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A STUDY ON A PARTIALLY NULL CURVE IN $\mathbb{E}_2^4$

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Abstract. In this paper, we study with Frenet equations which are given in [3] of a partially null curve in Semi- Euclidean 4-space $\mathbb{E}_2^4$ with index 2. By using the Frenet equations, we give some theorem and corollary. A characterization of a hyperbolic partially null curve in $\mathbb{E}_2^4$ is given. Additionally, we examine harmonic curvatures and curvatures of this curve in $\mathbb{E}_2^4$.

1. Introduction

Semi-Euclidean 4−space $\mathbb{E}_2^4$ with index 2 is the Euclidean 4−space $\mathbb{E}^4$ equipped with an indefinite flat metric g given by

$$g = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}_2^4$. A vector $v = (v_1, v_2, v_3, v_4)$ in $\mathbb{E}_2^4$ is a called a spacelike, a timelike or a null (lightlike), if respectively holds $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0 = (0, 0, 0, 0)$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v, v)|}$. Two vectors v and w in $\mathbb{E}_2^4$ are said to be orthogonal, if $g(v, w) = 0$.

An arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{E}_2^4$ can locally be spacelike, timelike or null, if respectively all of its velocity $\alpha'(s)$ are spacelike, timelike or null. Spacelike or timelike curve $\alpha(s)$ is said to be parametrized by arclength functions s, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. Let a, b be two spacelike vectors in $\mathbb{E}_2^4$, then, there is unique real number $0 \leq \delta \leq \pi$, called angle between a and b, such that $g(a, b) = \|a\| \, \|b\| \cos \delta$.

We also recall that the pseudosphere S_2^3 and the pseudohyperbolic space H_1^3 are the hyperquadrics in $\mathbb{E}_2^4$, defined respectively by:

$$\begin{aligned} S_2^3(c, r) &= \left\{ \alpha \epsilon \mathbb{E}_2^4 : g(\alpha - c, \alpha - c) = r^2 \right\}, \\ H_1^3(c, -r) &= \left\{ \alpha \epsilon \mathbb{E}_2^4 : g(\alpha - c, \alpha - c) = -r^2 \right\}, \end{aligned}$$

where center c and radius $r \epsilon \mathbb{R}^+$ [1].

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Let $a=(a_1,a_2,a_3,a_4)$, $b=(b_1,b_2,b_3,b_4)$ and $c=(c_1,c_2,c_3,c_4)$ be vectors in $\mathbb{E}_2^4$. The vector product in $\mathbb{E}_2^4$ is defined with the determinant

$$a\Lambda b\Lambda c=-\begin{vmatrix} -e_1 & -e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix},$$

where e_1,e_2,e_3, and e_4 are coordinate direction vectors. Also, Frenet apparatus of a partially null curve in $\mathbb{E}_2^4$ are

$$T=\alpha'(s),\ N(s)=\frac{\alpha''(s)}{\|\alpha''(s)\|},\ B_2=\frac{T\Lambda N\Lambda\alpha'''}{\|T\Lambda N\Lambda\alpha'''\|},\ B_1=N\Lambda T\Lambda B_2$$

[4]

2. A Partially Null Curve in $\mathbb{E}_2^4$

Denote by $\{T(s),N(s),B_1(s),B_2(s)\}$ the moving Frenet frame along the curve $\alpha=\alpha(s)$ in $\mathbb{E}_2^4$. Then T,N,B_1,B_2 are, respectively, the tangent, the principal normal ,the first binormal and the second binormal vector fields. Recall that spacelike curve with timelike principal normal and a null first and second binormal is called a partially null curve in $\mathbb{E}_2^4$. Then for a partially null curve α in $\mathbb{E}_2^4$, the following Frenet equations are given in [3]

$$\left\{\begin{array}{c} T'(s)=k_1(s)N(s), \\ N'(s)=k_1(s)T(s)+k_2(s)B_1(s), \\ B_1'(s)=k_3(s)B_1(s), \\ B_2'(s)=-\varepsilon_2k_2(s)N(s)-k_3(s)B_2(s). \end{array}\right\} \tag{2.1}$$

where T,N,B_1 and B_2 are mutually orthogonal vectors satisfying equations

$$\left\{\begin{array}{c} g(T,T)=\varepsilon_1=\pm1,\ g(N,N)=\varepsilon_2=\pm1,\ \text{whereby } \varepsilon_1\varepsilon_2=-1, \\ g(B_1,B_2)=1,\ g(B_1,B_1)=g(B_2,B_2)=0, \\ g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(N,B_2)=0. \end{array}\right\} \tag{2.2}$$

And here,

$$\left\{\begin{array}{c} k_1(s)=g(T'(s),N(s))\varepsilon_2, \\ k_2(s)=g(N'(s),B_2(s)), \\ k_3(s)=g(B_1'(s),B_2(s)) \end{array}\right.$$

are first, second and third curvature of the curve α, respectively. In the sequel, in [3] prove that $k_3(s)=0$ for each s. Consequently, there are only two curvatures $k_1(s)$ and $k_2(s)$ in this case. Thus, Frenet equations are as follows:

$$\left\{\begin{array}{c} T'(s)=k_1(s)N(s), \\ N'(s)=k_1(s)T(s)+k_2(s)B_1(s), \\ B_1'(s)=0, \\ B_2'(s)=-\varepsilon_2k_2(s)N(s). \end{array}\right\} \tag{2.3}$$

Theorem 1. [3]*Let α be a partially null curve in $\mathbb{E}_2^4$. $\{T, N, B_1, B_2\}$ is the Frenet frame of α. T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T', T) = g(N', N) = g(B_1', B_1) = g(B_2', B_2) = 0, \\ g(T', N) = -g(N', T), \\ g(T', B_1) = -g(T, B_1'), \\ g(N', B_1) = -g(N, B_1'), \\ g(B_1', B_2) = -g(B_1, B_2'), \\ g(T, B_2') = -g(T', B_2), \\ g(N', B_2) = -g(N, B_2'). \end{array}\right\} \tag{2.4}$$

Theorem 2. *Let α be a partially null curve in $\mathbb{E}_2^4$ with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ and $k_3(s) = 0$ for each s. $\{T, N, B_1, B_2\}$ is the Frenet frame of α. T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T', T) = g(T', B_1) = g(T', B_2) = g(N', B_1) = g(N', N) = 0, \\ g(B_1', T) = g(B_1', N) = g(B_1', B_1) = g(B_1', B_2) = 0, \\ g(B_2', T) = g(B_2', B_1) = g(B_2', B_2) = 0, \\ g(T', N) = \varepsilon_2 k_1, \\ g(N', T) = \varepsilon_1 k_1, \\ g(N', B_2) = -g(B_2', N). \end{array}\right\} \tag{2.5}$$

Corollary 1. *There is only one curvature k_1 in a previous theorem.*

Corollary 2. *i) If $\varepsilon_2 = 1$, then $g(T', N) = k_1$.*
ii) If $\varepsilon_2 = -1$, then $g(T', N) = -k_1$.
iii) If $\varepsilon_1 = 1$, then $g(N', T) = k_1$.
iv) If $\varepsilon_1 = -1$, then $g(N', T) = -k_1$.

Theorem 3. *Let α be a partially null curve in $\mathbb{E}_2^4$ with curvatures $k_1(s) \neq 0$, $k_2(s) \neq 0$ and $k_3(s) = 0$ for each s. $\{T, N, B_1, B_2\}$ is the Frenet frame of α. T, N, B_1, B_2 are, respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields. Then*

$$\left\{\begin{array}{c} g(T'', N) = g(T'', B_1) = g(N'', T) = g(N'', B_1) = g(N'', B_2) = 0, \\ g(B_1'', T) = g(B_1'', N) = g(B_1'', B_1) = g(B_1'', B_2) = g(B_2'', N) = g(B_2'', B_1) = 0, \\ g(T'', T) = -\varepsilon_2 k_1^2, \\ g(T'', B_2) = g(B_2'', T) = k_1 k_2, \\ g(N'', N) = -\varepsilon_1 k_1^2, \\ g(B_2'', B_2) = -\varepsilon_2 k_2^2. \end{array}\right.$$

Proof. By using Equations (2.2), (2.3) and (2.5), we obtain the proof of the theorem. □

Corollary 3. *There are only two curvatures k_1 and k_2 in a previous theorem.*

Corollary 4. *i) If* $\varepsilon_2 = 1$ *then* $g(T'', T) = -k_1^2$ *and* $g(B_2'', B_2) = -k_2^2$.
ii) If $\varepsilon_2 = -1$ *then* $g(T'', T) = k_1^2$ *and* $g(B_2'', B_2) = k_2^2$.
iii) If $\varepsilon_1 = 1$, *then* $g(N'', N) = -k_1^2$.
iv) If $\varepsilon_1 = -1$, *then* $g(N'', N) = k_1^2$.

3. A Hyperbolic Partially Null Curve in $\mathbb{E}_2^4$

Theorem 4. [2]*A partially null unit speed curve* $\alpha(s)$ *in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *for each* $s\epsilon I \subset \mathbb{R}$ *has* $k_3 = 0$ *for each* s.

Theorem 5. [2]*Let* $\alpha = \alpha(s)$ *be a unit speed partially null curve in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *for each* s. *If* α *lies on* $\mathbb{S}_2^3$ *Lorentzian hypersphere, then*

$$\frac{1}{k_2}\frac{d}{ds}\left(\frac{1}{k_1}\right) = constant.$$

Theorem 6. *Let* $\alpha = \alpha(s)$ *be a partially null curve in* $\mathbb{E}_2^4$ *with curvatures* $k_1 \neq 0$, $k_2 \neq 0$ *and* $k_3 = 0$ *for each* $s\epsilon I \subset \mathbb{R}$. *If* α *lies on a pseudohyperbolic space* $\mathbb{H}_1^3(c, -r)$, *then*

$$k_1 = \frac{\varepsilon_2}{r} = constant; \ \ if\ \varepsilon_2 = 1,$$

where radius $r\epsilon\mathbb{R}^+$ *and with center* c *in* $\mathbb{E}_2^4$.

Proof. Let us suppose that $\alpha = \alpha(s)$ lies on $\mathbb{H}_1^3$ with center c. By the definition, we have

$$g(\alpha - c, \alpha - c) = -r^2, \tag{3.1}$$

for every $s\epsilon I \subset \mathbb{R}$. Differentiating (3.1), four times with respect to s and using Frenet equations, we have, respectively,

$$\begin{cases} g(T, \alpha - c) = 0, \\ g(N, \alpha - c) = -\dfrac{\varepsilon_1}{k_1}, \\ g(B_1, \alpha - c) = 0, \\ g(B_2, \alpha - c) = 0. \end{cases}$$

Let us decompose $\alpha - c$ by

$$\alpha - c = -\frac{\varepsilon_1}{k_1}N.$$

Finally, if we calculate

$$g(\alpha - c, \alpha - c) = -r^2,$$

we easily obtain

$$k_1 = \frac{\varepsilon_2}{r} = constant,$$

where is $\varepsilon_2 = 1$. □

Corollary 5. *If* α *is a spacelike curve, that is,* $\varepsilon_2 = 1$ *then* $k_1 = \dfrac{1}{r}$.

4. Harmonic Curvatures of a Partially Null Curve in $\mathbb{E}_2^4$

Definition 1. [6]*Let α be a partially null curve in $\mathbb{E}_2^4$. The harmonic functions*

$$H_j : I \longrightarrow \mathbb{R} \quad , \quad j = 0, 1$$

defined by

$$\begin{cases} H_0 = 0, \ H_1 = \dfrac{k_1}{k_2}, \ (k_2 \neq 0), \end{cases}$$

are called the harmonic curvatures of α. Here, k_1 and k_2 are Frenet curvatures of α.

Now, the Theorem 2 and the Theorem 3 can be given in terms of harmonic curvatures as follows:

Theorem 7. *Let α be a partially null curve in $\mathbb{E}_2^4$. T, N, B_2 are, respectively, the tangent, the principal normal and the second binormal vector fields. Then*

$$\begin{cases} g(T', N) = \varepsilon_2 k_2 H_1, \quad g(T'', T) = -\varepsilon_2 k_2^2 H_1^2, \qquad g(B_2'', B_2) = -\varepsilon_2 \dfrac{k_1^2}{H_1^2}, \\ g(N', T) = \varepsilon_1 k_2 H_1, \quad g(N'', N) = -\varepsilon_1 k_2^2 H_1^2, \quad g(T'', B_2) = g(B_2'', T) = k_2^2 H_1, \end{cases}$$

where k_1, k_2 are curvatures and H_1 is harmonic curvature of the curve α.

Proof. By using the definition of the harmonic curvatures, we obtain the proof of the theorem. □

Corollary 6. *i) If $\varepsilon_1 = 1$ then $g(N', T) = k_2 H_1$ and $g(N'', N) = -k_2^2 H_1^2$.*
ii) If $\varepsilon_1 = -1$ then $g(N', T) = -k_2 H_1$ and $g(N'', N) = k_2^2 H_1^2$.
iii) If $\varepsilon_2 = 1$, then

$$g(T', N) = k_2 H_1, \quad g(T'', T) = -k_2^2 H_1^2, \quad g(B_2'', B_2) = -\frac{k_1^2}{H_1^2}.$$

iv) If $\varepsilon_2 = -1$, then

$$g(T', N) = -k_2 H_1, \quad g(T'', T) = k_2^2 H_1^2, \quad g(B_2'', B_2) = \frac{k_1^2}{H_1^2}.$$

Theorem 8. *Let α be a partially null curve in $\mathbb{E}_2^4$ where $\{T, N, B_1, B_2\}$ is the Frenet frame of α and k_1, k_2 , k_3 are curvatures of α. If $k_1 \neq 0, k_2 \neq 0$ and $k_3 = 0$ then*

$$\nabla_T^4 T - k_1^4 T - \frac{k_1^4}{H_1} B_1 = 0,$$

where $\nabla_T T = T'$ and ∇ is the Levi-Civita connection of $\mathbb{E}_2^4$.

Proof. Since $k_3 = 0$, from Equation (2.3), we have

$$\nabla_T T = k_1 N \Longrightarrow \nabla_T^2 T = k_1 \nabla_T N \Longrightarrow \nabla_T^3 T = k_1 \nabla_T^2 N \Longrightarrow \nabla_T^4 T = k_1 \nabla_T^3 N.$$

Since

$$\nabla_T N = k_1 T + k_2 B_1,$$

$$\nabla_T^2 N = k_1^2 N,$$

we have

$$\nabla_T^3 N = k_1^3 T + k_1^2 k_2 B_1.$$

In that case

$$\nabla_T^4 T = k_1 \nabla_T^3 N = k_1^4 T + k_1^3 k_2 B_1.$$

Thus we have

$$\nabla_T^4 T - k_1^4 T - \frac{k_1^4}{H_1} B_1 = 0,$$

where $k_2 = \dfrac{k_1}{H_1}$. □

References

[1] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983.

[2] Turgut, M. and Yılmaz S., Some characterizations of partially null curves in Semi-Euclidean Space, *International Mathematical Forum*, (2008), 32(3), 1569-1574.

[3] Petrovic-Torgasev M., İlarslan, K. and Nesovic E., On partially null and pseudo null curves in the Semi-Euclidean space R_2^4 , *J. Geom.,* (2005), 84, 106-116.

[4] Yılmaz, S. and Turgut M., On Frenet apparatus of partially null curves in Semi-Euclidean space, *Scientia Magna*, (2008), 4(2), 39-44.

[5] Petrovic-Torgasev, M. and Sucurovic E., Some characterizations of the spacelike, the timelike and the null curves on the pseudohyperbolic space H_0^2 in E_1^3, *Kragujevac J. Math.*, (2000), 22, 71-82.

[6] İyigün, E. and Arslan K., On harmonic curvatures of curves in Lorentzian n-space, *Commun. Fac. Sci.Univ. Ank. Series A1.,* (2005), 54(1), 29-34.

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