \mathbb{Z}_p, \mu, \partial)$ is a finite crossed module by p a prime, then we first observe that the conjugate classes of a direct products is the products of a class in each of the factors, thus we have in our case p^2 conjugate classes, for $\mathbb{Z}_p \times \mathbb{Z}_p$.

We make the assumption that the characters will be

$$\psi_{\psi_1\psi_2}(m, g) = \psi_{\psi_1}(m)\psi_{\psi_2}(g),$$

where ψ_1 and ψ_2 are irreducible characters of $\mathbb{Z}_p$. Now, we want to verify that these functions are p^2 irreducible characters by using Theorem 13 and by Theorem 14 we know that there are no other irreducible character and we are done.

$$< \psi_{\psi_1\psi_2}, \psi_{\psi_1\psi_2} >= \frac{1}{|\mathbb{Z}_p \times \mathbb{Z}_p|} \sum_{m,g\in\mathbb{Z}_p} \psi_{\psi_1\psi_2}(m, g)\overline{\psi_{\psi_1\psi_2}(m, g)}$$

$$= \frac{1}{p^2} \sum_{m,g\in\mathbb{Z}_p} \psi_1(m)\psi_2(g)\overline{\psi_1(m)}\,\overline{\psi_2(g)} = \frac{1}{p^2} \sum_{m,g\in\mathbb{Z}_p} e^{\frac{2\pi i}{p}\alpha m} e^{\frac{2\pi i}{p}\beta g} e^{\frac{-2\pi i}{p}\alpha' m} e^{\frac{-2\pi i}{p}\beta' g}$$

$$= \frac{1}{p^2} \sum_{m\in\mathbb{Z}_p} e^{\frac{2\pi i}{p}(\alpha-\alpha')m} \sum_{g\in\mathbb{Z}_p} e^{\frac{2\pi i}{p}(\beta-\beta')g} = \frac{1}{p^2} p\delta_{\alpha\alpha'} p\delta_{\beta\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}.$$

Here we have used that the sum over all the p-th roots of unity is zero and thus we have found all of characters.

Example 7. If $\chi = (\mathbb{Z}_p, \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p, \mu, \partial)$ is a finite crossed module, by p a prime, then conjugate classes of $(\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$ are represented by the products of the representatives of the classes in $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ and $\mathbb{Z}_p$. So, we have $p(p^2+p-1)$ classes in $(\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p$ represented by $((x \, mod \, p, y), z)$ and the center is $\{((pr, o), z) \mid r, z \in \mathbb{Z}_p\}$. We then proceed as we did in Example 6 and the characters as are

	$((pr,0),z)$	$((x \, mod \, p, y), z)$
$\psi_{\psi_1\psi_2\psi_3}$	$\lambda^{\psi_3(z)}$	$\lambda^{\psi_1(x)}\lambda^{\psi_2(y)}\lambda^{\psi_3(z)}$
$\phi_{\phi_1\phi_2}$	$p\lambda^{\phi_1(r)}\lambda^{\phi_2(z)}$	0

Example 8. If $\chi = (\mathbb{Z}_p, (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p, \mu, \partial)$ is a finite crossed module, then we know the characters of $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ from Example 7 and the conjugate classes, so we conclude that we have $p(p^2+p-1)$ classes represented by $(((x,0),z),w)$ and the center is $(((0,y),0),w)$. Thus the character table looks like, where, $\phi_1 \neq 0$

	$(((0,y),0),w)$	$(((x,0),z),w)$
$\psi_{\psi_1\psi_2\psi_3}$	$\lambda^{\psi_3(w)}$	$\lambda^{\psi_1(x)}\lambda^{\psi_2(z)}\lambda^{\psi_3(w)}$
$\phi_{\phi_1\phi_2}$	$p\lambda^{\phi_1(y)}\lambda^{\phi_2(z)}$	0

Example 9. Let $\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in F_p \right\} \leq GL_3(F_p)$ where F_p is the finite field of p elements and the operation is matrix multiplication. This group is of order p^3, since we have p-choices at three locations and also it is non-abelian. It is easily which this group has no element of order p^2 and every element in G raised to the power p is the identity. So G is a non-abelian group of order p^3 and every non-identity element has order p. Hence $G \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$. Now, since we know the irreducible characters of both $\mathbb{Z}_p$ and $\mathbb{Z}_p \times \mathbb{Z}_p$, we can look at the induced characters of these subgroups, but we are only interested in the case $\mathbb{Z}_p \times \mathbb{Z}_p$, since the degree of an induced representation is related to H and so we want $[G : H] = p$.
We need to find the conjugate classes of G. But if $g = ((x,y),z) \in G$, then g is equivalent to the matrix $\begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$, and that multiplication in G is with this presentation just matrix multiplication which gives the formula

$$((x,y),z) * ((x',y'),z') = ((x+x', y+y'+x'z), z+z').$$

Now, in order to find the conjugates of g by h, if $h = ((a, b), c)$, then $h^{-1}gh$ is

$$\begin{pmatrix} 1 & -c & ca-b \\ 0 & 1 & -a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z & y+za-xc \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Let x and z be non zero. We get p conjugates and if $x = z = 0$, then g is in the center of G. Hence we have $p^2 - 1$ different classes of this type, and $Z(G) = \{((0, y), 0) \mid y \in \mathbb{Z}_p\}$. If $H = \{((x, y), 0)\} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $[G : H] = p$. Let $\psi_{\varepsilon\eta}(x, y) = \lambda^{\varepsilon x}\lambda^{\eta y}$ be the characters of H. Then

$$\psi((x, y), z) = \frac{1}{|H|} \sum_{t\in G, t^{-1}gt\in H} \psi_{\varepsilon\eta}(t^{-1}gt).$$

Hence

$$\psi((x, y), z) = \frac{1}{p^2} \sum_{\substack{((a,b),c)\in G \\ ((x,y+za-xc),z)\in H}} \lambda^{\varepsilon x}\lambda^{\eta(y+za-xc)},$$

and with the definition of H we get that $\psi = 0$ if $z \neq 0$ and $\psi((x, y), 0) = \lambda^{\varepsilon x}\lambda^{\eta y}\sum_c \lambda^{-\eta cx}$. The sum $\sum_c \lambda^{-\eta xc}$ is a sum of all roots of unity and will vanish when $\eta x \neq 0$, but η is arbitrary and this implies that ψ vanishes whenever $x \neq 0$. But $\psi(g)$ is zero whenever $g \notin Z(G)$, so character ψ which takes its non-zero values on the center as $\psi_\eta((0, y), 0) = p\lambda^{\eta y}$. But

$$< \psi_\eta, \psi_{\eta'} >= \frac{1}{|G|}\sum_{g\in G} \psi_\eta(g)\overline{\psi_{\eta'}(g)} = \frac{1}{p^3}\sum_y p\lambda^{\eta y}p\lambda^{-\eta' y} = \frac{1}{p}\sum_y \lambda^{(\eta-\eta')y} = \delta_{\eta\eta'},$$

hence characters are irreducible.

Also, we have

$$< \psi_\eta, \phi_{\alpha\beta} >= \frac{1}{|G|}\sum_{g\in G} \psi_\eta(g)\overline{\phi_{\alpha\beta}(g)} = \frac{1}{p^3}\sum_{((x,y),z)\in Z(G)} p\lambda^{\eta y}\lambda^{-\alpha x}\lambda^{-\beta z} = \frac{1}{p^2}\sum_y \lambda^{\eta y}$$

and if $\eta \neq 0$, then $< \psi_\eta, \phi_{\alpha\beta} >= 0$ which is what we needed to prove that the characters are orthogonal. So, we have p^2 characters of degree 1 and $p-1$ characters of degree p.

Let $((0, y), 0)$ be the elements in $Z(G)$ and let $((x, 0), z)$ be the representatives of the non-trivial conjugate classes, and $\alpha, \beta, \eta \in \mathbb{Z}_p$ and $\eta \neq 0$. Then, the character table is

	$((0, y), 0)$	$((x, 0), y)$
$\phi_{\alpha\beta}$	1	$\lambda^{\alpha x+\beta z}$
ψ_η	$p\lambda^{\eta y}$	0

Now, we consider the cat^1-group of crossed module $(\mathbb{Z}_p, \mathbb{Z}_p, \mu, \partial)$. Since $(\mathbb{Z}_p \rtimes \mathbb{Z}_p, \mathbb{Z}_p, \sim, \approx, \sqsupset)$ where $t, s : \mathbb{Z}_p \rtimes \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$, $s(a, b) = b$, $t(a, b) = \partial(a)b$ and $i : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \rtimes \mathbb{Z}_p$, where

$i(b) = (1_{zp}, b)$, observe that we have $p(p^2+p-1)$ classes represented by $(((x,0),z),c)$ and the center is $(((0,y),0),c)$. Thus the character table is

	$(((0,y),0),c)$	$(((x,0),z),c)$
$\phi_{\alpha\beta\gamma}$	$\lambda^{\gamma c}$	$\lambda^{\alpha x}\lambda^{\beta z}\lambda^{\gamma c}$
$\Theta_{\varepsilon\eta}$	$p\lambda^{\varepsilon y}\lambda^{\eta z}$	0

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