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ON THE GEOMETRY OF PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY SASAKIAN MANIFOLD

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Abstract. In this paper, we study the pseudo-slant submanifolds of a nearly Sasakian manifold. We characteterize a totally umbilical properpseudo-slant submanifolds and find that a necessary and sufficient condition for such submanifolds totally geodesic. Also the integrability conditions of distributions of pseudo-slant submanifolds of a nearly Sasakian manifold are investigated.

1. Introduction

The differential geometry of slant submanifolds has shown an increasing development since B.Y. Chen defined slant submanifolds in complex manifolds as a natural generalization of both the invariant and anti-invariant submanifolds [3], [4]. Many research articles have been appeared on the existence of these submanifolds in different knows spaces. The slant submanifolds of an almost contact metric manifolds were defined and studied by A. Lotta [2]. After, such submanifolds were studied by J.L Cabrerizo et. al[6], in Sasakian manifolds . Recently, in [9],[10],[11],[13] M. Atçeken studied slant and pseudo-slant submanifold in $(LCS)_n-$manifold and other manifolds. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papagiuc [12]. Recently, A. Carrizo [5],[6] defined and studied bi-slant immersions in almost Hermitian manifolds and simultaneously gave the notion of pseudo-slant submanifolds in almost Hermitian manifolds. The contact version of pseudo-slant submanifolds has been defined and studied by V. A. Khan and M. A Khan [16].

The present paper is organized as follows.

In section 1, the notions and definitions of submanifolds of a Riemannian manifold were given for later use. In this paper, we study pseudo-slant submanifolds of

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a nearly Sasakian manifold. In section 2, we review basic formulas and definitions for a nearly Sasakian manifold and their submanifolds. In section 3, we recall the definition and some basic results of a pseudo-slant submanifold of almost contact metric manifold. We study characterization of totally umbilical proper-slant submanifolds and find that a necessary and sufficient condition for such submanifolds is to be totally geodesic. In section 4, the integrability conditions of distributions of pseudo-slant submanifolds of a nearly Sasakian manifold are investigated.

2. Preliminaries

In this section, we give some notations used throughout this paper. We recall some necessary fact and formulas from the theory of nearly Sasakian manifolds and their submanifolds.

Let $\widetilde{M}$ be an $(2m+1)-$dimensional almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g), that is, φ is a $(1,1)$ tensor field , ξ is a vector field; η is 1-form and g is a compatible Riemanian metric such that

$$\varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\varphi\xi = 0,\ \eta(\xi) = 1,\ \ \eta \circ \varphi = 0,\ \eta(X) = g(X, \xi) \tag{2.2}$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),\ \ g(\varphi X, Y) = -g(X, \varphi Y) \tag{2.3}$$

for any vector fields $X, Y \in \Gamma(T\widetilde{M})$, where $\Gamma(\widetilde{M})$ denotes the set of all vector fields on $\widetilde{M}$. If in addition to above relations,

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

then $\widetilde{M}$ is called a Sasakian manifold, where $\widetilde{\nabla}$ is the Levi-Civita connections of g.

The almost contact metric manifold $\widetilde{M}$ is called a nearly Sasakian manifold if it satisfy the following condition

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y \tag{2.5}$$

for any $X, Y \in \Gamma(T\widetilde{M})$.

Now, let M be a submanifold of an almost contact metric manifold $\widetilde{M}$, we denote the induced connections on M and the normal bundle $T^{\perp}M$ by ∇ and $\nabla^{\perp}$, respectively, then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.6}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The mean curvature vector H of M is given by

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i), \tag{2.9}$$

where m is the dimension of M and $\{e_1, e_2, ..., e_m\}$ is a local orthonormal frame of M. A submanifold M of a Riemannian manifold $\widetilde{M}$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$. A submanifold M is said to be totally geodesic if $h = 0$ and M is said to be minimal if $H = 0$.

Let M be a submanifold of an almost contact metric manifold $\widetilde{M}$. Then for any $X \in \Gamma(TM)$, we can write

$$\varphi X = TX + NX, \tag{2.11}$$

where TX is the tangential component and NX is the normal component of φX. Similarly, for $V \in \Gamma(T^{\perp}M)$, we can write

$$\varphi V = tV + nV, \tag{2.12}$$

where tV is the tangential component and nV is the normal component of φV.

Thus by using (2.1), (2.2), (2.11) and (2.12), we obtain

$$T^2 + tN = -I + \eta \otimes \xi, \quad NT + nN = 0, \tag{2.13}$$

$$Tt + tn = 0, \quad n^2 + Nt = -I \tag{2.14}$$

and

$$T\xi = 0 = N\xi, \quad \eta \circ T = 0 = \eta \circ N. \tag{2.15}$$

Furthermore, for any $X, Y \in \Gamma(TM)$, we have $g(TX, Y) = -g(X, TY)$ and $V, U \in \Gamma(T^{\perp}M)$, we get $g(U, nV) = -g(nU, V)$. These show that T and n are also skew-symmetric tensor fields. Moreover, for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, we have

$$g(NX, V) = -g(X, tV) \tag{2.16}$$

which gives the relation between N and t.

The covariant derivatives of the tensor field T, N, t and n are, respectively, defined by

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \tag{2.17}$$

$$(\nabla_X N)Y = \nabla^{\perp}_X NY - N\nabla_X Y, \tag{2.18}$$

$$(\nabla_X t)V = \nabla_X tV - t\nabla^{\perp}_X V \tag{2.19}$$

and

$$(\nabla_X n)V = \nabla^{\perp}_X nV - n\nabla^{\perp}_X V \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now, for any $X, Y \in \Gamma(TM)$, let us denote the tangential and normal parts of $(\widetilde{\nabla}_X \varphi)Y$ by $P_X Y$ and $F_X Y$, respectively. Then we decompose

$$(\widetilde{\nabla}_X \varphi)Y = P_X Y + F_X Y \tag{2.21}$$

thus, by an easy computation, we obtain the formulae

$$P_X Y = (\nabla_X T)Y - A_{NY}X - th(X, Y) \tag{2.22}$$

and

$$F_X Y = (\nabla_X N)Y + h(X, TY) - nh(X, Y). \tag{2.23}$$

Similarly, for any $V \in \Gamma(T^{\perp}M)$, denoting tangential and normal parts of $(\widetilde{\nabla}_X \varphi)V$ by $P_X V$ and $F_X V$, respectively, we obtain

$$P_X V = (\nabla_X t)V - A_{nV}X + TA_V X \tag{2.24}$$

and

$$F_X V = (\nabla_X n)V + h(tV, X) + NA_V X. \tag{2.25}$$

Now, for any $X, Y \in \Gamma(TM)$, from (2.5), we have

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X, Y)\xi - \eta(Y)X - \eta(X)Y \tag{2.26}$$

that is,

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = \widetilde{\nabla}_X \varphi Y - \varphi\widetilde{\nabla}_X Y + \widetilde{\nabla}_Y \varphi X - \varphi\widetilde{\nabla}_Y X.$$

By using (2.6), (2.7), (2.11) and (2.12), we get

$$\begin{aligned}
(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X &= \widetilde{\nabla}_X TY + \widetilde{\nabla}_X NY - \varphi(\nabla_X Y + h(X, Y)) \\
&\quad + \widetilde{\nabla}_Y TX + \widetilde{\nabla}_Y NX - \varphi(\nabla_Y X + h(X, Y)). \\
&= \nabla_X TY + h(X, TY) - A_{NY}X + \nabla^{\perp}_X NY \\
&\quad - T\nabla_X Y - N\nabla_X Y - th(X, Y) - nh(X, Y) \\
&\quad + \nabla_Y TX + h(Y, TX) - A_{NX}Y + \nabla^{\perp}_Y NX - T\nabla_Y X \\
&\quad - N\nabla_Y X - th(X, Y) - nh(X, Y).
\end{aligned} \tag{2.27}$$

Making use of (2.27) and (2.26), we obtain

$$\begin{aligned}
&\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y \\
&-th(X,Y) - nh(X,Y) + \nabla_Y TX + h(Y,TX) - A_{NX}Y \\
&+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - th(X,Y) - nh(X,Y) \\
&-2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0.
\end{aligned}$$

By taking tangential and normal parts of the above equation, respectively, we have equation

$$\begin{aligned}
(\nabla_X T)Y + (\nabla_Y T)X &= A_{NX}Y + A_{NY}X + 2th(X,Y) \\
&\quad +2g(X,Y)\xi - \eta(Y)X - \eta(X)Y \qquad (2.28)
\end{aligned}$$

and

$$(\nabla_X N)Y + (\nabla_Y N)X = -h(X,TY) - h(Y,TX) + 2nh(X,Y)\ . \qquad (2.29)$$

On the other hand, for $Y = \xi$ in (2.5) and by using (2.2), (2.6) and (2.7), we see

$$T\,[X,\xi] = (\nabla_\xi T)X - T\nabla_\xi X - 2th(X,\xi) - A_{NX}\xi + X - \eta(X)\xi \qquad (2.30)$$

and

$$N\,[X,\xi] = (\nabla_\xi N)X - N\nabla_\xi X - 2nh(X,\xi) + h(TX,\xi). \qquad (2.31)$$

In contact geometry, A. Lotta introduced slant submanifolds as follows [2]:

Definition 2.1. Let M be a submanifold of an almost contact metric manifold $(\bar{M},\varphi,\xi,\eta,g)$. Then M is said to be a slant submanifold if the angle $\theta(X)$ between φX and $T_M(p)$ is constant at any point $p \in M$ for any X linearly independent of ξ. Thus the invariant and anti-invariant submanifolds are special class of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. If the slant angle θ is neither zero nor $\frac{\pi}{2}$, then slant submanifold is said to be proper slant submanifold.

If M is a slant submanifold of an almost contact metric manifold, then the tangent bundle TM of M can be decomposed as

$$TM = D_\theta \oplus \xi,$$

where ξ denotes the distribution spanned by the structure vector field ξ and D_θ is the complementary distribution of ξ in TM, known as the slant distribution on M.

For a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the normal bundle $T^{\perp}M$ of M is decomposed as

$$T^{\perp}M = N(TM) \oplus \mu$$

where μ is the invariant normal subbundle with respect to φ orthogonal to $N(TM)$.

In an almost contact metric manifold. In fact, J. L Cabrerizo obtained the following theorem[6].

Theorem 2.2. [6]. *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant submanifold if and only if there exists a constant $\lambda \in [0,1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi) \tag{2.32}$$

Furthermore, the slant angle θ of M satisfies $\lambda = cos^2\theta$.

Hence, for a slant submanifold M of an almost contact metric manifold $\widetilde{M}$, the following relations are consequences of the above theorem.

$$g(TX, TY) = \cos^2\theta \{g(X,Y) - \eta(X)\eta(Y)\} \tag{2.33}$$

and

$$g(NX, NY) = \sin^2\theta \{g(X,Y) - \eta(X)\eta(Y)\}. \tag{2.34}$$

for any $X, Y \in \Gamma(TM)$.

Lemma 2.3. [6]. *Let D_θ be a distribution on M, orthogonal to ξ. Then, D_θ is a slant if and only if there is a constant $\lambda \in [0,1]$ such that*

$$(TP_2)^2 X = -\lambda X \tag{2.35}$$

for all $X \in \Gamma(D_\theta)$, where P_2 denotes the orthogonal projection on D_θ. Furthermore, the slant angle θ of M satisfies $\lambda = \cos^2\theta$.

3. Pseudo-slant submanifolds of a nearly Sasakian manifold

In this section, we will obtain the integrability condition of the distributions of pseudo-slant submanifold of a nearly Sasakian manifold.

Definition 3.1. We say that M is a pseudo-slant submanifold of an almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, g)$ if there exists two orthogonal distributions D_θ and $D^\perp$ on M such that
i. TM admits the orthogonal direct decomposition $TM = D^\perp \oplus D_\theta$, $\xi \in \Gamma(D_\theta)$
ii. The distribution $D^\perp$ is anti-invariant(totally-real) i.e., $\varphi D^\perp \subset (T^\perp M)$,
iii. The distribution D_θ is a slant with slant angle $\theta \neq 0, \frac{\pi}{2}$, that is, the angle between D_θ and $\varphi(D_\theta)$ is a constant θ[16].

From the definition, it is clear that if $\theta = 0$, then the pseudo-slant submanifold is a semi-invariant submanifold. On the other hand, if $\theta = \frac{\pi}{2}$, submanifold becomes an anti- invariant.

We suppose that M is a pseudo-slant submanifold of an almost contact metric manifold $\widetilde{M}$ and we denote the dimensions of distributions $D^{\perp}$and D_θ by d_1 and d_2, respectively, then we have the following cases:
i. If $d_2 = 0$ then M is an anti-invariant submanifold,
ii. If $d_1 = 0$ and $\theta = 0$, then M is an invariant submanifold,
iii. If $d_1 = 0$ and $\theta \in (0, \frac{\pi}{2})$then M is a proper slant submanifold
with slant angle θ,
iv. If $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold,
v. If $d_1.d_2 \neq 0$ and $\theta = 0$, then $\widetilde{M}$ is a semi-invariant submanifold,
vi. If $d_1.d_2 \neq 0$ and $\theta \in (0, \frac{\pi}{2})$, then M is a proper pseudo-slant submanifold.

If we denote the projections on $D^{\perp}$and D_θ by P_1 and P_2, respectively, then for any vector field $X \in \Gamma(TM)$, we can write

$$X = P_1X + P_2X + \eta(X)\xi. \tag{3.1}$$

On the other hand, applying φ on both sides of equation (3.1), we have

$$\varphi X = \varphi P_1X + \varphi P_2X$$

and

$$TX + NX = NP_1X + TP_2X + NP_2X, \quad TP_1X = 0, \tag{3.2}$$

from which

$$TX = TP_2X, \quad NX = NP_1X + NP_2X$$

and

$$\varphi P_1X = NP_1X, \ TP_1X = 0, \quad \varphi P_2X = TP_2X + NP_2X \tag{3.3}$$
$$TP_2X \in \Gamma(D_\theta).$$

For a pseudo-slant submanifold M of a nearly Sasakian manifold $\widetilde{M}$, the normal bundle $T^{\perp}M$ of a pseudo-slant submanifold M is decomposable as

$$T^{\perp}M = \varphi(D^{\perp}) \oplus N(D_\theta) \oplus \mu \quad \varphi(D^{\perp}) \perp N(D_\theta) \tag{3.4}$$

where μ is an invariant subbundle of $T^{\perp}M$.

Now, we construct an example of a pseudo-slant submanifold in an almost contact metric manifold.

Example 3.2. Let M be a submanifold of $\mathbb{R}^7$ defined by

$$\chi(u, v, s, z, w) = (\sqrt{3}u, v, v\sin\theta, v\cos\theta, s\cos z, -s\cos z, w).$$

We can easily to see that the tangent bundle of M is spanned by the tangent vectors

$$e_1 = \sqrt{3}\frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial y_1} + \sin\theta\frac{\partial}{\partial x_2} + \cos\theta\frac{\partial}{\partial y_2}, \quad e_5 = \xi = \frac{\partial}{\partial w},$$
$$e_3 = \cos z\frac{\partial}{\partial x_3} - \cos z\frac{\partial}{\partial y_3}, \ e_4 = -s\sin z\frac{\partial}{\partial x_3} + s\sin z\frac{\partial}{\partial y_3}.$$

For the almost contact structure of φ of $\mathbb{R}^7$, choosing

$$\varphi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}, \quad \varphi(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}, \ \varphi(\frac{\partial}{\partial w}) = 0, \quad 1 \leq i, \ j \leq 3$$

and $\xi = \frac{\partial}{\partial w}$, $\eta = dw$. For any vector field $W = \mu_i \frac{\partial}{\partial x_i} + \nu_j \frac{\partial}{\partial y_j} + \lambda \frac{\partial}{\partial w} \in T(\mathbb{R}^7)$, then we have

$$\varphi Z = \mu_i \frac{\partial}{\partial y_j} - \nu_j \frac{\partial}{\partial x_i}, \quad g(\varphi Z, \varphi Z) = \mu_i^2 + \nu_j^2,$$

$$g(Z, Z) = \mu_i^2 + \nu_j^2 + \lambda^2, \quad \eta(Z) = g(Z, \xi) = \lambda$$

and

$$\varphi^2 Z = -\mu_i \frac{\partial}{\partial x_i} - \nu_j \frac{\partial}{\partial y_j} - \lambda \frac{\partial}{\partial w} + \lambda \frac{\partial}{\partial w} = -Z + \eta(Z)\xi$$

for any i, $j = 1, 2, 3$. It follows that $g(\varphi Z, \varphi Z) = g(Z, Z) - \eta^2(Z)$. Thus (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^7$. Thus we have

$$\varphi e_1 = \sqrt{3} \frac{\partial}{\partial y_1}, \quad \varphi e_2 = -\frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial y_2} - \cos\theta \frac{\partial}{\partial x_2}$$

$$\varphi e_3 = \cos z \frac{\partial}{\partial y_3} + \cos z \frac{\partial}{\partial x_3}, \quad \varphi e_4 = -s \sin z \frac{\partial}{\partial y_3} - s \sin z \frac{\partial}{\partial x_3}.$$

By direct calculations, we can infer $D_\theta = span\{e_1, e_2\}$ is a slant distribution with slant angle $\cos\theta = \frac{g(e_2, \varphi e_1)}{\|e_2\| \|\varphi e_1\|} = \frac{\sqrt{2}}{2}$, $\theta = 45°$. Since

$$g(\varphi e_3, e_1) = g(\varphi e_3, e_2) = g(\varphi e_3, e_4) = g(\varphi e_3, e_5) = 0,$$

$$g(\varphi e_4, e_1) = g(\varphi e_4, e_2) = g(\varphi e_4, e_3) = g(\varphi e_4, e_5) = 0.$$

Thus e_3 and e_4 are orthogonal to M, $D^\perp = span\{e_3, e_4\}$ is an anti-invariant distribution. Thus M is a 5-dimensional proper pseudo-slant submanifold of $\mathbb{R}^7$ with its usual almost contact metric structure.

Theorem 3.3. *Let M be a pseudo-slant of a nearly Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^\perp$ is integrable if and only if*

$$A_{NX}Y = A_{NY}X \tag{3.5}$$

for any $X, Y \in \Gamma\ (D^\perp)$.

Proof. By using (2.3), (2.6) and (2.8), we can write

$$\begin{aligned} 2g(A_{\varphi Y}X, Z) &= g(h(Z, X), \varphi Y) + g(h(Z, X), \varphi Y) \\ &= g(\widetilde{\nabla}_X Z, \varphi Y) + g(\widetilde{\nabla}_Z X, \varphi Y) \\ &= -g(\varphi \widetilde{\nabla}_X Z, Y) - g(\varphi \widetilde{\nabla}_Z X, Y) \\ &= -g(\widetilde{\nabla}_X \varphi Z, Y) - g(\widetilde{\nabla}_Z \varphi X, Y) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Z + (\widetilde{\nabla}_Z \varphi)X, Y) \end{aligned}$$

for any $X, Y \in \Gamma\ (D^{\perp})$ and $Z \in \Gamma(TM)$. By using (2.5), (2.7) and (2.16) we have

$$\begin{aligned}
2g(A_{NY}X, Z) &= -g(\widetilde{\nabla}_X \varphi Z, Y) - g(\widetilde{\nabla}_Z \varphi X, Y) \\
&\quad + g(2g(Z, X)\xi - \eta(Z)X - \eta(X)Z, Y) \\
&= g(\widetilde{\nabla}_X Y, \varphi Z) + g(A_{NX}Z, Y) - \eta(Z)g(X, Y) \\
&= g(\widetilde{\nabla}_X Y, \varphi Z) + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z) \\
&= g(\nabla_X Y, TZ) + g(h(X, Y), NZ) \\
&\quad + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z) \\
&= -g(T\nabla_X Y, Z) + g(-th(X, Y), Z) \\
&\quad + g(A_{NX}Y, Z) - g(X, Y)g(\xi, Z).
\end{aligned}$$

This implies that

$$2A_{NY}X = A_{NX}Y - T\nabla_X Y - th(X, Y) - g(X, Y)\xi \tag{3.6}$$

interchanging X by Y in (3.6) , we have

$$2A_{NX}Y = A_{NY}X - T\nabla_Y X - th(X, Y) - g(X, Y)\xi \tag{3.7}$$

then from (3.6) and (3.7), we can derive.

$$\begin{aligned}
2A_{NY}X - 2A_{NX}Y &= A_{NX}Y - A_{NY}X + T\nabla_Y X - T\nabla_X Y \\
&= A_{NX}Y - A_{NY}X + T(\nabla_Y X - \nabla_X Y)
\end{aligned}$$

here

$$3(A_{NY}X - A_{NX}Y) = T\,[X, Y]\,.$$

For $[X, Y] \in \Gamma(D^{\perp})$, $\varphi\,[X, Y] = N\,[X, Y]$. Since the tangent component of $\varphi\,[X, Y]$ is the zero, the anti- invariant distribution $D^{\perp}$ is integrability if and only if (3.5) is satisfied. □

Theorem 3.4. *Let M be a pseudo-slant of a nearly Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if*

$$A_{NX}Y + T\nabla_X Y + th(Y, X) + g(X, Y)\xi = 0 \tag{3.8}$$

for any $X, Y \in \Gamma\ (D^{\perp})$.

Proof. For any $X, Y \in \Gamma\ (D^{\perp})$, from (2.5), we have

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X = 2g(X, Y)\xi$$

which is equivalent to

$$\widetilde{\nabla}_X \varphi Y - \varphi\widetilde{\nabla}_X Y + \widetilde{\nabla}_Y \varphi X - \varphi\widetilde{\nabla}_Y X - 2g(X, Y)\xi = 0.$$

By using (2.6), (2.7), (2.11) and (2.12), we can write

$$\begin{aligned}
&-A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y - 2th(Y, X) - A_{NX}Y \\
&+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - 2nh(Y, X) - 2g(X, Y)\xi = 0.
\end{aligned}$$

Then from the tangent components of the last equation and (3.5), we conclude that

$$2A_{NX}Y + T\nabla_X Y + T\nabla_Y X + 2th(Y,X) + 2g(X,Y)\xi = 0,$$

which implies

$$T\,[X,Y] = 2A_{NX}Y + 2T\nabla_X Y + 2th(Y,X) + 2g(X,Y)\xi.$$

This proves our assertion. □

Theorem 3.5. *Let M be a pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the slant distribution D_θ is integrable if and only if*

$$-2(\nabla_Y N)X + \nabla_Y^{\perp} NX - \nabla_X^{\perp} NY + h(Y,TX) - h(X,TY) + 2nh(X,TY) \in \mu \oplus N(D_\theta)$$

for any $Y, X \in \Gamma(D_\theta)$.

Proof. For any $Y, X \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y X, Z) - g(\widetilde{\nabla}_X Y, Z) \\ &= g(\varphi\widetilde{\nabla}_Y X, \varphi Z) - g(\varphi\widetilde{\nabla}_X Y, \varphi Z). \end{aligned}$$

Since of $\eta(Z) = 0$ and $\varphi Z = NZ$ for any $Z \in \Gamma(D^{\perp})$, we obtain

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\ \widetilde{\nabla}_X \varphi Y, NZ) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Y - (\widetilde{\nabla}_Y \varphi)X, NZ) \\ &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\widetilde{\nabla}_X \varphi Y, NZ) \\ &\quad + g((\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ). \end{aligned}$$

By using (2.5) in this equation, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y \varphi X, NZ) - g(\widetilde{\nabla}_X \varphi Y, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ) \\ &\quad + g(2g(Y,X)\xi - \eta(Y)X - \eta(X)Y, NZ). \end{aligned}$$

Also using (2.11) in this equation, we have

$$\begin{aligned} g([Y,X]\,,Z) &= g(\widetilde{\nabla}_Y TX, NZ) + g(\widetilde{\nabla}_Y NX, NZ) - g(\widetilde{\nabla}_X TY, NZ) \\ &\quad - g(\widetilde{\nabla}_X NY, NZ) - 2g((\widetilde{\nabla}_Y \varphi)X, NZ). \end{aligned}$$

From the Gauss and Weingarten formulas, the above equation takes the form

$$\begin{aligned} g([Y,X]\,,Z) &= -2g((\widetilde{\nabla}_Y \varphi)X, NZ) + g(\nabla_Y^{\perp} NX, NZ) - g(\nabla_X^{\perp} NY, NZ) \\ &\quad + g(h(Y,TX), NZ) - g(h(X,TY), NZ). \end{aligned} \tag{3.9}$$

Substituting $2g((\widetilde{\nabla}_Y \phi)X, NZ)$ into the (3.9), we get

$$\begin{aligned} 2g((\widetilde{\nabla}_Y \varphi)X, NZ) &= 2g((\nabla_Y T)X - A_{NX}Y - th(Y,X), NZ) \\ &\quad + 2g((\nabla_Y N)X + h(Y,TX) - nh(X,TY), NZ) \\ &= 2g((\nabla_Y N)X + h(Y,TX) - nh(X,TY), NZ). \end{aligned} \tag{3.10}$$

Substituting (3.10) in the equation (3.9), we have

$$\begin{aligned} g([Y,X],Z) &= g(-2(\nabla_Y N)X - 2h(Y,TX) + 2nh(X,TY), NZ) \\ &\quad +g(\nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + h(Y,TX) - h(X,TY), NZ), \\ &= g(-2(\nabla_Y N)X + \nabla^{\perp}_Y NX - \nabla^{\perp}_W NY \\ &\quad +h(Y,TX) - h(X,TY) + 2nh(X,TY), NZ). \end{aligned}$$

Thus we conclude $[Y,X] \in \Gamma(D_\theta)$ if and only if

$$-2(\nabla_Y N)X + \nabla^{\perp}_Y NX - \nabla^{\perp}_X NY + h(Y,TX) - h(X,TY) + 2nh(X,TY) \in \mu \oplus N(D_\theta).$$

□

Theorem 3.6. *Let M be a pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the slant distribution D_θ is integrable if and only if*

$$\begin{aligned} P_1\{(\nabla_Y T)X + \nabla_X TY - T\nabla_Y X - A_{NX}Y - A_{NY}X \\ -2th(X,Y) + \eta(Y)X + \eta(X)Y\} = 0 \end{aligned} \tag{3.11}$$

for any $X, Y \in \Gamma(D_\theta)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and we denote the projections on $D^{\perp}$ and D_θ by P_1 and P_2, respectively, then for any vector fields $X, Y \in \Gamma(D_\theta)$, by using equation (2.5), we obtain

$$(\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \phi)X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y,$$

that is,

$$\widetilde{\nabla}_X \varphi Y - \varphi \widetilde{\nabla}_X Y + \widetilde{\nabla}_Y \varphi X - \varphi \widetilde{\nabla}_Y X = 2g(X,Y)\xi - \eta(Y)X - \eta(X)Y.$$

By using equations (2.6), (2.7), (2.11) and (2.12), we can write

$$\begin{aligned} &\nabla_X TY + h(X,TY) - A_{NY}X + \nabla^{\perp}_X NY - T\nabla_X Y - N\nabla_X Y \\ &-th(X,Y) - nh(X,Y) + \nabla_Y TX + h(Y,TX) - A_{NX}Y \\ &+\nabla^{\perp}_Y NX - T\nabla_Y X - N\nabla_Y X - th(X,Y) - nh(X,Y) \\ &-2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0 \end{aligned}$$

From the tangential components of the last equation, we conclude that

$$\begin{aligned} &\nabla_X TY - T\nabla_X Y + (\nabla_Y T)X - A_{NX}Y - A_{NY}X \\ &-2th(X,Y) - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y = 0 \end{aligned}$$

which implies that

$$\begin{aligned} T[X,Y] &= \nabla_X TY - T\nabla_Y X + (\nabla_Y T)X - A_{NX}Y - A_{NY}X \\ &\quad -2th(X,Y) - 2g(X,Y)\xi + \eta(Y)X + \eta(X)Y. \end{aligned} \tag{3.12}$$

Applying P_1 to (3.12), we get (3.11) □

Theorem 3.7. *Let M be a proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then D_θ is integrable if and only if*

$$2g(\nabla_X Y, Z) = \Big\{ g(A_{NZ}X, TY) + g(A_{NZ}Y, TX) + g(\nabla^\perp_X NY, NZ) + g(\nabla^\perp_Y NX, NZ) \Big\}$$

for any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^\perp)$, by using (2.3), we have

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, \varphi Z) + \eta(\widetilde{\nabla}_X Y)\eta(Z).$$

Since $\eta(Z) = 0$, we get

$$g(\nabla_X Y, Z) = g(\varphi\widetilde{\nabla}_X Y, NZ),$$

from which

$$g(\nabla_X Y, Z) = g(\widetilde{\nabla}_X \varphi Y, NZ) - g((\widetilde{\nabla}_X \varphi)Y, NZ).$$

From the Gauss and Weingarten formulas and structure equation (2.5), we get

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\widetilde{\nabla}_X TY, NZ) + g(\widetilde{\nabla}_X NY, NZ) - g((\widetilde{\nabla}_X \varphi)Y, NZ) \\ &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y \varphi)X, NZ) \\ &\quad -2g(X, Y)g(\xi, NZ) + \eta(X)g(Y, NZ) + \eta(Y)g(X, NZ) \\ &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y \varphi)X, NZ) \end{aligned} \tag{3.13}$$

Interchanging X by Y in (3.13), we have

$$g(\nabla_Y X, Z) = g(h(Y, TX), NZ) + g(\nabla^\perp_Y NX, NZ) + g((\widetilde{\nabla}_X \varphi)Y, NZ). \tag{3.14}$$

From (3.13) and (3.14), we can derive

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= g(h(X, TY), NZ) + g(\nabla^\perp_X NY, NZ) + g((\widetilde{\nabla}_Y \varphi)X, NZ) \\ &\quad + g(h(Y, TX), NZ) + g(\nabla^\perp_Y NX, NZ) + g((\widetilde{\nabla}_X \varphi)Y, NZ) \\ &= g(h(X, TY), NZ) + g(h(Y, TX), NZ) \\ &\quad + g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ) \\ &\quad + g((\widetilde{\nabla}_Y \varphi)X + (\widetilde{\nabla}_X \varphi)Y, NZ) \end{aligned}$$

By using (2.5), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) + g(\nabla_Y X, Z) &= g(h(X, TY), NZ) + +g(h(Y, TX), NZ) \\ &\quad + g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ) \end{aligned}$$

Using the property of Lie bracket, we derive

$$\begin{aligned} 2g(\nabla_X Y, Z) + g([Y, X], Z) &= g(h(X, TY), NZ) + g(h(Y, TX), NZ) \\ &\quad + g(\nabla^\perp_Y NX, NZ) + g(\nabla^\perp_X NY, NZ), \end{aligned}$$

which implies that

$$2g(\nabla_X Y, Z) = \Big\{ g(A_{NZ}X, TY) + g(A_{NZ}Y, TX) + g(\nabla^{\perp}_X NY, NZ) + g(\nabla^{\perp}_Y NX, NZ) \Big\}.$$

This proves our assertion.

Theorem 3.8. *Let M be a totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then the endomorphism T is parallel on M if and only if M is anti-invariant submanifold of $\widetilde{M}$.*

Proof. If T is parallel, then from (2.10) and (2.28), we have

$$A_{NX}X + th(X, X) + g(X, X)\xi - \eta(X)X = 0 \tag{3.15}$$

Interchanging X by TX in (3.15), we drive

$$A_{NTX}TX + th(TX, TX) + g(TX, TX)\xi = 0. \tag{3.16}$$

Taking the inner product of (3.16) by ξ, we get

$$\begin{aligned} 0 &= g(A_{NTX}TX + th(TX, TX) + g(TX, TX)\xi, \xi) \\ &= g(h(TX, \xi), NTX) + g(TX, TX) \\ &= g(g(TX, \xi)H, NTX) + g(TX, TX) \\ &= g(TX, TX) = \cos^2\theta\{g(X, X) - \eta(X)\eta(X)\} \end{aligned}$$

for any vector field X on M. This implies that M is an anti-invariant submanifold. If M is an anti-invariant submanifold, then it is obvious that $\nabla T = 0$. □

Theorem 3.9. *Let M be totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then M is a totally geodesic submanifold if H, $\nabla^{\perp}_X H \in \Gamma(\mu)$.*

Proof. For any $X, Y \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_X \varphi Y = (\widetilde{\nabla}_X \varphi)Y + \varphi\widetilde{\nabla}_X Y. \tag{3.17}$$

Making use of (2.6), (2.7), (2.10), (2.11) and (3.17) equation takes the form

$$\begin{aligned} \nabla_X TY + g(X, TY)H - A_{NY}X + \nabla^{\perp}_X NY = {} & (\widetilde{\nabla}_X \varphi)Y + T\nabla_X Y \\ & + N\nabla_X Y + g(X, Y)\varphi H. \end{aligned}$$

Taking the inner product with φH the last equation, we obtain

$$g(\nabla^{\perp}_X NY, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y, \varphi H) + g(N\nabla_X Y, \varphi H) + g(X, Y)\left\|H\right\|^2$$

by using (2.7), we get

$$g(\widetilde{\nabla}_X NY, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y, \varphi H) + g(X, Y)\left\|H\right\|^2. \tag{3.18}$$

In same way, we have

$$g(\widetilde{\nabla}_Y NX, \varphi H) = g((\widetilde{\nabla}_Y \varphi)X, \varphi H) + g(X, Y)\left\|H\right\|^2. \tag{3.19}$$

Then from (3.18) and (3.19), we can derive

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = g((\widetilde{\nabla}_X \varphi)Y + (\widetilde{\nabla}_Y \varphi)X, \varphi H) + 2g(X,Y)\|H\|^2 .$$

From (2.5), we obtain

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = g(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y, \varphi H) + 2g(X,Y)\|H\|^2 .$$

Hence

$$g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) = 2g(X,Y)\|H\|^2 . \tag{3.20}$$

Now, for any $X \in \Gamma(TM)$, we have

$$\widetilde{\nabla}_X \varphi H = (\widetilde{\nabla}_X \varphi)H + \varphi \widetilde{\nabla}_X H$$

by means of (2.6), (2.7), (2.11), (2.12), (2.24) and (2.25), we obtain

$$-A_{\varphi H}X + \nabla^{\perp}_X \varphi H = P_X H + F_X H - TA_H X - NA_H X + n\nabla^{\perp}_X H. \tag{3.21}$$

Taking the inner product with NY and taking into account that $n\nabla^{\perp}_X H \in \Gamma(\mu)$, we see that

$$g(\nabla^{\perp}_X \varphi H, NY) = g(F_X H, NY) - g(NA_H X, NY). \tag{3.22}$$

From (2.8), (2.10), (2.34) and (3.22), we obtain

$$\begin{aligned} g(\nabla^{\perp}_X \varphi H, NY) &= -\sin^2\theta \Big\{ g(X,Y)\|H\|^2 - \eta(A_H X)\eta(Y) \Big\} \\ &\quad + g(F_X H, NY). \end{aligned}$$

Since ∇ is metric connection, NY and φH are mutually orthogonal, by using (2.2), (2.7), (2.8) and (2.10), we get

$$\begin{aligned} g(\widetilde{\nabla}_X NY, \varphi H) &= \sin^2\theta \{ g(X,Y) - \eta(X)\eta(Y) \} \|H\|^2 \\ &\quad - g(F_X H, NY). \end{aligned} \tag{3.23}$$

Similarly, we have

$$\begin{aligned} g(\widetilde{\nabla}_Y NX, \varphi H) &= \sin^2\theta \{ g(X,Y) - \eta(X)\eta(Y) \} \|H\|^2 \\ &\quad - g(F_Y H, NX). \end{aligned} \tag{3.24}$$

From (3.23) and (3.24), we obtain

$$\begin{aligned} g(\widetilde{\nabla}_X NY + \widetilde{\nabla}_Y NX, \varphi H) &= 2\sin^2\theta \{ g(X,Y) - \eta(X)\eta(Y) \} \|H\|^2 \\ &\quad - g(F_X H, NY) - g(F_Y H, NX). \end{aligned} \tag{3.25}$$

Thus (3.20) and (3.25) imply

$$\begin{aligned} 2g(X,Y)\|H\|^2 &= 2\sin^2\theta \{ g(X,Y) - \eta(X)\eta(Y) \} \|H\|^2 \\ &\quad - g(F_X H, NY) - g(F_Y H, NX). \end{aligned}$$

Thus we have

$$\cos^2\theta g(X,Y)\|H\|^2 + \sin^2\theta \eta(X)\eta(Y)\|H\|^2 = -\frac{1}{2}\{ g(F_X H, NY) + g(F_Y H, NX) \} .$$

In view of (2.20) and (2.25) the fact that $H \in \Gamma(\mu)$, then the above equation takes the form

$$\cos^2\theta g(X,Y)\|H\|^2 + \sin^2\theta\eta(X)\eta(Y)\|H\|^2 = -\sin^2\theta g(X,Y)\|H\|^2 + \sin^2\theta\eta(X)\eta(Y)\|H\|^2. \quad (3.26)$$

From (3.26), we conclude that $g(X,Y)\|H\|^2 = 0$, $\forall X, Y \in \Gamma(TM)$. Since M is a proper-slant, we obtain $H = 0$. This tells us that M is totally geodesic in $\widetilde{M}$. □

□

Theorem 3.10. *Let M be a totally umbilical pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then at least one of the following statements is true;*
i. *$dim(D^{\perp}) = 1$,*
ii. *$H \in \Gamma(\mu)$,*
iii. *M is a proper pseudo-slant submanifold.*

Proof. For any $X \in \Gamma(D^{\perp})$ from (2.5), we have

$$(\widetilde{\nabla}_X\varphi)X = g(X,X)\xi,$$

or,

$$\widetilde{\nabla}_X NX - \varphi(\nabla_X X + h(X,X)) - \|X\|^2\xi = 0.$$

From the last equation, we have

$$-A_{NX}X + \nabla^{\perp}_X NX - N\nabla_X X - th(X,X) - nh(X,X) - \|X\|^2\xi = 0.$$

The tangential components of (3.27), we obtain

$$A_{NX}X + th(X,X) + \|X\|^2\xi = 0$$

Taking the inner product by $Y \in \Gamma(D^{\perp})$, we have

$$g(A_{NX}X + th(X,X), Y) = 0.$$

This implies that

$$g(h(X,Y), NX) + g(th(X,X), Y) = 0.$$

Since M is totally umbilical submanifold, we obtain

$$g(g(X,Y)H, NX) + g(g(X,X)tH, Y) = 0$$

or,

$$g(X,Y)g(H,NX) + g(X,X)g(tH,Y) = 0,$$

which implies that

$$g(tH,Y)X - g(tH,X)Y = 0.$$

Here, tH is either zero or X and Y are linearly dependent. If $tH \neq 0$, then the vectors X and Y are linearly dependent and $\dim(D^{\perp}) = 1$.

On the other hand, $tH = 0$, i.e., $H \in \Gamma(\mu)$. Since dim $(D_\theta) \neq 0$, M is a pseudo-slant submanifold. Since $\theta \neq 0$ and $d_1.d_2 \neq 0$, M is a proper pseudo-slant submanifold. □

Theorem 3.11. *Let M be totally umbilical proper pseudo-slant submanifold of a nearly Sasakian manifold $\widetilde{M}$. Then following conditions are equivalent*
i. *$H \in \Gamma(\mu)$,*
ii. *$\varphi^2 X = -\nabla_{TX}\xi$,*
iii. *M is an anti-invariant submanifold,*
for any $X \in \Gamma(TM)$.

Proof. For any $X \in \Gamma(TM)$, from (2.5), we have

$$(\widetilde{\nabla}_X \varphi)X = g(X,X)\xi - \eta(X)X.$$

By means of (2.6), (2.7), (2.11) and (2.12), we obtain

$$\begin{aligned} 0 &= \widetilde{\nabla}_X TX + \widetilde{\nabla}_X NX - \varphi(\nabla_X X + h(X,X)) - g(X,X)\xi + \eta(X)X \\ &= \nabla_X TX + h(X,TX) - A_{NX}X + \nabla^{\perp}_X NX - T\nabla_X X - N\nabla_X X \\ &\quad -th(X,X) - nh(X,X) - g(X,X)\xi + \eta(X)X. \end{aligned} \tag{3.27}$$

The tangential components of (3.27), we obtain

$$\nabla_X TX - T\nabla_X X - th(X,X) - A_{NX}X - g(X,X)\xi + \eta(X)X = 0.$$

Since M is a totally umbilical submanifold, we can derive $A_{NX}X = g(H,NX)X$, then we have

$$\begin{aligned} &\nabla_X TX - T\nabla_X X - g(X,X)tH - g(H,NX)X \\ &\qquad -g(X,X)\xi + \eta(X)X = 0. \end{aligned} \tag{3.28}$$

If $H \in \Gamma(\mu)$, then from (3.28), we conclude that

$$\nabla_X TX - T\nabla_X X - g(X,X)\xi + \eta(X)X = 0. \tag{3.29}$$

Taking the inner product of (3.29) by ξ, we get

$$g(\nabla_X TX, \xi) = g(X,X) - \eta^2(X). \tag{3.30}$$

Interchanging X by TX in (3.30) and making use of (2.33), we derive

$$g(\nabla_{TX} T^2 X, \xi) = g(TX,TX),$$

or,

$$g(\nabla_{TX}\xi, T^2X) = -\cos^2\theta g(\phi X, \varphi X),$$

$$g(\nabla_{TX}\xi, -\cos^2\theta(X - \eta(X)\xi)) = -\cos^2\theta g(\varphi X, \varphi X),$$

that is,

$$\cos^2\theta\{g(\varphi X, \varphi X) - g(\nabla_{TX}\xi, (X - \eta(X)\xi)\} = 0.$$

Since M is a proper pseudo-slant submanifold, we have

$$g(\varphi X, \varphi X) - g(\nabla_{TX}\xi, (X - \eta(X)\xi)) = 0,$$

that is,

$$g(\varphi X, \varphi X) - g(\nabla_{TX}\xi, X) + \eta(X) g(\nabla_{TX}\xi, \xi) = 0. \tag{3.31}$$

Taking the covariant derivative of above equation with respect to TX for any $X \in \Gamma(TM)$, we obtain $g(\nabla_{TX}\xi, \xi) + g(\xi, \nabla_{TX}\xi) = 0$, which implies $g(\nabla_{TX}\xi, \xi) = 0$ and then (3.31) becomes

$$g(X, X) - \eta^2(X) - g(\nabla_{TX}\xi, X) = 0. \tag{3.32}$$

This proves **ii**. of the Theorem. So if (3.32) is satisfied, then (3.28) implies $H \in \Gamma(\mu)$.

Now, interchanging X by TX in (3.32), we derive

$$g(TX, TX) - g(\nabla_{T^2X}\xi, TX) = 0,$$

that is,

$$\cos^2\theta g(\varphi X, \varphi X) + \cos^2\theta g(\nabla_{(X-\eta(X)\xi)}\xi, TX) = 0,$$

from which

$$\cos^2\theta g(\varphi X, \varphi X) + \cos^2\theta g(\nabla_X\xi, TX) - \cos^2\theta \eta(X) g(\nabla_\xi\xi, TX) = 0.$$

Since $\nabla_\xi\xi = 0$, we obtain

$$\cos^2\theta \{g(\varphi X, \varphi X) + g(\nabla_X\xi, TX)\} = 0. \tag{3.33}$$

We note

$$g(\varphi X, \varphi X) + g(\nabla_X\xi, TX) \neq 0$$

from (3.33, we can derive if $\cos\theta = 0$, then M is an anti-invariant submanifold. $\square$

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