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A THEOREM ON WEIGHTED APPROXIMATION BY SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, pointwise approximation of functions $f \in L_{1,\varphi}(\mathbb{R})$ by the convolution type singular integral operators given in the following form:

$$L_\lambda(f;x) = \int_{\mathbb{R}} f(t)\, K_\lambda(t-x)\, dt,\ x \in \mathbb{R},\ \lambda \in \Lambda \subset \mathbb{R}_0^+,$$

is studied. Here, $L_{1,\varphi}(\mathbb{R})$ denotes the space of all measurable functions f for which $\left|\frac{f}{\varphi}\right|$ is integrable on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a corresponding weight function.

1. Introduction

The purpose of approximation theory is the approximation of functions by simply calculated functions. This theory is one of the most fundamental and important arm of mathematical analysis. The Weierstrass approximation theorem says that every continuous function defined on a closed and bounded interval of real numbers can be uniformly approximated by polynomials. Also, this well-known theorem plays significant role in the development of analysis. Then, Bernstein also proved Weierstrass's theorem by describing specific approximate polynomials known as Bernstein polynomials in the literature. Bernstein polynomials were changed by Kantorovich in order to approximate to the integrable functions. These polynomials and the generalizations were studied in [2], [8] and [11].

Taberski [21] studied the pointwise approximation of integrable functions and the approximation properties of derivatives of integrable functions in $L_1\langle -\pi, \pi\rangle$, where $\langle -\pi, \pi\rangle$ is an arbitrary closed, semi-closed or open interval, by a two parameter

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family of convolution type singular integral operators of the form:

$$T_\lambda(f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda(t-x)\,dt,\ x \in \langle -\pi,\pi\rangle,\ \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{1}$$

where $K_\lambda(t)$ is the kernel satisfying appropriate assumptions for all $\lambda \in \Lambda$ and Λ is a given set of non-negative indices with accumulation point λ_0.

Then, based on Taberski's indicated analysis, Gadjiev [10] and Rydzewska [16] proved some theorems concerning the pointwise convergence and the order of pointwise convergence of the operators of type (1) at a generalized Lebesgue point and $\mu-$generalized Lebesgue point of $f \in L_1(-\pi,\pi)$, respectively.

Further, the results of Taberski [21], Gadjiev [10] and Rydzewska [16] were extended by Karsli and Ibikli [12]. They proved some theorems for the more general integral operators defined by

$$T_\lambda(f;x) = \int_a^b f(t) K_\lambda(t-x)\,dt,\ x \in \langle a,b\rangle, \lambda \in \Lambda \subset \mathbb{R}_0^+.$$

Here, $f \in L_1\langle a,b\rangle$, where $\langle a,b\rangle$ is an arbitrary interval in $\mathbb{R}$ such as $[a,b]$, (a,b), $[a,b)$ or $(a,b]$. As concerns the study of integral operators in several settings, the reader may see also, e.g., [13], [18], [23], [24], [25], [26] and [27].

The main aim of this paper is to investigate the pointwise convergence of convolution type singular integral operators in the following form:

$$L_\lambda(f;x) = \int_{\mathbb{R}} f(t) K_\lambda(t-x)\,dt,\ x \in \mathbb{R},\ \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{2}$$

where $L_{1,\varphi}(\mathbb{R})$ is the space of all measurable functions f for which $\left|\frac{f}{\varphi}\right|$ is integrable on $\mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a corresponding weight function, at a common μ-generalized Lebesgue point of $\frac{f}{\varphi}$ and φ. In this paper, we studied a theorem of the Faddeev type similar to that of Taberski [19].

The paper is organized as follows: First, we introduce the fundamental definitions in the sequel of Introduction part. In Section 2, we prove the existence of the operators of type (2). Later, we present a theorem concerning the pointwise convergence of $L_\lambda(f;x)$ to $f(x_0)$ whenever x_0 is a common $\mu-$generalized Lebesgue point of $\frac{f}{\varphi}$ and φ.

Consequently, given that linear integral operators have become important tools in many areas, including the theory of Fourier series and Fourier integrals, approximation theory and summability theory, it is possible to use this article in the mathematical theorem.

Now, we introduce the main definitions used in this paper.

Definition 1. *A point $x_0 \in \langle a,b \rangle$ is called $\mu-$generalized Lebesgue point of the function $f \in L_1 \langle a,b \rangle$, if*

$$\lim_{h \to 0} \left(\frac{1}{\mu(h)} \int_0^h |f(t+x_0) - f(x_0)| \, dt \right) = 0,$$

where the function $\mu : \mathbb{R} \to \mathbb{R}$ is increasing and absolutely continuous on $[0, b-a]$ and $\mu(0) = 0$. Here, also holds when the integral is taken from $-h$ to 0 [12] and [16].

Definition 2. *(Class A_φ) Let $\Lambda \subset \mathbb{R}_0^+$ be an index set and $\lambda_0 \in \Lambda$ be an accumulation point of it. Let the weight function $\varphi : \mathbb{R} \to \mathbb{R}^+$ be bounded on arbitrary bounded subsets of $\mathbb{R}$ and satisfies the following inequality:*

$$\varphi(t+x) \leq \varphi(t)\varphi(x), \qquad x, t \in \mathbb{R}.$$

Suppose that there exists a function $K_\lambda^* : \mathbb{R} \to \mathbb{R}^+$ such that the following conditions hold there:

a) $\|\varphi K_\lambda^*\|_{L_1(\mathbb{R})} \leq M < \infty$, for all $\lambda \in \Lambda$.
b) For every $\xi > 0$,

$$\lim_{\lambda \to \lambda_0} \sup_{\xi \leq |t|} [\varphi(t) K_\lambda^*(t)] = 0.$$

c) For every $\xi > 0$,

$$\lim_{\lambda \to \lambda_0} \int_{\xi \leq |t|} \varphi(t) K_\lambda^*(t) dt = 0.$$

d)

$$\lim_{(x,\lambda) \to (x_0,\lambda_0)} \left| \frac{1}{\varphi(x_0)} \int_{\mathbb{R}} \varphi(t) K_\lambda(t-x) dt - 1 \right| = 0.$$

e) For any $\lambda \in \Lambda, K_\lambda(t)$ satisfies the following inequality:

$$|K_\lambda(t)| \leq K_\lambda^*(t)$$

and there exists $\delta_0 > 0$ such that $K_\lambda^*(t)$ is non-decreasing on $(-\delta_0, 0]$ and non-increasing on $[0, \delta_0)$ for any $\lambda \in \Lambda$.

If the above conditions are satisfied, then the function $K_\lambda : \mathbb{R} \to \mathbb{R}$ belongs to *class A_φ.*

Throughout this paper, we suppose that the kernel $K_\lambda(t)$ belongs to *class A_φ*.

2. Main Theorem

Definition 3. *Let $L_{1,\varphi}(\mathbb{R})$ is the space of all measurable functions for which $\left|\frac{f(t)}{\varphi(t)}\right|$ is integrable on $\mathbb{R}$. Here $\varphi:\mathbb{R}\to\mathbb{R}^+$ be a weight function and the norm in this space is given by the equality:*

$$\|f\|_{L_{1,\varphi}(\mathbb{R})}=\int\limits_{\mathbb{R}}\left|\frac{f(t)}{\varphi(t)}\right|dt.$$

Throughout this paper we suppose that the weight function $\varphi:\mathbb{R}\to\mathbb{R}^+$ [13].

The following lemma gives the existence of the operators defined by (2).

Lemma 1. *Let $\varphi:\mathbb{R}\to\mathbb{R}^+$ be a weight function. If $f\in L_{1,\varphi}(\mathbb{R})$, then $L_\lambda(f;x)$ defines a continuous transformation from $L_{1,\varphi}(\mathbb{R})$ to $L_{1,\varphi}(\mathbb{R})$.*

Proof. By the linearity of the operator $L_\lambda(f;x)$, it is sufficient to show that the expression

$$\|L_\lambda\|_1=\sup_{f\neq 0}\frac{\|L_\lambda(f;x)\|_{L_{1,\varphi}(\mathbb{R})}}{\|f\|_{L_{1,\varphi}(\mathbb{R})}}$$

remains bounded. Now, using Fubini's Theorem (see, e.g., [7]), we can write

$$\begin{aligned}
\|L_\lambda(f;x)\|_{L_{1,\varphi}(\mathbb{R})} &= \int\limits_{\mathbb{R}}\frac{1}{\varphi(x)}\left|\int\limits_{\mathbb{R}}f(t)\frac{\varphi(t)}{\varphi(t)}K_\lambda(t-x)dt\right|dx\\
&\leq \int\limits_{\mathbb{R}}\frac{1}{\varphi(x)}\left(\int\limits_{\mathbb{R}}\left|f(t+x)\frac{\varphi(t+x)}{\varphi(t+x)}K_\lambda(t)\right|dt\right)dx\\
&\leq \int\limits_{\mathbb{R}}|K_\lambda(t)|\left(\int\limits_{\mathbb{R}}\left|\frac{f(t+x)}{\varphi(t+x)}\right|\left|\frac{\varphi(t)\varphi(x)}{\varphi(x)}\right|dx\right)dt\\
&\leq \int\limits_{\mathbb{R}}\varphi(t)K^*_\lambda(t)dt\int\limits_{\mathbb{R}}\left|\frac{f(t+x)}{\varphi(t+x)}\right|dx\\
&\leq M\|f\|_{L_{1,\varphi}(\mathbb{R})}.
\end{aligned}$$

Thus, the proof is completed. □

The following theorem gives a pointwise convergence of the integral operators of type (2) at a common $\mu-$generalized Lebesgue point of $f\in L_{1,\varphi}(\mathbb{R})$ and the weight function $\varphi:\mathbb{R}\to\mathbb{R}^+$.

Theorem 1. *If x_0 is a common $\mu-$generalized Lebesque point of functions $f\in L_{1,\varphi}(\mathbb{R})$ and $\varphi:\mathbb{R}\to\mathbb{R}^+$, then*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)}L_\lambda(f;x)=f(x_0),$$

on any set Z on which the function

$$\sup_{t\in N_\delta(x_0)} \varphi(t)\left\{2K_\lambda^*(0)\mu(|x_0-x|)+\int\limits_{N_\delta(x_0)} K_\lambda^*(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right|dt\right\}$$

is bounded as (x,λ) tends to (x_0,λ_0), where $N_\delta(x_0)=(x_0-\delta,x_0+\delta)$.

Proof. Suppose that x_0 is a $\mu-$generalized Lebesque point of function $f\in L_{1,\varphi}(\mathbb{R})$.
Set $E=|L_\lambda(f;x)-f(x_0)|$. According to condition (d), we shall write

$$\begin{aligned}
E &= |L_\lambda(f;x)-f(x_0)| \\
&= \left|\int\limits_{\mathbb{R}} f(t)K_\lambda(t-x)dt-f(x_0)\right| \\
&\leq \int\limits_{\mathbb{R}}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&\quad+\left|\frac{f(x_0)}{\varphi(x_0)}\right|\left|\int\limits_{\mathbb{R}}\varphi(t)K_\lambda(t-x)dt-\varphi(x_0)\right| \\
&= I_1+I_2.
\end{aligned}$$

By condition (d) of class A_φ, $I_2\to 0$ as $(x,\lambda)\to(x_0,\lambda_0)$. Now, we investigate the integral I_1 i.e:

$$\begin{aligned}
I_1 &= \left\{\int\limits_{\mathbb{R}\setminus N_\delta(x_0)}+\int\limits_{N_\delta(x_0)}\right\}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&= I_{11}+I_{12}.
\end{aligned}$$

The following inequality holds for the integral I_{11} i.e:

$$\begin{aligned}
I_{11} &= \int\limits_{\mathbb{R}\setminus N(x_0)}\left|\frac{f(t)}{\varphi(t)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t)\left|K_\lambda(t-x)\right|dt \\
&\leq \int\limits_{\mathbb{R}\setminus N(x_0)}\left|\frac{f(t+x)}{\varphi(t+x)}-\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(t+x)\left|K_\lambda(t)\right|dt \\
&\leq \sup_{\xi\leq|t|}\left[\varphi(t)K_\lambda^*(t)\right]\varphi(x)\left\|f\right\|_{L_{1,\varphi}(\mathbb{R})}+\left|\frac{f(x_0)}{\varphi(x_0)}\right|\varphi(x)\int\limits_{\xi\leq|t|}\varphi(t)K_\lambda^*(t)dt.
\end{aligned}$$

According to conditions (c) and (d) of class A_φ, $I_{11} \to 0$ as $\lambda \to \lambda_0$. Next, we can show that I_{12} tends to zero as $(x, \lambda) \to (x_0, \lambda_0)$ on $N_\delta(x_0)$.

$$
\begin{aligned}
I_{12} &= \int\limits_{N_\delta(x_0)} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t) \left| K_\lambda(t-x) \right| dt \\
&= \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \varphi(t) \left| K_\lambda(t-x) \right| dt \\
&\leq \sup_{t \in N_\delta(x_0)} \varphi(t) \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \left| K_\lambda(t-x) \right| dt \\
&= \sup_{t \in N_\delta(x_0)} \varphi(t) \left\{ I_{121} + I_{122} \right\}.
\end{aligned}
$$

Let us consider first the integral I_{121}. By definition of $\mu-$generalized lebesgue point for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$
\int\limits_{x_0-h}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| dt < \varepsilon \mu(h)
$$

for all $0 < h \leq \delta < \delta_0$. Define the new function as

$$
F(t) = \int\limits_{t}^{x_0} \left| \frac{f(u)}{\varphi(u)} - \frac{f(x_0)}{\varphi(x_0)} \right| du. \tag{2.1}
$$

Then, for every t satisfying $0 < x_0 - t \leq \delta$ we have

$$
|F(t)| \leq \varepsilon \mu(x_0 - t). \tag{2.2}
$$

Hence, by (2.1) we can write

$$
\begin{aligned}
|I_{121}| &= \left| \int\limits_{x_0-\delta}^{x_0} \left| \frac{f(t)}{\varphi(t)} - \frac{f(x_0)}{\varphi(x_0)} \right| \left| K_\lambda(t-x) \right| dt \right| \\
&= \left| (LS) \int\limits_{x_0-\delta}^{x_0} \left| K_\lambda(t-x) \right| d\left[-F(t) \right] \right|,
\end{aligned}
$$

where (LS) denotes Lebesgue-Stieltjes integral. Applying integration by parts method to the Lebesgue-Stieltjes integral, we have

$$
|I_{121}| \leq |F(x_0-\delta)| \left| K_\lambda(x_0-\delta-x) \right| + \int\limits_{x_0-\delta}^{x_0} |F(t)| \left| (d_t \left| K_\lambda(t-x) \right|) \right|.
$$

According to (2.2) and condition (e) of class A_φ, we obtain

$$|I_{121}| \leq \varepsilon\mu(\delta)K_\lambda^*(x_0-\delta-x)+\varepsilon\int\limits_{x_0-\delta}^{x_0}\mu(x_0-t)\left|(d_tK_\lambda^*(t-x))\right|.$$

Now, we define the variations:

$$A(t)=\begin{cases}\bigvee\limits_{x_0-x-\delta}^{t}K_\lambda^*(s) & , \quad x_0-x-\delta<t\leq x_0-x\\ 0 & , \qquad t=x_0-x-\delta.\end{cases} \tag{2.3}$$

Taking above variations and applying integration by parts method to last inequality, we get

$$\begin{aligned}|I_{121}| &\leq \varepsilon\mu(\delta)K_\lambda^*(x_0-\delta-x)+\varepsilon\int\limits_{x_0-x-\delta}^{x_0-x}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= \varepsilon(i_1+i_2).\end{aligned}$$

Let us consider the integral i_2. Write

$$\begin{aligned}i_2 &= \int\limits_{x_0-x-\delta}^{x_0-x}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= \left\{\int\limits_{x_0-x-\delta}^{0}+\int\limits_{0}^{x_0-x}\right\}\{\mu(x_0-x-t)\}_t^{'}A(t)dt\\ &= i_{21}+i_{22}.\end{aligned}$$

From (2.3), we shall write

$$\begin{aligned}i_{21} &= \int\limits_{x_0-x-\delta}^{0}\left[\bigvee\limits_{x_0-x-\delta}^{t}K_\lambda^*(s)\right]\{\mu(x_0-x-t)\}_t^{'}dt\\ &= \int\limits_{x_0-x-\delta}^{0}\left[K_\lambda^*(t)-K_\lambda^*(x_0-x-\delta)\right]\{\mu(x_0-x-t)\}_t^{'}dt \qquad (2.4)\end{aligned}$$

and

$$\begin{aligned} i_{22} &= \int_{0}^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{t} K_\lambda^*(s) \right] \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &= \int_{0}^{x_0-x} \left[\bigvee_{x_0-x-\delta}^{0} K_\lambda^*(s) + \bigvee_{0}^{t} K_\lambda^*(s) \right] \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &= \int_{0}^{x_0-x} \left(2K_\lambda^*(0) - K_\lambda^*(x_0-x-\delta) - K_\lambda^*(t)\right) \{\mu(x_0-x-t)\}_t^{'} \, dt. \qquad (2.5) \end{aligned}$$

Combining (2.4) and (2.5), we obtain

$$\begin{aligned} i_2 &= i_{21} + i_{22} \\ &\leq -2K_\lambda^*(0)\mu(x_0-x) - K_\lambda^*(x_0-x-\delta)\mu(\delta) \\ &\quad + \int_{x_0-x-\delta}^{x_0-x} K_\lambda^*(t) \{\mu(x_0-x-t)\}_t^{'} \, dt. \end{aligned}$$

Thus

$$\begin{aligned} |I_{121}| &\leq \varepsilon\,(i_1 + i_2) \\ &\leq 2\varepsilon K_\lambda^*(0)\mu(x_0-x) + \varepsilon \int_{x_0-x-\delta}^{x_0-x} K_\lambda^*(t) \{\mu(x_0-x-t)\}_t^{'} \, dt \\ &\leq 2\varepsilon K_\lambda^*(0)\mu(x_0-x) + \varepsilon \int_{x_0-\delta}^{x_0} K_\lambda^*(t-x) \{\mu(x_0-t)\}_t^{'} \, dt. \qquad (2.6) \end{aligned}$$

We can use a similar method for estimating I_{122}. Then we find the inequality

$$|I_{122}| \leq \varepsilon \int_{x_0}^{x_0+\delta} K_\lambda^*(t-x) \{\mu(t-x_0)\}_t^{'} \, dt. \qquad (2.7)$$

Consequently, from (2.6) and (2.7), we can write the following inequality:

$$\begin{aligned} I_{12} &\leq \sup_{t\in N_\delta(x_0)} \varphi(t) \{I_{121} + I_{122}\} \\ &\leq \varepsilon \sup_{t\in N_\delta(x_0)} \varphi(t) \left[2K_\lambda^*(0)\mu(x_0-x) + \int_{x_0-\delta}^{x_0+\delta} K_\lambda^*(t-x) \left| \{\mu(|x_0-t|)\}_t^{'} \right| dt \right]. \end{aligned}$$

Note that in the above inequality we used the hypothesis of the theorem, i.e., boundedness of the following function:

$$\sup_{t\in N_\delta(x_0)} \varphi(t)\left[2K_\lambda^*(0)\mu(x_0-x)+\int_{x_0-\delta}^{x_0+\delta} K_\lambda^*(t-x)\left|\{\mu(|x_0-t|)\}_t^{'}\right| dt\right].$$

Since the remaining expression is bounded by the hypothesis, $I_{12}\to 0$ as $(x,\lambda)\to(x_0,\lambda_0)$. Thus, we obtain

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} L_\lambda(f;x)=f(x_0)$$

and the proof is completed. □

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