Available online: October 18, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 2, Pages 220–228 (2018)
DOI: 10.1501/Commua1_0000000876
ISSN 1303–5991

http://communications.science.ankara.edu.tr/index.php?series=A1

A Q-ANALOG OF THE BI-PERIODIC LUCAS SEQUENCE

ELIF TAN

Abstract. In this paper, we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

1. Introduction

It is well-known that the classical Fibonacci numbers F_n are defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2 \tag{1.1}$$

with the initial conditions $F_0 = 0$ and $F_1 = 1$. The Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. There are a lot of generalizations of Fibonacci and Lucas sequences. In [6], Edson and Yayenie introduced a generalization of the Fibonacci sequence, called as bi-periodic Fibonacci sequence, as follows:

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \; n \geq 2 \tag{1.2}$$

with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are nonzero numbers. Note that if we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence. These sequences are emerged as denominators of the continued fraction expansion of the quadratic irrational numbers. For detailed information related to these sequences,

Received by the editors: April 04, 2017, Accepted: June 06, 2017.

2010 *Mathematics Subject Classification.* 05A15, 05A30.

Key words and phrases. bi-periodic Fibonacci and Lucas sequences, bi-periodic incomplete Fibonacci and Lucas sequences, q-analog, matrix formula.

we refer to [6, 19, 8, 11, 12, 17, 18, 15, 16]. Yayenie [19] gave an explicit formula of q_n as:

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-i}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i} \tag{1.3}$$

where $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd.

Similar to (1.2), by taking initial conditions $p_0 = 2$ and $p_1 = a$, Bilgici [2] introduced the bi-periodic Lucas numbers as follows:

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2. \tag{1.4}$$

It should also be noted that, it gives the classical Lucas sequence in the case of $a = b = 1$ in $\{p_n\}$. In analogy with (1.3), Tan and Ekin [14] gave the explicit formula of the bi-periodic Lucas numbers as:

$$p_n = a^{\xi(n)} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\lfloor \frac{n}{2} \rfloor - i}, \ n \geq 1. \tag{1.5}$$

On the other hand, there are several different q-analogs for the Fibonacci and Lucas sequences [3, 4, 5, 13, 7, 1]. Particularly, Cigler [5] gave the (Carlitz-) q-Fibonacci and q-Lucas polynomials

$$f_n(x,s) = xf_{n-1}(x,s) + q^{n-2} s f_{n-2}(x,s); \ f_0(x,s) = 0, \ f_1(x,s) = 1, \tag{1.6}$$

$$l_n(x,s) = f_{n+1}(x,s) + sf_{n-1}(x,qs); \ l_0(x,s) = 2, \ l_1(x,s) = x, \tag{1.7}$$

respectively.

Additionally, Ramírez and Sirvent [10] introduced a q-analog of the bi-periodic Fibonacci sequence by

$$F_n^{(a,b)}(q,s) = \begin{cases} aF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n-1}^{(a,b)}(q,s) + q^{n-2} s F_{n-2}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.8}$$

with initial conditions $F_0^{(a,b)}(q,s) = 0$ and $F_1^{(a,b)}(q,s) = 1$. They derived the following equality to evaluate the q-bi-periodic Fibonacci sequence:

$$F_n^{(a,b)}(q,s) = \chi_n F_{n-1}^{(a,b)}(q,qs) - qsF_{n-2}^{(a,b)}\left(q, q^2 s\right), \tag{1.9}$$

where $\chi_n := a^{\xi(n+1)} b^{\xi(n)}$. Also, they gave the relationship between the q-bi-periodic Fibonacci sequence and the (Carlitz-) q-Fibonacci polynomials as:

$$F_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n+1)} f_n\left(\sqrt{ab}, s\right). \tag{1.10}$$

By using (1.10), they obtained the explicit formula of the q-bi-periodic Fibonacci sequence as:

$$F_n^{(a,b)}(q,s) = a^{\xi(n-1)} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} (ab)^{\lfloor \frac{n-1}{2} \rfloor - k} q^{k^2} s^k, \tag{1.11}$$

where $\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]_q!}{[k]_q![n-k]_q!}$ is the q-binomial coefficients with $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$.

Motivated by the Ramirez's results in [10], here we introduce a q-analog of the bi-periodic Lucas sequence, called as the q-bi-periodic Lucas sequence, and give some identities related to the q-bi-periodic Fibonacci and Lucas sequences. Also, we give a matrix representation for the q-bi-periodic Fibonacci sequence which allow us to obtain several properties of this sequence in a simple way. Moreover, by using the explicit formulas for the q-bi-periodic Fibonacci and Lucas sequences, we introduce q-analogs of the bi-periodic incomplete Fibonacci and Lucas sequences and give a relation between them.

2. A q-analog of the bi-periodic Lucas sequence

First, we consider the (Carlitz-) q-Lucas polynomials in (1.7), and define the q-bi-periodic Lucas sequence by means of the (Carlitz-) q-Lucas polynomials.

Definition 1. The q-bi-periodic Lucas sequence is defined by

$$L_n^{(a,b)}(q,s) = \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab}, s\right) \tag{2.1}$$

where $l_n(x,s)$ is the (Carlitz-) q-Lucas polynomials.

The terms of the q-bi-periodic Lucas sequence can be given as:

n	$L_n^{(a,b)}(q,s)$
0	2
1	a
2	$ab + sq + s$
3	$a^2b + as + asq + asq^2$
4	$a^2b^2 + abs + absq + absq^2 + absq^3 + s^2q^2 + s^2q^4$
5	$a^3b^2 + a^2bs + a^2bsq + a^2bsq^2 + a^2bsq^3 + a^2bsq^4$ $+as^2q^2 + as^2q^3 + as^2q^4 + as^2q^5 + as^2q^6$
$\vdots$	$\vdots$

Note that if we take $a = b = x$, we obtain the (Carlitz-) q-Lucas polynomials $l_n(x,s)$.

In the following lemma, we state the q-bi-periodic Lucas sequence in terms of the q-bi-periodic Fibonacci sequence.

Lemma 1. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs). \tag{2.2}$$

Proof. By using the definition of the q-bi-periodic Lucas sequence and the relations (1.7) and (1.10), we have

$$\begin{aligned} L_n^{(a,b)}(q,s) &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} l_n\left(\sqrt{ab},s\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(f_{n+1}\left(\sqrt{ab},s\right) + sf_{n-1}\left(\sqrt{ab},qs\right)\right) \\ &= \left(\sqrt{\frac{a}{b}}\right)^{\xi(n)} \left(\sqrt{\frac{b}{a}}\right)^{\xi(n)} \left(F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs)\right) \end{aligned}$$

which gives the desired result. □

Now we give an another relation between the q-bi-periodic Fibonacci sequence and q-bi-periodic Lucas sequence.

Theorem 1. *For any integer $n \geq 0$, we have*

$$\chi_n L_n^{(a,b)}(q,qs) = F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2 F_{n-2}^{(a,b)}\left(q,q^2 s\right) \tag{2.3}$$

where $\chi_n := a^{\xi(n+1)}b^{\xi(n)}$.

Proof. By using the definition of the q-bi-periodic Fibonacci sequence in (1.8) and the relations (2.2) and (1.9), we get

$$\begin{aligned} \chi_n L_n^{(a,b)}(q,qs) &= \chi_n\left(F_{n+1}^{(a,b)}(q,qs) + qsF_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qsF_n^{(a,b)}\left(q,q^2s\right) + \chi_n qsF_{n-1}^{(a,b)}\left(q,q^2s\right) \\ &= F_{n+2}^{(a,b)}(q,s) - qs\left(F_n^{(a,b)}\left(q,q^2s\right) - \chi_n F_{n-1}^{(a,b)}\left(q,q^2s\right)\right) \\ &= F_{n+2}^{(a,b)}(q,s) - q^{n+1}s^2F_{n-2}^{(a,b)}\left(q,q^2s\right). \end{aligned}$$

□

If we take $a = b = x$ in (2.3), it reduces to the relation between q-bi-periodic Fibonacci sequence and Lucas polynomials

$$xl_n(x,qs) = f_{n+2}(x,s) - q^{n+1}s^2 f_{n-2}\left(x,q^2s\right)$$

which can be found in [5, Equation (3.15)].

In the following theorem, we give the explicit expression of the q-bi-periodic Lucas sequence $L_n^{(a,b)}(q,s)$. Since we define the incomplete sequences by using its explicit formula, the following theorem play a key role for our further study in the next section.

Theorem 2. *For any integer $n \geq 0$, we have*

$$L_n^{(a,b)}(q,s) = a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k. \tag{2.4}$$

Proof. By using the relations (2.2) and (1.11), we have

$$\begin{aligned}
L_n^{(a,b)}(q,s) &= F_{n+1}^{(a,b)}(q,s) + sF_{n-1}^{(a,b)}(q,qs) \\
&= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \\
&\quad + a^{\xi(n-2)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \begin{bmatrix} n-2-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - 1 - k} q^{k^2+k} s^{k+1} \\
&= a^{\xi(n)} \left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2} s^k \right. \\
&\quad \left. + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k \right) \\
&= a^{\xi(n)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(q^k \begin{bmatrix} n-k \\ k \end{bmatrix} + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} \right) (ab)^{\lfloor \frac{n}{2} \rfloor - k} q^{k^2-k} s^k.
\end{aligned}$$

By using the identity

$$q^k \begin{bmatrix} n-k \\ k \end{bmatrix} + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} = \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix},$$

we obtain the desired result. □

If we take $a = b = x$ in the above theorem, it reduces to the (Carlitz-) q-Lucas polynomials

$$l_n(x,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} q^{k^2-k} s^k x^{n-2k}$$

which can be found in [5, Equation (3.14)].

Now we give a matrix representation for the q-bi-periodic Fibonacci sequence which can be proven by induction. By using matrix formula, one can obtain several properties of this sequence.

Theorem 3. *For $n \geq 1$, let define the matrix* $C(\chi_n, s) := \begin{pmatrix} 0 & 1 \\ s & \chi_n \end{pmatrix}$. *Then we have*

$$\begin{aligned} M_n(\chi_n, s) &:= C\left(\chi_n, q^{n-1}s\right) C\left(\chi_{n-1}, q^{n-2}s\right) \cdots C(\chi_2, qs)\, C(\chi_1, s) \\ &= \begin{pmatrix} sF_{n-1}^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s) \\ sF_n^{(a,b)}(q, qs) & \left(\frac{b}{a}\right)^{\xi(n)} F_{n+1}^{(a,b)}(q, s) \end{pmatrix}. \end{aligned} \tag{2.5}$$

In the following theorem, we give the q-Cassini formula for the q-bi-periodic Fibonacci sequence by taking the determinant of the both sides of the equation (2.5).

Theorem 4. *For any integer $n > 0$, we have*

$$\begin{aligned} &\left(\frac{b}{a}\right)^{\xi(n)} F_{n-1}^{(a,b)}(q, qs)\, F_{n+1}^{(a,b)}(q, s) - \left(\frac{b}{a}\right)^{\xi(n+1)} F_n^{(a,b)}(q, s)\, F_n^{(a,b)}(q, qs) \\ = \; &(-1)^n s^{n-1} q^{\frac{n(n-1)}{2}}. \end{aligned} \tag{2.6}$$

Note that by taking $a = b = x$, we obtain the result in [5, Equation (3.12)].

Theorem 5. *For any integer $n > 0$, we have*

$$F_{2n}^{(a,b)}(q, s) = \left(\frac{a}{b}\right)^{\xi(n)} q^n s F_{n-1}^{(a,b)}\left(q, q^{n+1}s\right) F_n^{(a,b)}(q, s) + F_n^{(a,b)}(q, q^n s)\, F_{n+1}^{(a,b)}(q, s). \tag{2.7}$$

Proof. Since $M_{m+n}(\chi_n, s) = M_m(\chi_n, q^n s)\, M_n(\chi_n, s)$, if we equate the corresponding entries of each matrices and take $m = n$ in the resulting equality, we get the desired result. □

One can get several properties of the q-bi-periodic Fibonacci sequence by taking proper powers of the matrix in (2.5).

3. $q-$bi-periodic incomplete Fibonacci and Lucas sequences

In this section, we define q-bi-periodic incomplete Fibonacci and Lucas sequences. Let n be a positive integer and l be an integer.

Ramirez [9] defined the bi-periodic incomplete Fibonacci numbers by using the explicit formula of the bi-periodic Fibonacci sequences in (1.3) as:

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-1-i}{i} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - i}, \; 0 \leq l \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

Similarly, by using the explicit formula of the bi-periodic Lucas sequence in (1.5), Tan and Ekin [14] defined the bi-periodic incomplete Lucas numbers as:

$$p_n(l) = a^{\xi(n)} \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - i}, \; 0 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Analogously, by using the explicit formulas of the q-bi-periodic Fibonacci sequence in (1.11) and the q-bi-periodic Lucas sequence in (2.4), we define the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows.

Definition 2. For any non negative integer n, the q-bi-periodic incomplete Fibonacci and Lucas sequences are defined by

$$F_{n,l}^{(a,b)}(q,s) = a^{\xi(n-1)} \sum_{k=0}^{l} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - k} q^{k^2} s^k, \ 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor \tag{3.1}$$

and

$$L_{n,l}^{(a,b)}(q,s) = a^{\xi(n)} \sum_{k=0}^{l} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - k} q^{k^2-k} s^k, \ 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor, \tag{3.2}$$

respectively.

If we take $l = \left\lfloor \frac{n-1}{2} \right\rfloor$ in (3.1), we obtain the q-bi-periodic Fibonacci sequence, and if we take $l = \left\lfloor \frac{n}{2} \right\rfloor$ in (3.2), we obtain the q-bi-periodic Lucas sequence.

Next, we give non-homogenous recurrence relation for the q-bi-periodic incomplete Fibonacci sequence.

Theorem 6. *For $0 \le l \le \frac{n-2}{2}$, the non-linear recurrence relation of the q-bi-periodic incomplete Fibonacci sequence is*

$$F_{n+2,l+1}^{(a,b)}(q,s) = \begin{cases} aF_{n+1,l+1}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is even} \\ bF_{n+1,l+1}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s), & \text{if } n \text{ is odd} \end{cases}. \tag{3.3}$$

The relation (3.3) can be transformed into the non-homogeneous recurrence relation

$$F_{n+2,l}^{(a,b)}(q,s) = aF_{n+1,l}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s) - a \begin{bmatrix} n-1-l \\ l \end{bmatrix} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - l} q^{n+l^2} s^{l+1} \tag{3.4}$$

for even n, and

$$F_{n+2,l}^{(a,b)}(q,s) = bF_{n+1,l}^{(a,b)}(q,s) + q^n s F_{n,l}^{(a,b)}(q,s) - \begin{bmatrix} n-1-l \\ l \end{bmatrix} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - l} q^{n+l^2} s^{l+1} \tag{3.5}$$

for odd n.

Proof. If n is even, then $\left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$. By using the Definition (3.1), we can write the RHS of (3.3) as

$$a^{1+\xi(n)} \sum_{k=0}^{l+1} \begin{bmatrix} n-k \\ k \end{bmatrix} (ab)^{\left\lfloor \frac{n}{2} \right\rfloor - k} q^{k^2} s^k$$

$$+q^n s a^{\xi(n-1)} \sum_{k=0}^{l} \begin{bmatrix} n-1-k \\ k \end{bmatrix} (ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor - k} q^{k^2} s^k$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k} q^{k^2}s^k + q^n\ a\sum_{k=0}^{l}\begin{bmatrix} n-1-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n-1}{2}\rfloor -k} q^{k^2}s^{k+1}$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k} q^{k^2}s^k$$

$$+q^n\ a\sum_{k=1}^{l+1}\begin{bmatrix} n-k \\ k-1 \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k} q^{(k-1)^2}s^k$$

$$= a\sum_{k=0}^{l+1}\left(\begin{bmatrix} n-k \\ k \end{bmatrix} + q^{n-2k+1}\begin{bmatrix} n-k \\ k-1 \end{bmatrix}\right)(ab)^{\lfloor \frac{n}{2}\rfloor -k} q^{k^2}s^k\ \ (ab)^{\lfloor \frac{n}{2}\rfloor -k} - 0$$

$$= a\sum_{k=0}^{l+1}\begin{bmatrix} n-k+1 \\ k \end{bmatrix}(ab)^{\lfloor \frac{n}{2}\rfloor -k} q^{k^2}s^k\ \ (ab)^{\lfloor \frac{n}{2}\rfloor -k}$$

$$= F_{n+2,l+1}^{(a,b)}(q,s).$$

Also from equation (3.3), we have

$$\begin{aligned}
F_{n+2,l}^{(a,b)}(q,s) &= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l-1}^{(a,b)}(q,s) \\
&= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l}^{(a,b)}(q,s) + q^n s(F_{n,l-1}^{(a,b)}(q,s) - F_{n,l}^{(a,b)}(q,s)) \\
&= aF_{n+1,l}^{(a,b)}(q,s) + q^n sF_{n,l}^{(a,b)}(q,s) - a\begin{bmatrix} n-1-l \\ l \end{bmatrix}(ab)^{\lfloor \frac{n-1}{2}\rfloor -l} q^{n+l^2}s^{l+1}.
\end{aligned}$$

If n is odd, the proof is completely analogous. □

Note that the q-bi-periodic Lucas sequence does not satisfy a recurrence like (3.3), since $F_{n+1}^{(a,b)}(q,s)$ and $F_{n+1}^{(a,b)}(q,qs)$ do not satisfy the same recurrence relation.

Finally we give the relationship between the q-bi-periodic incomplete Fibonacci and Lucas sequences as follows:

Theorem 7. *For* $0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor$, *we have*

$$L_{n,l}^{(a,b)}(q,s) = F_{n+1,l}^{(a,b)}(q,s) + F_{n-1,l-1}^{(a,b)}(q,qs). \tag{3.6}$$

Proof. It can be proved easily by using the definitions (3.1) and (3.2). □

4. Acknowledgement

This research is supported by Ankara University Scientific Research Project Unit (BAP). Project No:17H0430002.

References

[1] Andrews, G. E., Fibonacci numbers and the Rogers-Ramanujan identities, *Fibonacci Quart.* 42(1) (2004), 3–19.
[2] Bilgici, G., Two generalizations of Lucas sequence, *Applied Mathematics and Computation* 245 (2014) 526–538.
[3] Carlitz, L., Fibonacci notes 4: q-Fibonacci Polynomials. *The Fibonacci Quarterly* 13(2) (1975), 97-102.
[4] Cigler, J., A new class of q-Fibonacci polynomials, *Electron. J. Comb.*10 (2003), ArticleR19.
[5] Cigler, J., Some beautiful q-analogues of Fibonacci and Lucas polynomials, arXiv:11042699.
[6] Edson, M. and Yayenie, O., A new generalizations of Fibonacci sequences and extended Binet's Formula, *Integers* 9, (A48) (2009), 639-654.
[7] Jia, C. Z., Liu, H. M., and Wang, T. M., q-Analogs of Generalized Fibonacci and Lucas Polynomials, *The Fibonacci Quarterly* (451) 2007, 26-34.
[8] Panario, D., Sahin, M. and Wang, Q., A family of Fibonacci-like conditional sequences, *Integers* 13 (2013).
[9] Ramírez, J. L., Bi-periodic incomplete Fibonacci sequences, *Ann. Math. Inform.* 42(2013), 83–92.
[10] Ramírez, J. L. and Sirvent, V. F., A q-Analoque of the Biperiodic Fibonacci Sequence, *Journal of Integer Sequences*, Vol. 19 (2016).
[11] Sahin, M., The generating function of a family of the sequences in terms of the continuant, *Applied Mathematics and Computation* 217 (2011) 5416–5420.
[12] Sahin, M., The Gelin-Cesaro identity in some conditional sequences, *Hacet. J. Math. Stat.* 40(6) (2011), 855-861.
[13] Schur, I., Ein beitrag zur additiven zahlentheorie. Sitzungsber, A*kad. Wissencsh. Berlin, Phy.-Math. Klasse* (1917), 302-321.
[14] Tan, E. and Ekin, A. B., Bi-periodic Incomplete Lucas Sequences, *Ars Combinatoria.* 123 (2015), 371-380.
[15] Tan, E. and Ekin, A. B., Some Identities On Conditional Sequences By Using Matrix Method. *Miskolc Mathematical Notes.* 18 (1) (2017), 469–477.
[16] Tan,E., On bi-periodic Fibonacci and Lucas numbers by matrix method, *Ars Combinatoria.* 133 (2017), 107-113.
[17] Tan, E., Yilmaz, S. and Sahin, M., On a new generalization of Fibonacci quaternions, *Chaos, Solitons and Fractals* 82 (2016) 1–4.
[18] Tan, E., Yilmaz, S. and Sahin, M., A note on bi-periodic Fibonacci and Lucas quaternions, *Chaos, Solitons and Fractals* 85 (2016) 138–142.
[19] Yayenie, O., A note on generalized Fibonacci sequeence, *Appl. Math. Comput.* 217 (2011) 5603-5611.

E-mail address: etan@ankara.edu.tr

Current address: Department of Mathematics, Faculty of Sciences, Ankara University, 06100 Tandogan Ankara, Turkey

ORCID Address: http://orcid.org/0000-0002-8381-8750