TITLE: On the Construction of Generalized Bobillier Formula

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ON THE CONSTRUCTION OF GENERALIZED BOBILLIER FORMULA

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ABSTRACT. In this study, we consider the generalized complex number system $C_p = \{x + iy \ : \ x, y \in R, \ i^2 = p \in R\}$ corresponding to elliptical complex number, parabolic complex number and hyperbolic complex number systems for the special cases of $p < 0$, $p = 0$, $p > 0$, respectively. This system is used to derive Bobillier Formula in the generalized complex plane. In accordance with this purpose we obtain this formula by two different methods for one-parameter planar motion in C_p; the first method depends on using the geometrical interpretation of the generalized Euler-Savary formula and the second one uses the usual relations of the velocities and accelerations.

1. INTRODUCTION

A system of rigid elements (linkages) connected to transmit motion is called mechanism. In other words, a mechanism is a combination of links which can transform a determined motion. A planar mechanism is a mechanical system. So that the trajectories of points in all the bodies of the system is constrained for lying on planes parallel to a grand plane. The rotational axes of hinged joints that connect the bodies in the system are perpendicular to this ground plane, [18]. Kinematics concerned with the characteristics of movement without considering the concepts of mass and force is sub-branch of mechanics and analyzes the displacement of one point or a point system (object) with respect to time.

The Euler-Savary formula which is one of the most used formulas in kinematics and fills an important place in many fields such as mathematics, engineering and astronomy was found by Euler in 1765 and Savary in 1845, [21]. This formula giving the relationship between the curvatures of trajectory curves drawn by the points of the moving plane in the fixed plane was studied by [3, 4, 5, 24] in two and three dimensional Euclidean spaces. The Euler-Savary formula in complex plane was

given by Masal et al., [22]. Then the Euler-Savary formulas in the hyperbolic plane and Galilean plane were obtained in [1, 7], respectively. Moreover, Akbıyık and Yüce considered a base curve, a rolling curve and a roulette on complex plane and obtained Euler-Savary formula which gives the relation between the curvatures of these three curves, [2]. In addition to these, there are too many studies on the Euler-Savary formula including its applications to kinematics, mechanical engineering of robotics and sciences of machines and mechanisms.

The Euler-Savary formulas in Affine Cayley-Klein plane and in generalized complex number plane $C_J = \left\{ x + Jy \ : \ x, y \in R, \quad J^2 = \mathrm{p}, \quad \mathrm{p} \in \{-1, 0, 1\} \right\} \subset C_\mathrm{p}$ were given by [14] and [16], respectively.

In 1988, Fayet defined a formula giving the relation of the curvatures of second order of one-parameter planar motion and generalizing the Euler-Savary formula. Since this formula analytically solves the problem that the Bobillier's construction solved graphically it is called the Bobillier formula, [11].

In [12], it was demonstrated that the Bobillier formula can be obtained without the use of the Euler-Savary formula by Fayet. In addition the Bobillier formula was obtained by conventional procedures in [23].

Ersoy and Bayrak studied the Bobillier formula for the one-parameter planar motion in the complex plane, [9]. Also, they investigated the Bobillier formula in Lorentzian sense and saw that the same results can be achieved without the use of the Euler-Savary formula for the Bobillier formula. In doing so, they considered that the cases of polar curves to be timelike or spacelike, separately, [10].

In Galilean plane, Gürses et al. gave the Bobillier formula by using the geometrical interpretation of Euler-Savary formula in Galilean plane. Moreover, they obtained Bobillier formula without considering the Galilean Euler-Savary formula, [13]. Moreover, Dündar et al. gave Bobillier formula for the elliptical harmonic motion, [6].

The generalized complex numbers play the same role for Cayley-Klein geometry like that the ordinary numbers play in the Euclidean geometry, [27, 28]. The Cayley-Klein plane geometries including Euclidean, Galilean, Minkowskian and Bolyai-Lobachevskian were introduced first by F. Klein and A. Cayley, [19, 20]. After Cayley and Klein, I. M. Yaglom distinguished these geometries with choosing one of three ways of measuring length (parabolic, elliptic or hyperbolic) between two points on a line and one of the three ways of measuring angles (parabolic, elliptic or hyperbolic) between two lines. This gives nine ways of measuring lengths and angles, [28].

In the light of these studies, we construct the Bobillier formula with a new generalization for generalized complex plane C_p. We introduce this generalization in two different ways; using the Euler-Savary formula in the generalized complex plane and an alternative way towards to it.

2. Preliminaries

The generalized complex numbers or binary numbers are introduced as follows

$$z = x + iy \quad (x, y \in R), \qquad i^2 = i\mathrm{q} + \mathrm{p} \quad (\mathrm{q}, \mathrm{p} \in R).$$

The double, dual and ordinary numbers are the particular members of two parameter family of complex number systems. Moreover, the generalized complex number systems are isomorphic to the double, ordinary and dual complex numbers when $\mathrm{p} + \mathrm{q}^2/4$ is positive, negative and zero, respectively, [27].

Unless otherwise stated we assume that $i^2 = \mathrm{p}$ and $\mathrm{q} = 0$ $(\mathrm{p} \in R)$. This complex number system is denoted by $C_\mathrm{p} = \left\{ x + iy \ : \ x, y \in R, \ i^2 = \mathrm{p}, \ \mathrm{p} \in R \right\}$.

The set C_p is called the generalized complex plane.

For $z_1 = (x_1 + iy_1), z_2 = (x_2 + iy_2) \in C_\mathrm{p}$ the addition, subtraction and product are defined by

$$z_1 \pm z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

and

$$M^\mathrm{p}(z_1, z_2) = (x_1 x_2 + \mathrm{p} y_1 y_2) + i(x_1 y_2 + x_2 y_1),$$

respectively. The product definition yields the ordinary, Study and Clifford products as p is equal to -1, 0 and 1

$$M^{-1}(z_1, z_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \ \text{ for } \ \mathrm{p} = -1,$$
$$M^0(z_1, z_2) = (x_1 x_2) + i(x_1 y_2 + x_2 y_1) \ \text{ for } \ \mathrm{p} = 0,$$
$$M^1(z_1, z_2) = (x_1 x_2 + y_1 y_2) + i(x_1 y_2 + x_2 y_1) \ \text{ for } \ \mathrm{p} = 1,$$

respectively, [27].

For $\mathbf{z}_1 = (x_1, y_1), \quad \mathbf{z}_2 = (x_2, y_2) \in C_\mathrm{p}$, the scalar product is as follows

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_\mathrm{p} = \mathrm{Re}\left(M^\mathrm{p}(\mathbf{z}_1, \overline{\mathbf{z}_2})\right) = \mathrm{Re}\left(M^\mathrm{p}(\overline{\mathbf{z}_1}, \mathbf{z}_2)\right) = x_1 x_2 - \mathrm{p} y_1 y_2 \qquad (1)$$

and the magnitude of $\mathbf{z} = (x, y) \in C_\mathrm{p}$ is given by the non-negative real number

$$\|\mathbf{z}\|_\mathrm{p} = \sqrt{|M^\mathrm{p}(\mathbf{z}, \overline{\mathbf{z}})|} = \sqrt{|x^2 - \mathrm{p} y^2|} \qquad (2)$$

where the over bar denotes the complex conjugation. The equalities

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_\mathrm{p} = \|\mathbf{z}_1\|_\mathrm{p} \|\mathbf{z}_2\|_\mathrm{p} \cos \mathrm{p}\theta_\mathrm{p}, \qquad (3)$$

and for $\mathrm{p} \neq 0$

$$\|\mathbf{z}_1 \wedge_\mathrm{p} \mathbf{z}_2\|_\mathrm{p} = \sqrt{|-\mathrm{p}|} \|\mathbf{z}_1\|_\mathrm{p} \|\mathbf{z}_2\|_\mathrm{p} \sin \mathrm{p}\theta_\mathrm{p} \qquad (4)$$

and for $\mathrm{p} = 0$, (from the special definition in Galilean plane)

$$\|\mathbf{z}_1 \wedge_\mathrm{p} \mathbf{z}_2\|_\mathrm{p} = \|\mathbf{z}_1\|_\mathrm{p} \|\mathbf{z}_2\|_\mathrm{p} \sin \mathrm{p}\theta_\mathrm{p} \qquad (5)$$

are valid in the generalized complex plane where θ_p is the p$-$rotation angle between the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$.

In the Euclidean plane, a circle is defined in two ways. The first definition is the locus of points a fixed distance from a given fixed point. The second is the set

of points from which a segment AB is seen at a constant directed angle θ_{p}. Using these definitions, since a unit circle is the geometric locus of points z satisfying $\|z\|_{\mathrm{p}} = 1$, the unit circles in the generalized complex planes are drawn as in Figure 1.Let us interpret Figure 1. If we take p < 0, then the unit circle corresponds to
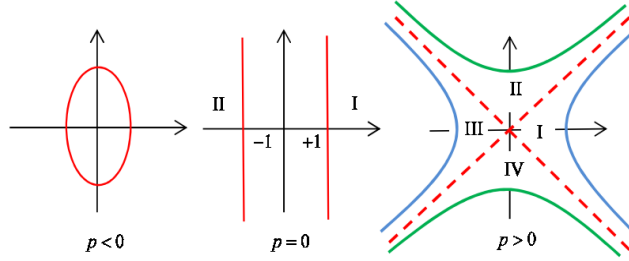

FIGURE 1. Unit circles in C_{p}.

unit ellipse of the form $x^2 + |\mathrm{p}|\, y^2 = 1$ and the complex number system C_{p} (p < 0) corresponds to elliptical complex number system. In this case, if p $= -1$, the unit circle in the generalized complex plane corresponds to a standard unit circle defined by $x^2 + y^2 = 1$ and the generalized complex plane corresponds to the Euclidean plane.

If we consider p $= 0$, the above **circle** definitions yield different sets of points in Galilean geometry. According to the first definition, the equation $\|z\|_0^2 = x^2$ is hold and the unit **circle** is as $x = \pm 1$, (Figure 1) and for the first circle we cannot talk of curvature. According to the second definition, the set of points is called a **cycle**. The **cycle** (inflection cycle, red lines) on Galilean plane is the (Euclidean) parabola, [28]. Therefore, the space C_0 is the parabolic complex number system and the generalized complex plane corresponds to the Galilean plane. Moreover, as can be seen from Figure 1 the parabolic complex plane is divided into two area by the imaginary axis.

Finally, when p > 0, unit circles in C_{p} (p > 0) are the hyperbolas of the form $\left| x^2 - \mathrm{p}y^2 \right| = 1$ whose asymptotes are $y = \pm x / \sqrt{\mathrm{p}}$ (red dashed line in Figure 1). Thus, the space C_{p} (p > 0) corresponds to hyperbolic complex number system. Especially, if we take p $= 1$, the generalized complex plane is referred to as a well-known hyperbolic plane and the asymptotes of the unit circles separate the hyperbolic planes into four regions, [17].

p−trigonometric functions (p−cosine $(\cos p)$, p−sine $(\sin p)$ and p−tangent $(\tan p)$ are, respectively, defined by

$$\cos p\theta_p = \begin{cases} \cos\left(\theta_p\sqrt{|p|}\right), & p < 0 \\ 1, & p = 0 \quad (branch\,I) \\ \cosh\left(\theta_p\sqrt{p}\right), & p > 0 \quad (branch\,II) \end{cases} ,$$

$$\sin p\theta_p = \begin{cases} \frac{1}{\sqrt{|p|}}\sin\left(\theta_p\sqrt{|p|}\right), & p < 0 \\ \theta_p, & p = 0 \quad (branch\,I) \\ \frac{1}{\sqrt{p}}\sinh\left(\theta_p\sqrt{p}\right), & p > 0 \quad (branch\,II) \end{cases} , \tag{6}$$

and

$$\tan p\theta_p = \frac{\sin p\theta_p}{\cos p\theta_p}. \tag{7}$$

Additionally, the derivatives of p−trigonometric functions are as follows,

$$\frac{d}{d\theta_p}(\cos p\theta_p) = p\sin p\theta_p, \qquad \frac{d}{d\theta_p}(\sin p\theta_p) = \cos p\theta_p \tag{8}$$

and the generalized Euler's formula is as follows

$$e^{i\theta_p} = \cos p\theta_p + i\sin p\theta_p. \tag{9}$$

The polar and exponential forms of any generalized complex number z are

$$z = r_p(\cos p\theta_p + i\sin p\theta_p) = r_p e^{i\theta_p}$$

where $r_p = \|z\|_p$ and θ_p are the p−magnitude and argument of z, respectively, [17].

Let K_p and K'_p be the moving and fixed generalized complex planes and $\{O; \mathbf{t}_1, \mathbf{t}_2\}$ and $\{O'; \mathbf{t}'_1, \mathbf{t}'_2\}$ be the perpendicular coordinate system of these planes, respectively. Moreover, we consider that the one-parameter planar motion K_p/K'_p in the generalized complex plane C_p. The geometric loci of the pole points in the fixed and moving generalized complex planes K'_p and K_p are called fixed pole curve (Q') and moving pole curve (Q), respectively.

A point X taken in the moving generalized complex plane K_p draws a trajectory which has curvature center X' in the fixed generalized complex plane K'_p. Conversely, in the reverse motion K'_p/K_p in C_p the point X' in K'_p draws a trajectory with curvature center X in K_p. This relation between the points X and X' is given by Euler-Savary Formula with the following equation

$$\frac{1}{r'} - \frac{1}{r} = \mathrm{Im}\left(e^{i\theta_p}\right)\left(\frac{1}{a'} - \frac{1}{a}\right) \tag{10}$$

where r and r' are the radii of curvature of the pole curves (Q) and (Q') of one-parameter planar motion K_p/K'_p and a, a' represent the distances from the rotation pole to the point X, X', respectively. This formula for homothetic motion on the

plane $C_J = \left\{ x + Jy : x, y \in R, \ J^2 = \mathrm{p}, \ \mathrm{p} \in \{-1, 0, 1\} \right\} \subset C_\mathrm{p}$ was given by Gürses et al.

3. On the Construction of Generalized Bobillier Formula

In this section, we use a method to derive generalized Bobillier formula for generalized complex plane C_p. For this, we regard to the generalized Euler-Savary formula given by the equation (10).

Let K_p and K'_p be the moving and fixed generalized complex planes and $\{O; \mathbf{t}_1, \mathbf{t}_2\}$ and $\{O'; \mathbf{t}'_1, \mathbf{t}'_2\}$ be the perpendicular coordinate system of these planes, respectively. Moreover, we assume that the points N_p^1, N_p^2 and N_p^3 are the fixed points in the moving generalized complex plane K_p and the points γ_p^1, γ_p^2 and γ_p^3 are centers of curvature of trajectory drawn by these fixed points in the fixed generalized complex plane K'_p.

The normals of trajectories drawn by these points pass from the instantaneous rotation center I and there is a p$-$rotation pole at each t moment in C_p. The pole curves rolls upon each other without sliding during the motion of the generalized complex planes $K_\mathrm{p}/K'_\mathrm{p}$. So, the pole curves (Q), (Q') are tangent to each other and have the same velocity at each t moment. Thus, the real axis is the common tangent and the imaginary axis is the common normal for the pole curves in C_p.

If θ_p is the p$-$rotation angle of motion of the plane with respect to K'_p at each t moment, then each point N_p^i makes a rotation motion with $\dot{\theta}_\mathrm{p}$ angular velocity at the instantaneous center I.

Let

$$X_\mathrm{p}^1 = \frac{\mathbf{IN}_\mathrm{p}^1}{\left\| \mathbf{IN}_\mathrm{p}^1 \right\|}, \qquad \mathbf{X}_\mathrm{p}^2 = \frac{\mathbf{IN}_\mathrm{p}^2}{\left\| \mathbf{IN}_\mathrm{p}^2 \right\|}, \qquad \mathbf{X}_\mathrm{p}^3 = \frac{\mathbf{IN}_\mathrm{p}^3}{\left\| \mathbf{IN}_\mathrm{p}^3 \right\|} \qquad (11)$$

be the unit vectors in the direction of the pole rays IN_p^1, IN_p^2 and IN_p^3, respectively, (see Figure 2 for the special cases of $p < 0, p = 0, p > 0$).

The $p-$distances of the points N_p^k and γ_p^k from the origin I are ρ_k and ρ'_k $(k = 1, 2, 3)$, respectively, then from the equations (3) and (4) we can write

$$\left| \left\langle \mathbf{IN}_\mathrm{p}^1, \mathbf{X}_\mathrm{p}^1 \right\rangle_\mathrm{p} \right| = \rho_1, \qquad \left| \left\langle \mathbf{I}\gamma_\mathrm{p}^1, \mathbf{X}_\mathrm{p}^1 \right\rangle_\mathrm{p} \right| = \rho'_1. \qquad (12)$$

Similarly,

$$\left| \left\langle \mathbf{IN}_\mathrm{p}^2, \mathbf{X}_\mathrm{p}^2 \right\rangle_\mathrm{p} \right| = \rho_2, \qquad \left| \left\langle \mathbf{I}\gamma_\mathrm{p}^2, \mathbf{X}_\mathrm{p}^2 \right\rangle_\mathrm{p} \right| = \rho'_2$$

and

$$\left| \left\langle \mathbf{IN}_\mathrm{p}^3, \mathbf{X}_\mathrm{p}^3 \right\rangle_\mathrm{p} \right| = \rho_3, \qquad \left| \left\langle \mathbf{I}\gamma_\mathrm{p}^3, \mathbf{X}_\mathrm{p}^3 \right\rangle_\mathrm{p} \right| = \rho'_3.$$

An inflection point may be defined to be a point whose trajectory momentarily has an infinite radius of curvature. Such points also have zero acceleration normal to their trajectory, [3, 25, 26]. Let the inflection points be N_p^{1*}, N_p^{2*} and N_p^{3*}. The

locus of such points is a circle in the moving generalized complex plane K_p called as an inflection circle. So, from the equation (3) and (4) we can write

$$\left|\left\langle \mathbf{IN}_\mathrm{p}^{1\,*}, \mathbf{X}_\mathrm{p}^1 \right\rangle_\mathrm{p}\right| = \rho_1^*, \qquad \left|\left\langle \mathbf{IN}_\mathrm{p}^{2\,*}, \mathbf{X}_\mathrm{p}^2 \right\rangle_\mathrm{p}\right| = \rho_2^*, \qquad \left|\left\langle \mathbf{IN}_\mathrm{p}^{3\,*}, \mathbf{X}_\mathrm{p}^3 \right\rangle_\mathrm{p}\right| = \rho_3^* \quad (13)$$
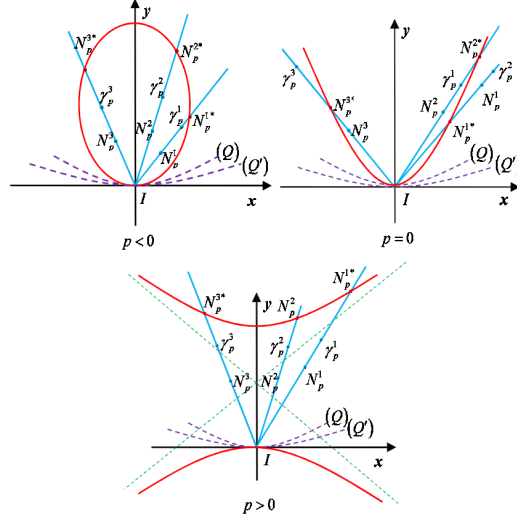

FIGURE 2. The special cases of the inflection circles of motion in the generalized complex plane.

On the other hand the points move around an trajectory whose instantaneous center is I during the motion $K_\mathrm{p}/K_\mathrm{p}'$. So, the amount of displacement of the point I is equal to the product of the diameter h with the amount of angular displacement.

We take that the inflection points N_p^{1*}, N_p^{2*} and N_p^{3*} are on the direction of (I, X_p^1), (I, X_p^2) and (I, X_p^3), respectively. Moreover, we assume that, the images of these inflection points are Q_p^1, Q_p^2 and Q_p^3 where $IQ_\mathrm{p}^k = \frac{1}{\rho_k^*} X_\mathrm{p}^k$, $(1 \leq k \leq 3)$, (Figure 3,4,5). Then, from the equations (3) and (4) we have

$$\left|\left\langle \mathbf{IQ}_\mathrm{p}^1, \mathbf{X}_\mathrm{p}^1 \right\rangle_\mathrm{p}\right| = \frac{1}{\rho_1^*}, \qquad \left|\left\langle \mathbf{IQ}_\mathrm{p}^2, \mathbf{X}_\mathrm{p}^2 \right\rangle_\mathrm{p}\right| = \frac{1}{\rho_2^*}, \qquad \left|\left\langle \mathbf{IQ}_\mathrm{p}^3, \mathbf{X}_\mathrm{p}^3 \right\rangle_\mathrm{p}\right| = \frac{1}{\rho_3^*}. \qquad (14)$$

So, the following equations hold;

$$\begin{aligned}
\mathbf{IQ}_\mathrm{p}^1 \sin \mathrm{p}\theta_\mathrm{p}^1 &= \tfrac{1}{\rho_1^*} \mathbf{X}_\mathrm{p}^1 \sin \mathrm{p}\theta_\mathrm{p}^1 = \tfrac{1}{h} \mathbf{X}_\mathrm{p}^1, \\
\mathbf{IQ}_\mathrm{p}^2 \sin \mathrm{p}\theta_\mathrm{p}^2 &= \tfrac{1}{\rho_2^*} \mathbf{X}_\mathrm{p}\mathrm{p}^2 \sin \mathrm{p}\theta_\mathrm{p}^2 = \tfrac{1}{h} \mathbf{X}_\mathrm{p}^2, \\
\mathbf{IQ}_\mathrm{p}^3 \sin \mathrm{p}\theta_\mathrm{p}^3 &= \tfrac{1}{\rho_3^*} \mathbf{X}_\mathrm{p}^3 \sin \mathrm{p}\theta_\mathrm{p}^3 = \tfrac{1}{h} \mathbf{X}_\mathrm{p}^3.
\end{aligned} \qquad (15)$$

where the diameter of the inflection cycle is h. The last three equations show that

$$\left| \langle \mathbf{IQ}_{\mathrm{p}}^1, \mathbf{X}_{\mathrm{p}}^1 \rangle_{\mathrm{p}} \right| \sin \mathrm{p}\theta_{\mathrm{p}}^1 = \left| \langle \mathbf{IQ}_{\mathrm{p}}^2, \mathbf{X}_{\mathrm{p}}^2 \rangle_{\mathrm{p}} \right| \sin \mathrm{p}\theta_{\mathrm{p}}^2 = \left| \langle \mathbf{IQ}_{\mathrm{p}}^3, \mathbf{X}_{\mathrm{p}}^3 \rangle_{\mathrm{p}} \right| \sin \mathrm{p}\theta_{\mathrm{p}}^3 = \frac{1}{h}. \quad (16)$$

So, the set of the points Q_{p} is a straight line D parallel to axis x. Thus, the line D is an image of the inflection circle of the generalized complex plane by means of an inversion at the rotation center I, (Figure 3,4,5).

Since the points $Q_{\mathrm{p}}^1, Q_{\mathrm{p}}^2, Q_{\mathrm{p}}^3$ are linear, the vectors $\left(\mathbf{IQ}_{\mathrm{p}}^1 - \mathbf{IQ}_{\mathrm{p}}^2 \right)$ and $\left(\mathbf{IQ}_{\mathrm{p}}^2 - \mathbf{IQ}_{\mathrm{p}}^3 \right)$ are linearly dependent. So, the $p-$cross product of these vectors is

$$\left(\mathbf{IQ}_{\mathrm{p}}^1 - \mathbf{IQ}_{\mathrm{p}}^2 \right) \wedge_{\mathrm{p}} \left(\mathbf{IQ}_{\mathrm{p}}^2 - \mathbf{IQ}_{\mathrm{p}}^3 \right) = 0.$$

Thus, we can write

$$\left(\mathbf{IQ}_{\mathrm{p}}^1 \wedge_{\mathrm{p}} \mathbf{IQ}_{\mathrm{p}}^2 \right) + \left(\mathbf{IQ}_{\mathrm{p}}^3 \wedge_{\mathrm{p}} \mathbf{IQ}_{\mathrm{p}}^1 \right) + \left(\mathbf{IQ}_{\mathrm{p}}^2 \wedge_{\mathrm{p}} \mathbf{IQ}_{\mathrm{p}}^3 \right) = 0.$$

If we consider the equation (20), we can easily write

$$\frac{1}{\rho_1^* \rho_2^*} \left(\mathbf{X}_{\mathrm{p}}^1 \wedge_{\mathrm{p}} \mathbf{X}_{\mathrm{p}}^2 \right) + \frac{1}{\rho_1^* \rho_3^*} \left(\mathbf{X}_{\mathrm{p}}^3 \wedge_{\mathrm{p}} \mathbf{X}_{\mathrm{p}}^1 \right) + \frac{1}{\rho_2^* \rho_3^*} \left(\mathbf{X}_{\mathrm{p}}^2 \wedge_{\mathrm{p}} \mathbf{X}_{\mathrm{p}}^3 \right) = 0.$$

Considering that $\rho_1^* \rho_2^* \rho_3^* \neq 0$, the last equation and the equation (5), we obtain that

$$\rho_1^* \sin \mathrm{p}\theta_{\mathrm{p}}^{23} + \rho_2^* \sin \mathrm{p}\theta_{\mathrm{p}}^{31} + \rho_3^* \sin \mathrm{p}\theta_{\mathrm{p}}^{12} = 0 \qquad (17)$$

where $\frac{1}{\rho_k^*} = \frac{1}{\rho_k} - \frac{1}{\rho'_k}$ and θ_{p}^{lm} are the $p-$rotation angles between X_{p}^l and X_{p}^m for the permutations of the indices $k, l, m = 1, 2, 3 \; ; \; 2, 3, 1 \; ; \; 3, 1, 2$.

This formula is called of the generalized Bobillier formula in C_{p}. The special cases of the generalized Bobillier formula with respect to the sign of real number p are as follows;

Case 1. If we take $p < 0$, the generalized complex number system C_{p} corresponds to the elliptical complex number system. From the equation (6), the generalized Bobillier formula is equal to

$$\rho_1^* \sin \mathrm{p} \left(\theta_{\mathrm{p}}^{23} \sqrt{|\mathrm{p}|} \right) + \rho_2^* \sin \mathrm{p} \left(\theta_{\mathrm{p}}^{31} \sqrt{|\mathrm{p}|} \right) + \rho_3^* \sin \mathrm{p} \left(\theta_{\mathrm{p}}^{12} \sqrt{|\mathrm{p}|} \right) = 0.$$

Especially, if we take $p = -1$, this formula becomes

$$\rho_1^* \sin \theta^{23} + \rho_2^* \sin \theta^{31} + \rho_3^* \sin \theta^{12} = 0.$$

Under the circumstances, the generalized Bobillier formula is reduced to the Bobillier formula on the complex plane or Euclidean plane. Figure 3 represents the illustration of the Bobillier construction in the complex plane, [9, 12].
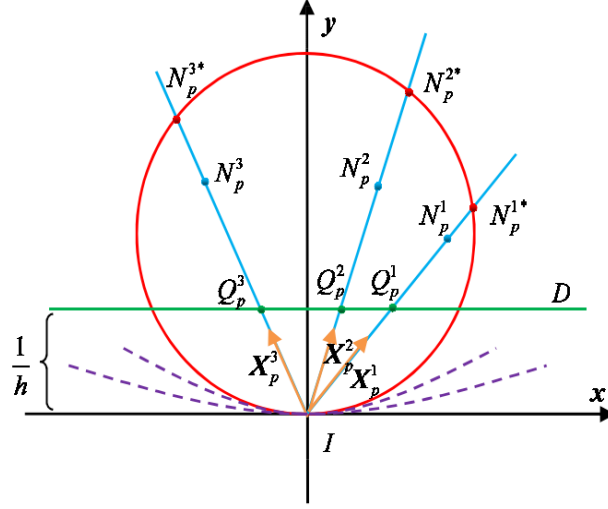
FIGURE 3. Inflection Circle in Complex Plane.

Case 2. If we consider $p = 0$, the generalized complex number system C_{p} is equal to the parabolic complex number system. By considering the equation (6), we get

$$\rho_1^* \theta^{23} + \rho_2^* \theta^{31} + \rho_3^* \theta^{12} = 0.$$

This formula is the Bobillier formula for Shear motion in the Galilean plane. The inflection cycle is illustrated as for Shear motion in Galilean plane, [13].

Case 3. When $p > 0$, the generalized complex number system C_{p} is referred to as the hyperbolic complex number system. In addition, considering the equation (6), we obtain that the generalized Bobillier formula is reduced to

$$\rho_1^* \sinh\left(\theta_{\mathrm{p}}^{23} \sqrt{\mathrm{p}}\right) + \rho_2^* \sinh\left(\theta_{\mathrm{p}}^{31} \sqrt{\mathrm{p}}\right) + \rho_3^* \sinh\left(\theta_{\mathrm{p}}^{12} \sqrt{\mathrm{p}}\right) = 0.$$

Especially, if we take $p = 1$, we obtain that the Bobillier formula as

$$\rho_1^* \sinh \theta^{23} + \rho_2^* \sinh \theta^{31} + \rho_3^* \sinh \theta^{12} = 0.$$

Thus, this formula is equal to the Bobillier formula at the Lorentzian plane, [10]. The figure of the (hyperbolic) inflection circle for $p = 1$ is as follows

4. AN ALTERNATIVE WAY FOR THE GENERALIZED BOBILLIER FORMULA

In this section, we will obtain the generalized Bobillier formula in the generalized complex plane without the use of the generalized Euler-Savary formula. So, the generalized Bobillier formula will be obtained considering the velocities and accelerations of planar motion in C_{p}.
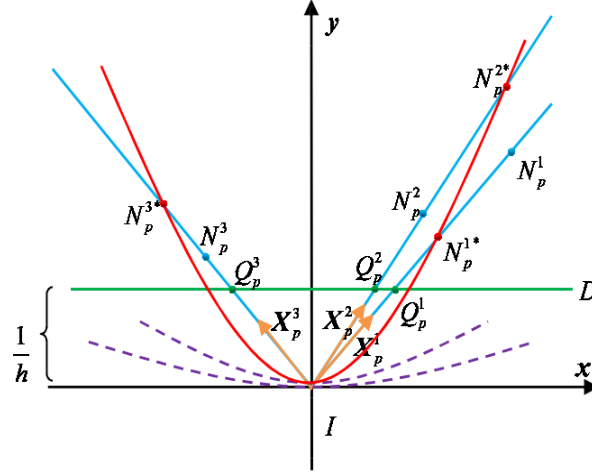
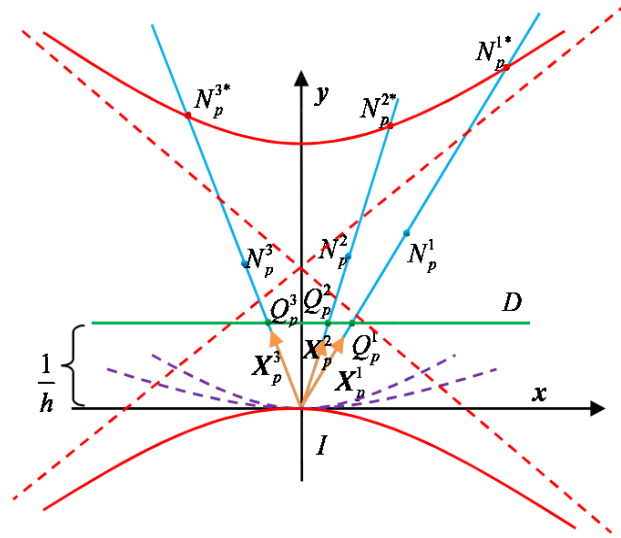FIGURE 4. Inflection Circle in Galilean Plane.

FIGURE 5. Inflection Circle in Hyperbolic Plane.

Firstly, we calculate the trajectory velocities and accelerations of the points in the moving generalized complex plane K_{p}. Let $V_{\mathrm{p}}^a \left(N_{\mathrm{p}}^1 \right)$ and $J_{\mathrm{p}}^a \left(N_{\mathrm{p}}^1 \right)$ be absolute

velocity and acceleration vectors of the point N_p^1, respectively. Moreover, if w_p is the angular velocity of motion $K_\mathrm{p}/K_\mathrm{p}'$, $w_p = \frac{d\theta_\mathrm{p}^1}{dt}$ where θ_p^1 is $p-$rotation angle.

The sliding velocity vector of the inflection point N_p^{1*} is perpendicular to the vector which connects the centre to this point. Moreover, the sliding velocity vector is perpendicular to the angular velocity vector. So, we can write

$$\mathbf{V}_\mathrm{p}^f\left(N_\mathrm{p}^1\right) = \mathbf{w}_\mathrm{p} \wedge_\mathrm{p} \mathbf{IN}_\mathrm{p}^1 \tag{18}$$

where $w_\mathrm{p} = w_\mathrm{p} z_\mathrm{p}$ is angular velocity vector of motion and z_p is the unit vector in the direction of the angular velocity vector.

In addition, from the equation (1) there is the relation between the velocity vectors

$$\mathbf{V}_\mathrm{p}^a\left(N_\mathrm{p}^1\right) = \mathbf{V}_\mathrm{p}^r\left(I_\mathrm{p}\right) + \mathbf{V}_\mathrm{p}^f\left(N_\mathrm{p}^1\right) \tag{19}$$

where V_p^a, V_p^f and V_p^r are the absolute, sliding and relative velocity vectors, respectively. If we consider the equations (18) and (19) it is obtained that

$$\mathbf{V}_\mathrm{p}^a\left(N_\mathrm{p}^1\right) = \mathbf{V}_\mathrm{p}^r\left(I_\mathrm{p}\right) + \mathbf{w}_\mathrm{p} \wedge_\mathrm{p} \mathbf{IN}_\mathrm{p}^1. \tag{20}$$

Differentiating the equation (20) with respect to t, we get

$$\mathbf{J}_\mathrm{p}^a\left(N_\mathrm{p}^1\right) = \mathbf{J}_\mathrm{p}^r\left(I_\mathrm{p}\right) + \dot{w}_\mathrm{p}\mathbf{z}_\mathrm{p} \wedge_\mathrm{p} \mathbf{IN}_\mathrm{p}^1 + w_\mathrm{p}^2 \mathbf{IN}_\mathrm{p}^1 \tag{21}$$

where the first term is the path wise tangential acceleration component, the second term is the centripetal component and the third term can be shown to be a pure imaginary component, [25]. Considering this analysis, the absolute velocity vector of the inflection point is linearly dependent with the absolute acceleration vector of the inflection point since the normal component of acceleration is zero. So we can write

$$\mathbf{V}_\mathrm{p}^a\left(N_\mathrm{p}^{1*}\right) \wedge_\mathrm{p} \mathbf{J}_\mathrm{p}^a\left(N_\mathrm{p}^{1*}\right) = \mathbf{0}. \tag{22}$$

Here considering $V_\mathrm{p}^r = 0$ for the inflection point N_p^{1*} and using the equations (20) (21) and (22) we can easily find

$$w_\mathrm{p}\left\langle \mathbf{J}_\mathrm{p}^r\left(I_\mathrm{p}\right), \mathbf{IN}_\mathrm{p}^1\right\rangle_\mathrm{p} + w_\mathrm{p}^3\rho_1^* = 0$$

and finally it is said that

$$\rho_1^* = -\frac{\left\langle \mathbf{J}_\mathrm{p}^r\left(I_\mathrm{p}\right), \mathbf{X}_\mathrm{p}^1\right\rangle_\mathrm{p}}{w_\mathrm{p}^2}. \tag{23}$$

With similar process for the inflection points N_p^{2*} and N_p^{3*} the following equations hold;

$$\rho_2^* = -\frac{\left\langle \mathbf{J}_\mathrm{p}^r\left(I_\mathrm{p}\right), \mathbf{X}_\mathrm{p}^2\right\rangle_\mathrm{p}}{w_\mathrm{p}^2} \tag{24}$$

and

$$\rho_3^* = -\frac{\left\langle \mathbf{J}_\mathrm{p}^r\left(I_\mathrm{p}\right), \mathbf{X}_\mathrm{p}^3\right\rangle_\mathrm{p}}{w_\mathrm{p}^2}. \tag{25}$$

On the other hand the linear connection between X_{p}^1, X_{p}^2 and X_{p}^3 may be written as follows

$$\lambda_1 \mathbf{X}_{\mathrm{p}}^1 + \lambda_2 \mathbf{X}_{\mathrm{p}}^2 + \lambda_3 \mathbf{X}_{\mathrm{p}}^3 = 0$$

where $\lambda_1, \lambda_3, \lambda_3 \in R$. From the last equation by successive $p-$cross product with X_{p}^1, X_{p}^2 and X_{p}^3, we get

$$\lambda_1 = \sin \mathrm{p}\theta_{\mathrm{p}}^{23}, \quad \lambda_2 = \sin \mathrm{p}\theta_{\mathrm{p}}^{31}, \quad \lambda_3 = \sin \mathrm{p}\theta_{\mathrm{p}}^{12}$$

where $\theta_{\mathrm{p}}^{23} = \theta_{\mathrm{p}}^3 - \theta_{\mathrm{p}}^2$, $\theta_{\mathrm{p}}^{31} = \theta_{\mathrm{p}}^1 - \theta_{\mathrm{p}}^3$ and $\theta_{\mathrm{p}}^{12} = \theta_{\mathrm{p}}^2 - \theta_{\mathrm{p}}^1$. So, we obtain that

$$\sin p\theta_{\mathrm{p}}^{23} X_{\mathrm{p}}^1 + \sin p\theta_{\mathrm{p}}^{31} X_{\mathrm{p}}^2 + \sin p\theta_{\mathrm{p}}^{12} X_{\mathrm{p}}^3 = 0.$$

If we make scalar product of the last equation with $\dfrac{\mathbf{J}_{\mathrm{p}}^r(I_{\mathrm{p}})}{w_{\mathrm{p}}^2}$, the following formula is hold

$$\rho_1^* \sin p\theta_{\mathrm{p}}^{23} + \rho_2^* \sin p\theta_{\mathrm{p}}^{31} + \rho_3^* \sin p\theta_{\mathrm{p}}^{12} = 0. \tag{26}$$

This is the formula given in the equation (17). So, this direct way gives us the Bobillier formula without using the generalized Euler-Savary formula.

5. CONCLUSION

The angle between the tangent of pole curve at the instantaneous pole center of coupler with respect to the base of a four bar linkage and of the cranks is equal to the angle between the other crank and the collineation axis. This expression is can be verified by the Bobillier's construction by graphically. This has been a major interest to physicists, mathematicians and engineers. At the same time this problem can be solved by an analytical method called the Bobillier formula which is more practical to use. Various geometric and analytical methods have been developed for the Bobillier formula for Euclidean, Lorentzian and Galilean planar motion. Then the following question can be asked. "Is it possible to give one generalized formula for the Bobillier's construction in generalized complex plane including all planes?" Thus, we find that one formula for all cases of p. Moreover, we check and interpret this formula for the special cases of p. As a consequence we think that this study would be useful at the disciplines of mathematics, engineering and astronomy.

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