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ON A NEW VARIATION OF INJECTIVE MODULES

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Abstract. In this paper, we provide various properties of GE and GEE-modules, a new variation of injective modules. We call M a *GE-module* if it has a g-supplement in every extension N and, we call also M a *GEE-module* if it has ample g-supplements in every extension N. In particular, we prove that every semisimple module is a GE-module. We show that a module M is a GEE-module if and only if every submodule is a GE-module. We study the structure of GE and GEE-modules over Dedekind domains. Over Dedekind domains the class of GE-modules lies between $\overline{WS}$-coinjective modules and Zöschinger's modules with the property (E). We also prove that, if a ring R is a local Dedekind domain, an R-module M is a GE-module if and only if $M \cong (R^*)^n \oplus K \oplus N$, where R^* is the completion of R, K is injective and N is a bounded module.

1. Introduction

Throughout the whole text, all rings are associative with unit and all modules are unital left modules. Let M be such a module. We shall write $M \subseteq N$ if M is a submodule of N. A nonzero submodule $L \subseteq M$ is said to be *essential* in M, denoted as $L \trianglelefteq M$, if $L \cap N \neq 0$ for every nonzero submodule $N \subseteq M$ ([10]). Dually, a proper submodule S of M is called *small* (in M), denoted as $S \ll M$, if $M \neq S + L$ for every proper submodule L of M ([13, 19.1]). Let U, $V \subseteq M$. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$. V is a supplement of U in M if and only if $M = U + V$ and $U \cap V \ll V$. A submodule S of a module M has *ample supplements* in M if every submodule T of M such that $M = S + T$ contains a supplement of S in M (see [13, pages 348 and 354]). Following Zöschinger's paper [15], we consider the following properties for a module M:

(E) M has a supplement in every extension.
(EE) M has ample supplements in every extension.

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Linearly compact modules (in particular, Artinian modules) have the property (EE). Here a module M is said to be *linearly compact* if for every index set I, elements m_i in M and submodules N_i $(i \in I)$ such that the cosets $m_i + N_i$ satisfy the finite intersection property, $\cap_I(m_i + N_i)$ is non-empty (see [13, 29.7]). Since every direct summand is a supplement, modules with the property (E) are a generalization of injective modules. Zöschinger studied modules with the property (E) and determined their structure over Dedekind domains. In recent years, many papers dealing with generalizations of injective modules via supplements have been published. In [12], a module M is called E^* in case M has a supplement in every extension N with $\frac{N}{M}$ coatomic. Here a module K is called *coatomic* if every proper submodule of K is contained in a maximal submodule of K. A module M is called a *CE-module* if M has a supplement in every cofinite extension N (that is, $\frac{N}{M}$ is finitely generated) (see [7]). Since finitely generated modules are coatomic, E^*-modules are CE-modules.

In [6], the authors studied a new variation of small submodules. A submodule S is called *generalized small* in M, denoted by $S \ll_g M$, (according to [14], essential small) if $M = S + T$ with $T \trianglelefteq M$ implies $T = M$. Every small submodule is generalized small. On the other hand, proper generalized small submodules of an uniform module M are small. Since supplements can be characterized by small submodules, a submodule V of a module M is called *g-supplement* of a submodule U in M if $M = U + V$ and $U \cap V \ll_g V$ (see [6]). A submodule U of M has *ample g-supplements* if, whenever $U + V = M$, V contains a g-supplement of U in M. For the properties of g-supplements, we refer to [6] and [14]. So it is natural to introduce another variation of injective modules that we called *GE-modules.* We call M a *GE-module* if it has a g-supplement in every extension N. We call also M a *GEE-module* if it has ample g-supplements in every extension N.

In this paper, we obtain various properties of GE and GEE-modules. We prove that every semisimple module is a GE-module. The class of GE-modules is closed under direct summands. We show that a module M is a GEE-module if and only if every submodule of M is a GE-module. This implies that every submodule of a GEE-module is g-supplemented. Let R be a Dedekind domain. Over the ring R, every left GE-module is $\overline{WS}$-coinjective. Every g-small submodule of an R-module M is coatomic. This fact allows us to give the following structure of GE over a local Dedekind domain R: an R-module M is a GE-module if and only if $M \cong (R^*)^n \oplus K \oplus N$, where R^* is the completion of R, K is injective and N is a bounded module. We also prove that over a semilocal Dedekind domain a torsion GE-module has the property (E).

2. GE-Modules

Every module with the property (E) is a GE-module, but it is not generally true that every GE-module has the property (E). To see this, we need these following facts. The *socle* of a module M, denoted by $Soc(M)$, it will be the sum of all simple submodules of M. Note that $Soc(M)$ is the largest semisimple submodule of M.

Lemma 1. *For a submodule S of a module M, the following are equivalent.*

(1) *S is a generalized small submodule of M;*
(2) *If $M = S + K$, there is a decomposition $M = K \oplus L$ such that L is semisimple;*
(3) *If $M = S + K$ with $Soc(M) \subseteq K$, then $K = M$.*

Proof. (1) $\Longrightarrow$ (2) This follows from [14, Proposition 2.3].

(2) $\Longrightarrow$ (3) Let $M = S + K$ with $Soc(M) \subseteq K$. By the assumption, we have $M = K \oplus L$ for some semisimple submodule L of M. Since $L \subseteq Soc(M) \subseteq K$, $M = K \oplus L$ implies that $L = 0$. Therefore, we can write $K = M$.

(3) $\Longrightarrow$ (1) Let $M = S + T$ for some essential submodule T of M. Then, $Soc(M) \subseteq T$. (3) implies that $T = M$. □

The following result is an immediate consequence of Lemma 1.

Corollary 2. *Every submodule of a semisimple module is g-small in that module.*

In order to give an example to seperate modules with the property (E) from GE-modules, we have the following simple fact which plays a key role in our work.

Proposition 3. *Let M be a semisimple module. Then, M is a GE-module.*

Proof. Let $M \subseteq N$. Suppose that $N = M + K$ for some submodule K of N. Since M is semisimple, there exists a semisimple submodule L of M such that $M = (M \cap K) \oplus L$. Note that $N = M + K = (M \cap K) \oplus L + K = K \oplus L$.

By Lemma 1, M is a generalized small submodule of N. This means that N is a g-supplement of M in N. Hence, M is a GE-module. □

By $Rad(M)$ we denote the sum of all small submodules of a module M or, equivalently the intersection of all maximal submodules of M.

Example 4. *Consider the $\mathbb{Z}$-module $N = \prod_{p \in \mathbb{P}} \frac{\mathbb{Z}}{p\mathbb{Z}}$, where $\mathbb{P}$ is the set of all prime elements of $\mathbb{Z}$. Let $M = Soc(N) = \bigoplus_{p \in \mathbb{P}} \frac{\mathbb{Z}}{p\mathbb{Z}}$. It follows from Proposition 3 that M is a GE-module. By* [3, Lemma 2.9]*, there exists a submodule T of N such that $\frac{T}{M} \cong \mathbb{Q}$. If M has a supplement K in T, we have $T = M \oplus K$ since $Rad(N) = 0$. Therefore, K is injective and so $K = Rad(K) \subseteq Rad(N) = 0$, a contradiction. Thus, M hasn't the property (E).*

Since every submodule of a semisimple module is semisimple, we obtain that any submodule of a semisimple module M is a GE-module by Proposition 3. In generally, a submodule of a GE-module need not be a GE-module. To see this, it is enough to consider the left $\mathbb{Z}$-modules $\mathbb{Z} \subseteq \mathbb{Q}$ (see Example 16). But we have:

Proposition 5. *Every direct summand of a GE-module is a GE-module.*

Proof. Let M be a GE-module and N be a direct summand of M. Then, we can write $M = N \oplus K$ for some submodule K of M. For any extension L of N, we consider the external direct product of the modules L and K. Put $T = L \oplus K$. Let consider the monomorphism $\xi : M \longrightarrow T$ by $\xi(m) = \xi(l + k) = (l, k)$ for all $m = l + k \in N \oplus K = M$. Since M is a GE-module, we get that $\xi(M)$ is a GE-module. In particular, we can write $T = \xi(M) + V$ and $\xi(M) \cap V <<_g V$ for some submodule V of T. Therefore, we obtain that $L = N + \pi(V)$, where $\pi : T \longrightarrow L$ is the natural projection. Since $ker(\pi) \subseteq \xi(M)$, we have $N \cap \pi(V) \ll_g \pi(V)$ by [6, Lemma 1(3)]. Hence, $\pi(V)$ is a g-supplement of N in L. □

We do not know whether a factor module of a GE-module is a GE-module. Now we prove that every factor module of a GE-module is a GE-module, under a certain condition: namely, when R is a left hereditary ring.

Let R be a ring. R is called a *left hereditary ring* if every factor module of an injective R-module is injective. In the following, we show that every factor module of a GE-module over a left hereditary ring is a GE-module. By $E(M)$, we denote the injective hull of a module M.

Proposition 6. *Let R be a left hereditary ring and M be a GE-module. Then, every factor module of M is a GE-module.*

Proof. Let M be a GE-module and K be any submodule of M. Suppose that N is an extension of the factor module $\frac{M}{K}$. Since R is left hereditary, we deduce that $\frac{E(M)}{K}$ is injective as a factor module of the injective module $E(M)$. Therefore, there exists a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_1} & M & \xrightarrow{\pi} & \frac{M}{K} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle i_2} & & \\
0 & \longrightarrow & K & \xrightarrow{\xi} & L & \xrightarrow{\phi} & N & \longrightarrow & 0
\end{array}$$

i.e., $\xi id = \vartheta i_1$ and $\phi\vartheta = i_2\pi$, where $\vartheta : M \longrightarrow L$ is a monomorphism by [9, Lemma 2.16]. Since $\vartheta(M) \cong M$ is a GE-module, there exists a submodule T of L such that T is a g-supplement of $\vartheta(M)$ in L. Now $N = \phi(L) = \phi(\vartheta(M)) + \phi(T) = \frac{M}{K} + \phi(T)$ and $\frac{M}{K} \cap \phi(T) = i_2(\pi(M)) \cap \phi(T) = \phi(\vartheta(M) \cap T) \ll_g \phi(T)$ by [6, Lemma 1(3)]. This means that $\phi(T)$ is a g-supplement of $\frac{M}{K}$ in N. Thus, $\frac{M}{K}$ is a GE-module. □

Recall that a ring R is a *left V-ring* if every simple R-module is injective. By [12, Proposition 5], the notions of injective modules and modules with the property (E) coincide over such a ring. In the following example, we shall show that this fact is not true for GE-modules over left V-rings.

Example 7. *(see* [11, Example 2.5]*) Consider the non-Noetherian commutative ring A which is the direct product $\prod_{i\geq 1}^{\infty} F_i$, where $F_i = F$ is any field. Suppose* *that R is the subring of the ring A consisting of all sequences $(r_n)_{n\in\mathbb{N}}$ such that there exist $r \in F, m \in \mathbb{N}$ with $r_n = r$ for all $n \geq m$. Then, R is a V-ring. Let M be the left R-module R. Since R is a V-ring, $Soc(M)$ is the direct sum of simple injective R-modules. It follows from Proposition 3 that $Soc(M)$ is a GE-module. On the other hand, it is not a direct summand of M. This means that $Soc(M)$ is not injective.*

A ring R is a *left SSI-ring* if every semisimple left R-module is injective ([4]).

Proposition 8. *Let R be a ring with the property that every left GE-module over R is injective. Then R is a left SSI-ring.*

Proof. Let M be a semisimple left R-module. It follows from Proposition 3 that M is a GE-module. So M is injective by assumption. Hence R is a left SSI-ring. □

Corollary 9. *Let R be a commutative ring. Then, R is semisimple if and only if over the ring every left GE-module is injective.*

Proof. ($\Longrightarrow$) It is clear since every left R-module is injective.

($\Longleftarrow$) Proposition 8 implies that R is a left SSI-ring. Hence R is semisimple by [4, Corollary of Proposition 1]. □

Theorem 10. *For a module M, the following statements are equivalent.*

(1) *M is a GEE-module;*
(2) *Every submodule of M is a GE-module;*
(3) *Every submodule of M is a GEE-module.*

Proof. (1) $\Longrightarrow$ (2) Let T be any submodule of M and N be any extension of T. We shall show that T has a g-supplement in N. By W, we denote the external direct product of M and N. Put $F = \frac{W}{H}$, where the submodule $H = \{(a, -a) \in W \mid a \in T\} \subseteq W$. For these inclusion homomorphism $\mu_1 : T \longrightarrow N$ and $\mu_2 : T \longrightarrow M$, we can draw the pushout in the following:

$$\begin{tikzcd} T \arrow[r, "\mu_1"] \arrow[d, "\mu_2"] & N \arrow[d, "\beta"] \\ M \arrow[r, "\alpha"] & F \end{tikzcd}$$

where α and β are monomorphisms. It is easy to see that $F = Im(\alpha) + Im(\beta)$ and $\beta^{-1}(Im(\alpha)) = T$. Since α is a monomorphism, we have $M \cong Im(\alpha)$. By the assumption, $Im(\alpha)$ is a GEE-module. Then, it follows immediately that $Im(\alpha)$ has a g-supplement V in F with $V \subseteq Im(\beta)$, i.e. $F = Im(\alpha) + V$ and $Im(\alpha) \cap V \ll_g V$. Therefore, $N = T + \beta^{-1}(V)$ and $T \cap \beta^{-1}(V) \ll_g \beta^{-1}(V)$ by [6, Lemma 1(3)]. Hence, $\beta^{-1}(V)$ is a g-supplement of T in N.

(2) $\Longrightarrow$ (3) Let $K \subseteq M$. For an extension N of K, assume $N = K + L$ for some submodule L of N. By the hypothesis, $K \cap L$ has a g-supplement, say T, in L.

Note that $N = K + L = K + (K \cap L + T) = K + T$ and $K \cap T = K \cap (L \cap T) = (K \cap L) \cap T <<_g T$ Thus, K is a GEE-module.

(3) $\Longrightarrow$ (1) Clear. $\square$

A module M is said to be *g-supplemented* if every submodule of M has a g-supplement in M ([6]). The following fact is a direct consequence of Theorem 10.

Corollary 11. *Let M be a GEE-module. Then, every submodule of M is g-supplemented.*

Proof. Let $U \subseteq K \subseteq M$ be modules. Since M is a GEE-module, it follows from Theorem 10 that U is a GE-module. In particular, U has a g-supplement in K. So K is g-supplemented. $\square$

3. GE-Modules over Dedekind Domains

In this section, we study the structure of GE and GEE-modules over Dedekind domains.

We start with the following:

Theorem 12. *Let R be an arbitrary ring and let M be a GE-module over the ring R. Suppose that $Soc(M) = 0$. Then, M has a supplement in every essential extension.*

Proof. Let $M \trianglelefteq N$. Since M is a GE-module, there exists a submodule K of N such that $N = M + K$ and $M \cap K <<_g K$. Assume that $M \cap K + X = K$ for some submodule X of K. By Lemma 1, we have the decomposition $K = X \oplus Y$, where Y is a semisimple submodule of K.

Next, we shall prove that $Y = 0$. Suppose that $Y \neq 0$. Since N is an essential extension of M, we get $Y \cap M \neq 0$. Therefore, we can write $Y = Y \cap M \oplus Z$ for some semisimple submodule Z of the semisimple module Y. Note that $0 = Z \cap (Y \cap M) = Z \cap M$ and thus, $Z = 0$ since $M \trianglelefteq N$. So $Y = Y \cap M \subseteq M$. This implies that $Y \subseteq Soc(M) = 0$, a contradiction. Hence, we obtain that $X = K$. This means that K is a supplement of M in N. $\square$

Let R be a commutative domain and M be an R-module. We denote by $Tor(M)$ the set of all elements m of M for which there exists a non-zero element r of R such that $rm = 0$, i.e. $Ann(m) \neq 0$. Then $Tor(M)$, which is a submodule of M, is called *the torsion submodule* of M. If $M = Tor(M)$, then M is called a *torsion module* and M is called *torsion-free* provided $Tor(M) = 0$. Note that $Tor(\frac{M}{Tor(M)}) = 0$ for every module M over a commutative domain R.

Corollary 13. *Let R be a Dedekind domain. If an R-module M is a GE-module, then $\frac{M}{Tor(M)}$ has a supplement in every essential extension.*

Proof. Let M be a GE-module. It follows from Proposition 6 that $\frac{M}{Tor(M)}$ is a GE-module as a factor module of M. Since $Soc(\frac{M}{Tor(M)}) \subseteq Tor(\frac{M}{Tor(M)}) = 0$, applying Theorem 12, we get that $\frac{M}{Tor(M)}$ has a supplement in every essential extension. $\square$

Let M be an R-module and let U and V be any submodules of M with $M = U + V$. If $U \cap V$ is a small submodule of M, then V is said to be a *weak supplement* of U in M. Clearly, every supplement submodule is weak supplement. M is said to be *(weakly) supplemented* if every submodule of M has a (weak) supplement in M.

Proposition 14. *Let M be a GE-module and $M \subseteq N$ with $N = Rad(N)$. Then, M has a weak supplement in N.*

Proof. Since M is a GE-module, there exists a submodule K of N such that $M + K = N$ and $M \cap K <<_g K$. By [6, Lemma 1 (2)], we obtain that $M \cap K <<_g N$. Let $M \cap K + X = N$ for some submodule X of N. It follows from Lemma 1 that we can write $N = X \oplus Y$, where Y is a semisimple submodule of N. Then, $Y \subseteq Soc(N) = Soc(Rad(N))$, and so $Soc(N) << N$ according to [5, 2.8(9)]. Applying [13, 19.3 (4)], we deduce that Y is a small submodule of N. Since $N = X \oplus Y$, we get $X = N$. Hence, K is a weak supplement of M in N. $\square$

In [2], a module M over a Dedekind domain is called $\overline{WS}$*-coinjective* if it has a weak supplement in the injective hull $E(M)$. The following result shows that GE-modules over Dedekind domains are $\overline{WS}$-coinjective.

Corollary 15. *Let M be a GE-module over a Dedekind domain. Then, M is $\overline{WS}$-coinjective.*

Proof. Since $Rad(E(M)) = E(M)$, it follows from Proposition 14 that M has a weak supplement in $E(M)$. $\square$

A $\overline{WS}$-coinjective module need not be a GE-module in general.

Example 16. *Let M denote $\mathbb{Z}$ as a $\mathbb{Z}$-module. Since $E(M) = \mathbb{Q}$ and $M \ll \mathbb{Q}$, we obtain that M is $\overline{WS}$-coinjective. Suppose that M is a GE-module. Since $Tor(M) = 0$, it follows from Corollary 13 that M has a supplement in every essential extension. Therefore, M is divisible by* [15, Lemma 5.5]*. This is a contradiction. Hence M is not a GE-module.*

Hence we have the following strict containments of classes of modules:

$$\{\text{modules with the property } (E)\} \subset \{GE\text{-modules}\} \subset \{\overline{WS}\text{-coinjective modules}\}$$

A module M is called *radical supplemented* if $Rad(M)$ has a supplement in M ([15]).

Corollary 17. *Let M be a GE-module over a Dedekind domain. Then, $Tor(M)$ is radical supplemented.*

Proof. It follows from Corollary 15 and [2, Theorem 4.1 and Corollary 4.3]. $\square$

Lemma 18. *Let M be a module and K be a generalized small submodule of M. Suppose that $M = K \oplus L$ for some submodule L of M. Then, K is semisimple.*

Proof. Since K is a generalized small submodule of M, by Lemma 1, we have $M = K \oplus L = L \oplus N$, where N is semisimple. Note that $K \cong \frac{M}{L} \cong N$ and thus, K is semisimple. □

An R-module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. It is well known that M is coatomic if and only if $Rad(\frac{M}{K}) = \frac{M}{K}$ implies that $K = M$. Note that coatomic modules have small radical. Over Dedekind domains a small submodule of a module M is coatomic. Now we obtain the following:

Corollary 19. *Let R be a Dedekind domain and M be an R-module. If K is a generalized small submodule of M, then K is coatomic.*

Proof. Let $Rad(\frac{K}{L}) = \frac{K}{L}$ for some submodule L of M. By [1, Lemma 4.4], $\frac{K}{L}$ is injective, and so there exists a submodule $\frac{N}{L}$ of $\frac{M}{L}$ such that $\frac{M}{L} = \frac{K}{L} \oplus \frac{N}{L}$. It follows from Lemma 18 that $\frac{K}{L}$ is semisimple. Since semisimple modules have zero radical, we get $\frac{K}{L} = Rad(\frac{K}{L}) = 0$. This means that $L = K$. □

Now we have the following implications on submodules over a Dedekind domain:

$$\textit{small} \Longrightarrow \textit{generalized small} \Longrightarrow \textit{coatomic}$$

A module M over a commutative domain R is said to be *bounded* if $rM = 0$ for some nonzero $r \in R$. Note that a bounded module over Dedekind domains has the property (E) as it can be deduced from the following lemma.

Lemma 20. *(Corollary of* [15, Lemma 1.4]*) Over a Noetherian integral domain with Krull-Dimension* 1*, every bounded module M has the property (E).*

Theorem 21. *Let R be a local Dedekind domain and M be an R-module. Then, the following statements are equivalent:*

(1) *M is a GE-module;*
(2) *M has the property (E);*
(3) *$M \cong (R^*)^n \oplus K \oplus N$, where R^* is the completion of R, K is injective and N is a bounded module.*

Proof. (2) $\Longleftrightarrow$ (3) follows from [15, Theorem 3.5]. Clearly, we have (2) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). Let $M \subseteq N$. By the assumption, M has a g-supplement, say K, in N. So, we can write $N = M + K$ and $M \cap K <<_g K$. Put $U = M \cap K$. It follows from Corollary 19 that U is coatomic. Since coatomic modules are radical supplemented, U has a weak supplement in every extension by [15, Lemma 3.3]. Let V be a weak supplement of U in K. Then, $K = U + V$ and $U \cap V << K$.

Next, we shall show that V is a supplement of M in N. Now, we have $N = M + K = M + (U + V) = M + V$ and $U \cap V = (M \cap K) \cap V = M \cap V << K$. Since U is a generalized small submodule of K, the equation $K = U + V = M \cap K + V$

implies that $K = V \oplus V'$ for some semisimple submodule V' of K according to Lemma 1. Thus, $M \cap V$ is small in V by [13, 19.3 (5)]. Hence, V is a supplement of M in N. This completes the proof. □

Corollary 22. *Let M be a GEE-module over a local Dedekind domain. Then, M has the property (EE).*

Proof. Let U be any submodule of M. By Theorem 10, we obtain that U is a GE-module. It follows from Theorem 21 that U has the property (E). Hence, M has the property (EE) by [15, Lemma 1.2]. □

Let R be a Dedekind domain and M be an R-module. We denote by Ω the set of all maximal (i.e., nonzero prime) ideals of R. Suppose that $\mathfrak{p}$ is any element of Ω. We denote by $T_{\mathfrak{p}}(M)$, which is a submodule of M, the set of all elements m of M for which there exists a positive integer n such that $\mathfrak{p}^n m = 0$. Then $T_{\mathfrak{p}}(M)$ is called the $\mathfrak{p}$-*primary component* of M. For a torsion module M over a Dedekind domain, we have the decomposition $M = \bigoplus_{\mathfrak{p}\in\Omega} T_{\mathfrak{p}}(M)$.

A commutative ring R is called *semilocal* if R has finitely many maximal ideals.

Proposition 23. *Let R be a semilocal Dedekind domain and M be a torsion R-module. If M is a GE-module, then it has the property (E).*

Proof. Suppose N is an extension of M. By the hypothesis, we have $N = M + K$ and $M \cap K <<_g K$ for some submodule K of N. Applying Corollary 19, we obtain that $M \cap K$ is coatomic. Then, $Rad(M \cap K)$ is a small submodule of $M \cap K$.

Assume that Ω is the set of all maximal ideals $\mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_n$ of the semilocal ring R. By ([8, Proposition 3.7]), $T_{\mathfrak{p}_i}(M \cap K)$ is bounded for every element $\mathfrak{p}_i$ in Ω. By Lemma 20, $T_{\mathfrak{p}_i}(M \cap K)$ has the property (E) for $\mathfrak{p}_i$ in Ω. Hence, $M \cap K$ has the property (E) as a finite direct sum of modules with (E).

Let K' be a supplement of $M \cap K$ in K. Therefore, $N = M + K = M + (M \cap K + K') = M + K'$ and $M \cap K' = (M \cap K) \cap K' << K'$. That is, K' is a supplement of M in N. Hence, we deduce that M has the property (E). □

Note that the condition "semilocal" in the above proposition is necessary. For this, see Example 4.

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