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GROWTH ESTIMATES OF ENTIRE FUNCTIONS WITH THE HELP OF THEIR RELATIVE L*-TYPES AND RELATIVE L*-

WEAK TYPES SANJIB KUMAR DATTA AND TANMAY BISWAS

ABSTRACT. In this paper we attempt to prove some results related to the growth rates of entire functions on the basis of relative L^* -type and relative

 L^* -weak type of an entire function with respect to another entire function.

1. Introduction

Let \mathbb{C} be the set of all finite complex numbers. For any entire function $f = \sum_{n=0}^{\infty} a_n z^n$ defined on \mathbb{C} , the function $M_f(r)$ is defined as

$$M_f(r) = \max_{|z|=r} |f(z)|$$

To start our paper we just recall the following definitions:

Definition 1. The order ρ_f and lower order λ_f of an entire function f are defined as

$$\rho_f = \lim_{r \to \infty} \sup \frac{\log \log M_f(r)}{\log r} \text{ and } \lambda_f = \lim_{r \to \infty} \inf \frac{\log \log M_f(r)}{\log r}$$

An entire function for which order and lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

Definition 2. The type σ_f and lower type $\overline{\sigma}_f$ of an entire function f such that $0 < \sigma_f < 1$ are defined as

$$\sigma_f = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{r^{\rho_f}} \text{ and } \overline{\sigma}_f = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{r^{\rho_f}}.$$

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Datta and Jha [6] introduced the definition of *weak type* of an entire function of finite positive *lower order* in the following way:

Definition 3. [6] The weak type τ_f and the growth indicator $\overline{\tau}_f$ of an entire function f of finite positive lower order λ_f are defined by

$$\overline{\tau}_f = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{r^{\lambda_f}} \text{ and } \tau_f = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{r^{\lambda_f}}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.*, $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant *a*.

Somasundaram and Thamizharasi [9] introduced the notions of *L*-order and *L*-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly. The more generalized concept for *L*-order and *L*-type for entire functions are L^* -order and L^* -type. Their definitions are as follows:

Definition 4. [9] The L^* -order $\rho_f^{L^*}$ and the L^* -lower order $\lambda_f^{L^*}$ of an entire function f are defined as

$$\rho_f^{L^*} = \lim_{r \to \infty} \sup \frac{\log \log M_f(r)}{\log \left[r e^{L(r)} \right]} \text{ and } \lambda_f^{L^*} = \lim_{r \to \infty} \inf \frac{\log \log M_f(r)}{\log \left[r e^{L(r)} \right]}.$$

An entire function for which L^* -order and L^* -lower order are the same is said to be of regular L^* -growth. Functions which are not of regular L^* -growth are said to be of irregular L^* -growth.

Definition 5. [9] The L*-type $\sigma_f^{L^*}$ and L*-lower type $\overline{\sigma}_f^{L^*}$ of an entire function f such that $0 < \rho_f^{L^*} < 1$ are defined as

$$\sigma_f^{L^*} = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \text{ and } \overline{\sigma}_f^{L^*} = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

In order to determine the growth of two entire functions of same non zero finite L^* -lower order, one may define the L^* -weak type in the following way:

Definition 6. The L*-weak type $\tau_f^{L^*}$ of an entire function f such that $0 < \lambda_f^{L^*} < \infty$ are defined as

$$\tau_f^{L^*} = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{\left[r e^{L(r)} \right]^{\lambda_f^{L^*}}}.$$

Likewise the growth indicator $\overline{\tau}_f^{L^*}$ of an entire function f such that $0 < \lambda_f^{L^*} < \infty$ can be defined in the following manner :

$$\overline{\tau}_f^{L^*} = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{\left[r e^{L(r)} \right]^{\lambda_f^{L^*}}}.$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous and its inverse M_g^{-1} : $(|g(0)|, \to \infty$ exists and is such that

$$\lim_{s \to \infty} M_g^{-1} = \infty$$

In the line of Somasundaram and Thamizharasi [9] and Bernal [1] one may define the relative L^* -order of an entire function in the following manner :

Definition 7. { [5], [7] } The relative L^* -order $\rho_g^{L^*}(f)$ and relative L^* -lower order $\lambda_g^{L^*}(f)$ of an entire function f with respect to another entire function g are defined as

$$\rho_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\log [r e^{L(r)}]} \text{ and } \lambda_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\log [r e^{L(r)}]}.$$

In order to determine the relative growth of two entire functions having same non zero finite relative L^* -order with respect to another entire function, one may define the concept of relative L^* -type and relative L^* -lower type in the following manner:

Definition 8. The relative L^* -type $\sigma_g^{L^*}(f)$ and relative L^* -lower type $\overline{\sigma}_g^{L^*}(f)$ of an entire function f with respect to g such that $0 < \rho_g^{L^*}(f) < 1$ are defined as follows:

$$\sigma_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\rho_g^{L^*}(f)}} \text{ and } \overline{\sigma}_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\rho_g^{L^*}(f)}}$$

Analogously, in order to determine the relative growth of two entire functions having same non zero finite relative L^* -lower order with respect to another entire function, one can define the relative L^* -weak type in the following way:

Definition 9. The relative L^* -weak type $\tau_g^{L^*}(f)$ of an entire function f with respect to g of finite positive relative L^* -lower order $\lambda_g^{L^*}(f)$ is defined as:

$$\tau_g^{L^*}(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\lambda_g^{L^*}(f)}}.$$

Similarly the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an entire function f with respect to another entire function $g \ 0 < \lambda_g^{L^*}(f) < 1$ can be defined in the following manner:

$$\overline{\tau}_g^{L^*}(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{\left\lceil r e^{L(r)} \right\rceil^{\lambda_g^{L^*}(f)}}.$$

In the paper we study some relative growth properties of entire functions with respect to another entire function on the basis of relative L^* -type and relative L^* -weak type. In fact some works on different relative growth indicators have also been explored by Datta et al {[3], [4]}. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [10].

2. Some Examples

In this section we present some examples in connection with definitions given in the previous section.

Example 1. (Order and L^* Order) Given any natural number n, let f(z) = exp(nz). Then $M_f(r) = exp(nr)$. Therefore

$$\rho_f = \lim_{r \to \infty} \sup \frac{\log \log M_f(r)}{\log r} = 1 \text{ and } \lambda_f = \lim_{r \to \infty} \inf \frac{\log \log M_f(r)}{\log r} = 1.$$

Further we take $L(r) = \log r$; then

$$\rho_f^{L^*} = \lambda_f^{L^*} = \frac{1}{2}.$$

Example 2. (Type, Weak type, L* Type and L* weak Type) Let us consider f(z) = exp(nz) for any natural number n. Then

$$\sigma_f = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{r^{\rho_f}} = \frac{nr}{r} = n \text{ and } \overline{\sigma}_f = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{r^{\rho_f}} = \frac{nr}{r} = n,$$

since $\rho_f = 1$. Similarly

$$\tau_f = \lim_{r \to \infty} \inf \frac{\log M_f(r)}{r^{\lambda_f}} = \frac{nr}{r} = n \text{ and } \overline{\tau}_f = \lim_{r \to \infty} \sup \frac{\log M_f(r)}{r^{\lambda_f}} = \frac{nr}{r} = n.$$

as $\lambda_f = 1$. Further if we take $L(r) = \log r$, then $\rho_f^{L^*} = \lambda_f^{L^*} = \frac{1}{2}$ and therefore we get that

$$\sigma_f^{L^*} = \overline{\sigma}_f^{L^*} = \tau_f^{L^*} = \overline{\tau}_f^{L^*} = \infty.$$

Example 3. (*Relative Order and relative* L^* *Order*) *Suppose* f = g = expz. *Therefore*

$$\rho_g(f) = \lambda_g(f) = 1.$$

Further if we take L(r) = logr, then

$$\rho_g^{L^*}(f) = \lambda_g^{L^*}(f) = \frac{1}{2}$$

Example 4. (Relative Type, relative weak type etc.) Suppose f = g = expz. Therefore

$$\sigma_g(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{r^{\rho_g(f)}} = 1 \text{ and } \overline{\sigma}_g(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{r^{\rho_g(f)}} = 1,$$

since $\rho_g(f) = 1$. Likewise

$$\tau_g(f) = \lim_{r \to \infty} \sup \frac{\log M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} = 1 \text{ and } \overline{\tau}_g(f) = \lim_{r \to \infty} \inf \frac{\log M_g^{-1} M_f(r)}{r^{\lambda_g(f)}} = 1,$$

as $\lambda_g(f) = 1$. Further if we take $L(r) = \log r$, then $\rho_g^{L^*}(f) = \lambda_g^{L^*}(f) = \frac{1}{2}$ and therefore we obtain that

$$\sigma_g^{L^*}(f) = \overline{\sigma}_g^{L^*}(f) = \tau_g^{L^*}(f) = \overline{\tau}_g^{L^*}(f) = 1.$$

3. Lemmas

First of all let us recall the following theorem due to Datta et al. [2] :

Theorem A Let f and g be any two entire functions such that $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and $0 \leq \lambda_g \leq \rho_g \leq \infty$. Then

$$\frac{\lambda_f^{L^*}}{\rho_g} \le \lambda_g^{L^*}(f) \le \min\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \le \max\left\{\frac{\lambda_f^{L^*}}{\lambda_g}, \frac{\rho_f^{L^*}}{\rho_g}\right\} \le \rho_g^{L^*}(f) \le \frac{\rho_f^{L^*}}{\lambda_g}.$$

Now From the conclusion of the above theorem, we present the following two lemmas which will be needed in the sequel.

Lemma 1. ([2]) Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then

$$\rho_g^{L^*}(f) = \frac{\rho_f^{L^*}}{\rho_g} \text{ and } \lambda_g^{L^*}(f) = \frac{\lambda_f^{L^*}}{\lambda_g}.$$

Lemma 2. ([2]) Let f be an entire function of regular L^* -growth with non zero finite L^* -order and g be an entire function with $0 \le \lambda_g \le \rho_g \le \infty$. Then

$$\rho_g^{L^*}(f) = \frac{\lambda_f^{L^*}}{\lambda_g} \text{ and } \lambda_g^{L^*}(f) = \frac{\rho_f^{L^*}}{\rho_g}.$$

4. MAIN RESULTS

In this section we state the main results of the paper.

Theorem 1. Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then

$$\begin{split} \left[\overline{\sigma}_{f}^{L^{*}} \\ \overline{\sigma}_{g}^{-} \right]^{\frac{1}{\rho_{g}}} &\leq \overline{\sigma}_{g}^{L^{*}} \leq \min\left\{ \left[\frac{\overline{\sigma}_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\rho_{g}}} , \left[\frac{\sigma_{f}^{L^{*}}}{\sigma_{g}} \right]^{\frac{1}{\rho_{g}}} \right\} \\ &\leq \max\left\{ \left[\frac{\overline{\sigma}_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\rho_{g}}} , \left[\frac{\sigma_{f}^{L^{*}}}{\sigma_{g}} \right]^{\frac{1}{\rho_{g}}} \right\} \leq \sigma_{g}^{L^{*}}(f) \leq \left[\frac{\sigma_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\rho_{g}}} \end{split}$$

Proof. Let us consider that $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\sigma_f^{L^*}$ and $\overline{\sigma}_f^{L^*}$, we have for all sufficiently large values of r that

$$M_f(r) \le \exp\left[\left(\sigma_f^{L^*} + \varepsilon\right) \left[re^{L(r)}\right]^{\rho_f^{L^*}}\right],\tag{1}$$

$$M_f(r) \ge \exp\left[\left(\overline{\sigma}_f^{L^*} - \varepsilon\right) \left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]$$
(2)

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \ge \exp\left[\left(\sigma_f^{L^*} - \varepsilon\right) \left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]$$
(3)

$$M_f(r) \le \exp\left[\left(\overline{\sigma}_f^{L^*} + \varepsilon\right) \left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]$$
(4)

Similarly from the definitions of σ_g and $\overline{\sigma}_g,$ it follows for all sufficiently large values of r that

$$\begin{aligned}
M_g(r) &\leq \exp\left[(\sigma_g + \varepsilon) \cdot r^{\rho_g}\right] \\
i.e., r &\leq M_g^{-1}\left[\exp\left[(\sigma_g + \varepsilon) \cdot r^{\rho_g}\right]\right] \\
i.e., M_g^{-1}(r) &\geq \left[\left(\frac{\log r}{(\sigma_g + \varepsilon)}\right)^{\frac{1}{\rho_g}}\right],
\end{aligned}$$
(5)

$$\begin{aligned}
M_g(r) &\geq \exp\left[\left(\overline{\sigma}_g - \varepsilon\right) \cdot r^{\rho_g}\right] \\
i.e., r &\geq M_g^{-1}\left[\exp\left[\left(\overline{\sigma}_g - \varepsilon\right) \cdot r^{\rho_g}\right]\right] \\
M_g^{-1}(r) &\leq \left[\left(\frac{\log r}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right]
\end{aligned} \tag{6}$$

and for a sequence of values of r tending to infinity, we obtain that

$$\begin{aligned}
M_{g}(r) &\geq \exp\left[\left(\sigma_{g} - \varepsilon\right) \cdot r^{\rho_{g}}\right] \\
i.e., r &\geq M_{g}^{-1}\left[\left(\sigma_{g} - \varepsilon\right) \cdot r^{\rho_{g}}\right] \\
i.e., M_{g}^{-1}(r) &\leq \left[\left(\frac{\log r}{\left(\sigma_{g} - \varepsilon\right)}\right)^{\frac{1}{\rho_{g}}}\right],
\end{aligned} \tag{7}$$

$$M_{g}(r) &\leq \exp\left[\left(\overline{\sigma}_{g} + \varepsilon\right) \cdot r^{\rho_{g}}\right]$$

$$M_{g}(r) \leq \exp\left[\left(\sigma_{g} + \varepsilon\right) \cdot r^{\rho_{g}}\right]$$

i.e., $r \leq M_{g}^{-1}\left[\exp\left[\left(\overline{\sigma}_{g} + \varepsilon\right) \cdot r^{\rho_{g}}\right]\right]$
i.e., $M_{g}^{-1}(r) \geq \left[\left(\frac{\log r}{(\overline{\sigma}_{g} + \varepsilon)}\right)^{\frac{1}{\rho_{g}}}\right].$
(8)

Now from (3) and in view of (5), we get for a sequence of values of r tending to infinity that

$$M_{g}^{-1}M_{f}(r) \geq M_{g}^{-1}\left[\exp\left[\left(\sigma_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]\right]$$
$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\left(\frac{\log\exp\left[\left(\sigma_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]}{\left(\sigma_{g}+\varepsilon\right)}\right)^{\frac{1}{\rho_{g}}}\right]$$
$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\frac{\left(\sigma_{f}^{L^{*}}-\varepsilon\right)}{\left(\sigma_{g}+\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}$$
$$i.e., \frac{M_{g}^{-1}M_{f}(r)}{\left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}} \geq \left[\frac{\left(\sigma_{f}^{L^{*}}-\varepsilon\right)}{\left(\sigma_{g}+\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}}.$$
(9)

As $\varepsilon(>0)$ is arbitrary, in view of Lemma 1 it follows that

$$\lim_{r \to \infty} \sup \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}(f)}} \ge \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e., \sigma_g^{L^*}(f) \ge \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
(10)

Analogously from (2) and in view of (8), for a sequence of values of r tending to infinity we get that

$$M_{g}^{-1}M_{f}(r) \geq M_{g}^{-1}\left[\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]\right]$$

$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]}{(\overline{\sigma}+\varepsilon)}\right)^{\frac{1}{\rho_{g}}}\right]$$

$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\frac{(\overline{\sigma}_{f}^{L^{*}}-\varepsilon)}{(\overline{\sigma}_{g}+\varepsilon)}\right]^{\frac{1}{\rho_{g}}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}$$

$$i.e., \frac{M_{g}^{-1}M_{f}(r)}{\left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}} \geq \left[\frac{(\overline{\sigma}_{f}^{L^{*}}-\varepsilon)}{(\overline{\sigma}_{g}+\varepsilon)}\right]^{\frac{1}{\rho_{g}}}$$
(11)

Since $\varepsilon(>0)$ is arbitrary, we get from above and Lemma (1) that

$$\lim_{r \to \infty} \sup \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)} \right]^{\rho_g^{L^*}(f)}} \ge \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g} \right]^{\frac{1}{\rho_g}}$$
$$i.e., \sigma_g^{L^*}(f) \ge \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g} \right]^{\frac{1}{\rho_g}}.$$
(12)

Again in view of (6), we have from (1) for all sufficiently large values of r that

$$M_g^{-1}M_f(r) \leq M_g^{-1}\left[\exp\left[\left(\sigma_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]\right]$$

$$i.e., M_g^{-1}M_f(r) \leq \left[\left(\frac{\log \exp\left[\left(\sigma_f^{L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\rho_f^{L^*}}\right]}{(\overline{\sigma}_g - \varepsilon)}\right)^{\frac{1}{\rho_g}}\right]$$

$$i.e., M_g^{-1}M_f(r) \leq \left[\frac{\left(\sigma_f^{L^*} + \varepsilon\right)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}$$

$$i.e., \frac{M_g^{-1}M_f(r)}{\left[re^{L(r)}\right]^{\frac{\rho_f^{L^*}}{\rho_g}}} \le \left[\frac{(\sigma_f^{L^*} + \varepsilon)}{(\overline{\sigma}_g - \varepsilon)}\right]^{\frac{1}{\rho_g}}.$$
(13)

As $\varepsilon(>0)$ is arbitrary, in view of Lemma 1 it follows that

$$\lim_{r \to \infty} \sup \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}(f)}} \le \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e., \sigma_g^{L^*}(f) \le \left[\frac{\sigma_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$
(14)

Again from (2) and in view of (5), we get for all sufficiently large values of r that

$$M_{g}^{-1}M_{f}(r) \geq M_{g}^{-1}\left[\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]\right]$$
$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]}{\left(\sigma_{g}+\varepsilon\right)}\right)^{\frac{1}{\rho_{g}}}\right]$$
$$i.e., M_{g}^{-1}M_{f}(r) \geq \left[\frac{\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)}{\left(\sigma_{g}+\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}$$
$$i.e., \frac{M_{g}^{-1}M_{f}(r)}{\left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}} \geq \left[\frac{\left(\overline{\sigma}_{f}^{L^{*}}-\varepsilon\right)}{\left(\sigma_{g}+\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}}.$$
(15)

As $\varepsilon(>0)$ is arbitrary, it follows from above and Lemma 1 that

$$\lim_{r \to \infty} \inf \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}(f)}} \ge \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e., \sigma_g^{L^*}(f) \ge \left[\frac{\overline{\sigma}_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
(16)

Also in view of (7), we get from (1) for a sequence of values of r tending to infinity that

$$M_{g}^{-1}M_{f}(r) \leq M_{g}^{-1}\left[\exp\left[\left(\sigma_{f}^{L^{*}}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]\right]$$

$$i.e., M_{g}^{-1}M_{f}(r) \leq \left[\left(\frac{\log\exp\left[\left(\sigma_{f}^{L^{*}}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]}{\left(\sigma_{g}-\varepsilon\right)}\right)^{\frac{1}{\rho_{g}}}\right]$$

$$i.e., M_{g}^{-1}M_{f}(r) \leq \left[\frac{\left(\sigma_{f}^{L^{*}}+\varepsilon\right)}{\left(\sigma_{g}-\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}$$

$$i.e., \frac{M_{g}^{-1}M_{f}(r)}{\left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}} \leq \left[\frac{\left(\sigma_{f}^{L^{*}}+\varepsilon\right)}{\left(\sigma_{g}-\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}}.$$
 (17)

Since $\varepsilon (> 0)$ is arbitrary, we get from Lemma (1) and above that

$$\lim_{r \to \infty} \sup \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}(f)}} \le \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e., \overline{\sigma}_g^{L^*}(f) \le \left[\frac{\sigma_f^{L^*}}{\sigma_g}\right]^{\frac{1}{\rho_g}}.$$
(18)

Similarly from (4) and in view of (6), it follows for a sequence of values of r tending to infinity that г -

$$M_{g}^{-1}M_{f}(r) \leq M_{g}^{-1}\left[\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]\right]$$

i.e., $M_{g}^{-1}M_{f}(r) \leq \left[\left(\frac{\log\exp\left[\left(\overline{\sigma}_{f}^{L^{*}}+\varepsilon\right)\left[re^{L(r)}\right]^{\rho_{f}^{L^{*}}}\right]}{\left(\overline{\sigma}_{g}-\varepsilon\right)}\right)^{\frac{1}{\rho_{g}}}\right]$
i.e., $M_{g}^{-1}M_{f}(r) \leq \left[\frac{\left(\overline{\sigma}_{f}^{L^{*}}+\varepsilon\right)}{\left(\overline{\sigma}_{g}-\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}} \cdot \left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}$
i.e., $\frac{M_{g}^{-1}M_{f}(r)}{\left[re^{L(r)}\right]^{\frac{\rho_{f}^{L^{*}}}{\rho_{g}}}} \leq \left[\frac{\left(\overline{\sigma}_{f}^{L^{*}}+\varepsilon\right)}{\left(\overline{\sigma}_{g}-\varepsilon\right)}\right]^{\frac{1}{\rho_{g}}}$
(19)

Since $\varepsilon(>0)$ is arbitrary, we get from Lemma (1) and above that

$$\lim_{r \to \infty} \sup \frac{M_g^{-1} M_f(r)}{\left[r e^{L(r)}\right]^{\rho_g^{L^*}(f)}} \le \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}$$
$$i.e., \overline{\sigma}_g^{L^*}(f) \le \left[\frac{\overline{\sigma}_f^{L^*}}{\overline{\sigma}_g}\right]^{\frac{1}{\rho_g}}.$$
(20)
ws from (10), (12), (14), (16), (18) and (20).

Thus the theorem follows from (10), (12), (14), (16), (18) and (20).

Theorem 2. Let f be an entire function of regular L^* -growth with non zero finite L*-order and g be an entire function with $0 \le \lambda_g \le \rho_g \le \infty$. Then

$$\begin{bmatrix} \overline{\tau}_{f}^{L^{*}} \\ \overline{\overline{\tau}_{g}} \end{bmatrix}^{\frac{1}{\lambda_{g}}} \leq \overline{\sigma}_{g}^{L^{*}}(f) \leq \min \left\{ \begin{bmatrix} \overline{\tau}_{f}^{L^{*}} \\ \overline{\tau}_{g} \end{bmatrix}^{\frac{1}{\lambda_{g}}}, \begin{bmatrix} \overline{\overline{\tau}}_{f}^{L^{*}} \\ \overline{\overline{\tau}_{g}} \end{bmatrix}^{\frac{1}{\lambda_{g}}} \right\} \\ \leq \max \left\{ \begin{bmatrix} \underline{\tau}_{f}^{L^{*}} \\ \overline{\tau}_{g} \end{bmatrix}^{\frac{1}{\lambda_{g}}}, \begin{bmatrix} \overline{\overline{\tau}}_{f}^{L^{*}} \\ \overline{\overline{\tau}_{g}} \end{bmatrix}^{\frac{1}{\lambda_{g}}} \right\} \leq \sigma_{g}^{L^{*}}(f) \leq \begin{bmatrix} \overline{\overline{\tau}}_{f}^{L^{*}} \\ \overline{\tau}_{g} \end{bmatrix}^{\frac{1}{\lambda_{g}}}.$$

Proof. Suppose $\varepsilon(>0)$ is arbitrary number. Now from the definitions of $\overline{\tau}_f^{L^*}$ and $\tau_f^{L^*}$, we have for all sufficiently large values of r that

$$M_{f}(r) \leq \exp\left[\left(\overline{\tau}_{f}^{L^{*}} + \varepsilon\right) \left[re^{L(r)}\right]^{\lambda_{f}^{L^{*}}}\right],$$
$$M_{f}(r) \geq \exp\left[\left(\tau_{f}^{L^{*}} - \varepsilon\right) \left[re^{L(r)}\right]^{\lambda_{f}^{L^{*}}}\right],$$

and also for a sequence of values of r tending to infinity, we get that

$$M_f(r) \geq \exp\left[\left(\overline{\tau}_f^{L^*} - \varepsilon\right) \left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right],$$

$$M_f(r) \leq \exp\left[\left(\tau_f^{L^*} + \varepsilon\right) \left[re^{L(r)}\right]^{\lambda_f^{L^*}}\right].$$

Similarly from the definitions of $\overline{\tau}_g$ and $\tau_g,$ it follows for all sufficiently large values of r that

$$\begin{split} M_g(r) &\leq \exp\left[\left(\overline{\tau}_g + \varepsilon\right) \cdot r^{\lambda_g}\right] \\ i.e., r &\leq M_g^{-1} \left[\exp\left[\left(\overline{\tau}_g + \varepsilon\right) \cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\overline{\tau}_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right], \\ M_g(r) &\geq \exp\left[\left(\tau_g - \varepsilon\right) \cdot r^{\lambda_g}\right] \\ i.e., r &\geq M_g^{-1} \left[\exp\left[\left(\tau_g - \varepsilon\right) \cdot r^{\lambda_g}\right]\right] \\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\tau_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right] \end{split}$$

and for a sequence of values of r tending to infinity, we obtain that

$$\begin{split} M_g(r) &\geq \exp\left[\left(\overline{\tau}_g - \varepsilon\right) \cdot r^{\lambda_g}\right],\\ i.e., r &\geq M_g^{-1}\left[\exp\left[\left(\overline{\tau}_g - \varepsilon\right) \cdot r^{\lambda_g}\right]\right]\\ i.e., M_g^{-1} &\leq \left[\left(\frac{\log r}{(\overline{\tau}_g - \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]\\ M_g(r) &\leq \exp\left[\left(\tau_g + \varepsilon\right) \cdot r^{\lambda_g}\right]\\ i.e., r &\leq M_g^{-1}\left[\exp\left[\left(\tau_g + \varepsilon\right) \cdot r^{\lambda_g}\right]\right]\\ i.e., M_g^{-1} &\geq \left[\left(\frac{\log r}{(\tau_g + \varepsilon)}\right)^{\frac{1}{\lambda_g}}\right]. \end{split}$$

Now using the same technique of Theorem (1), one can easily prove the conclusion of the present theorem by the help of Lemma (2) and the above inequalities. Therefore the remaining part of the proof of the present theorem is omitted.

Theorem 3. Let f be an entire function with $0 \leq \lambda_f^{L^*} \leq \rho_f^{L^*} \leq \infty$ and g be an entire function of regular growth with non zero finite order. Then

$$\begin{bmatrix} \frac{\tau_f^{L^*}}{\overline{\tau}_g} \end{bmatrix}^{\frac{1}{\lambda_g}} \leq \tau_g^{L^*}(f) \leq \min\left\{ \begin{bmatrix} \frac{\tau_f^{L^*}}{\tau_g} \end{bmatrix}^{\frac{1}{\lambda_g}}, \begin{bmatrix} \overline{\tau}_f^{L^*} \\ \overline{\tau}_g \end{bmatrix}^{\frac{1}{\lambda_g}} \right\} \\ \leq \max\left\{ \begin{bmatrix} \frac{\tau_f^{L^*}}{\tau_g} \end{bmatrix}^{\frac{1}{\lambda_g(m,p)}}, \begin{bmatrix} \overline{\tau}_f^{L^*} \\ \overline{\tau}_g \end{bmatrix}^{\frac{1}{\lambda_g}} \right\} \leq \overline{\tau}_g^{L^*}(f) \leq \begin{bmatrix} \overline{\tau}_f^{L^*} \\ \overline{\tau}_g \end{bmatrix}^{\frac{1}{\lambda_g}}$$

Theorem 4. Let f be an entire function of regular L^* -growth with with non zero finite L^* -order and g be an entire function with $0 \le \lambda_g \le \rho_g \le \infty$. Then

$$\begin{split} \left[\frac{\overline{\sigma}_{f}^{L^{*}}}{\sigma_{g}} \right]^{\frac{1}{\rho_{g}}} &\leq \tau_{g}^{L^{*}}(f) \leq \min \left\{ \left[\frac{\overline{\sigma}_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\rho_{g}}}, \left[\frac{\sigma_{f}^{L^{*}}}{\sigma_{g}} \right]^{\frac{1}{\rho_{g}}} \right\} \\ &\leq \max \left\{ \left[\frac{\overline{\sigma}_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\rho_{g}}}, \left[\frac{\sigma_{f}^{L^{*}}}{\sigma_{g}} \right]^{\frac{1}{\rho_{g}}} \right\} \leq \overline{\tau}_{g}^{L^{*}}(f) \leq \left[\frac{\sigma_{f}^{L^{*}}}{\overline{\sigma}_{g}} \right]^{\frac{1}{\lambda_{g}}} \end{split}$$

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