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CONVOLUTION PROPERTIES FOR SALAGEAN-TYPE ANALYTIC FUNCTIONS DEFINED BY $q-$DIFFERENCE OPERATOR

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Abstract. In this paper, we define Salagean-type analytic functions by using concept of $q-$derivative operator. We investigate convolution properties and coefficient estimates for Salagean-type analytic functions denoted by $\mathcal{S}_q^{m,\lambda}[A,B]$.

1. Introduction

Let $\mathcal{A}$ be the class of functions f defined by

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1}$$

that are analytic in the open unit disc $U=\{z:|z|<1\}$ and Ω be the family of functions w which are analytic in U and satisfy the conditions $w(0)=0$, $|w(z)|<1$ for all $z\in U$. If f_1 and f_2 are analytic functions in U, then we say that f_1 is subordinate to f_2 written as $f_1\prec f_2$ if there exists a Schwarz function $w\in\Omega$ such that $f_1(z)=f_2(w(z)), z\in U$. We also note that if f_2 univalent in U, then $f_1\prec f_2$ if and only if $f_1(0)=f_2(0)$, $f_1(U)\subset f_2(U)$ (see [5]).

Let $f_1(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $f_2(z)=z+\sum_{n=2}^{\infty}b_nz^n$ be elements in $\mathcal{A}$. Then the Hadamard product or convolution of these functions is defined by

$$f_1(z)*f_2(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

Next, for arbitrary fixed numbers A,B, $-1\leq B<A\leq 1$, denote by $\mathcal{P}[A,B]$ the family of functions $p(z)=1+p_1z+p_2z^2+\cdots$, analytic in U such that $p\in\mathcal{P}[A,B]$

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if and only if

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

for some functions $w \in \Omega$ and every $z \in U$. This class was introduced by Janowski [8].

In 1909 and 1910 Jackson [6, 7] initiated a study of $q-$difference operator D_q defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad \text{for} \quad z \in B \backslash \{0\}, \tag{2}$$

where B is a subset of complex plane $\mathbb{C}$, called $q-$geometric set if $qz \in B$, whenever $z \in B$. Obviously, $D_q f(z) \to f'(z)$ as $q \to 1^-$. The $q-$difference operator (2) is also called Jackson $q-$difference operator. Note that such an operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [1, 3, 4, 9]).

Since

$$D_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$, it follows that for any $f \in \mathcal{A}$, we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1},$$

where $q \in (0, 1)$. Clearly, as $q \to 1^-$, $[n]_q \to n$. For notations, one may refer to [4].

The Salagean differential operator R^m was introduced by Salagean [10] in 1998. Since then, many mathematicians used the idea of Salagean differential operator in their papers (see [2]). $q-$Salagean differential operator is defined as below:

Definition 1. *The $q-$analogue of Salagean differential operator $R_q^m f(z) : \mathcal{A} \to \mathcal{A}$ is formed by*

$$\begin{aligned} R_q^0 f(z) &= f(z) \\ R_q^1 f(z) &= z D_q(f(z)) \\ &\vdots \\ R_q^m f(z) &= z D_q^1(R_q^{m-1} f(z)). \end{aligned}$$

From definition $R_q^m f(z)$, we obtain

$$R_q^m f(z) = z + \sum_{n=2}^{\infty} [n]_q^m a_n z^n, \tag{3}$$

where $[n]_q^m = (\frac{1-q^n}{1-q})^m$, $q \in (0, 1)$, $m \in \mathbb{N}$. Clearly, as $q \to 1^-$, the equation (3) reduces to Salagean differential operator.

Motivated by $q-$Salagean differential operator, we define the class of Salagean-type analytic functions denoted by $\mathcal{S}_q^{m,\lambda}[A,B]$.

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_q^{m,\lambda}[A,B]$ such that*

$$1+\frac{e^{i\lambda}}{\cos\lambda}\left(\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-1\right)\prec\frac{1+Az}{1+Bz},$$

where $q\in(0,1), |\lambda|<\frac{\pi}{2}, m\in\mathbb{N}, z\in U$.

Also, we note that $\mathcal{C}_q^{m,\lambda}[A,B]$ is the class of functions $f\in\mathcal{A}$ satisfying $zD_qf\in\mathcal{S}_q^{m,\lambda}[A,B]$.

In this paper, we investigate the necessary and sufficient convolution conditions and coefficient estimates for the class $\mathcal{S}_q^{m,\lambda}[A,B]$ associated with the $q-$derivative operator.

2. Main Results

We first begin with necessary and sufficient convolution conditions of our class $\mathcal{S}_q^{m,\lambda}[A,B]$.

Theorem 3. *The function f defined by (1) is in the class $\mathcal{S}_q^{m,\lambda}[A,B]$ if and only if*

$$\frac{1}{z}\left[R_q^m f(z) * \frac{z-Lqz^2}{(1-z)(1-qz)}\right]\neq 0 \tag{4}$$

for all $L=\frac{e^{-i\theta}+(A-B)\cos\lambda e^{-i\lambda}+B}{(A-B)\cos\lambda e^{-i\lambda}}$, where $\theta\in[0,2\pi]$, $q\in(0,1)$, $|\lambda|<\frac{\pi}{2}$ and also $L=1$.

Proof. First suppose $f\in\mathcal{S}_q^{m,\lambda}[A,B]$, then we have

$$1+\frac{e^{i\lambda}}{\cos\lambda}\left(\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-1\right)\prec\frac{1+Az}{1+Bz}, \tag{5}$$

therefore we get

$$\frac{R_q^{m+1}f(z)}{R_q^m f(z)}\prec\frac{1+((A-B)\cos\lambda e^{-i\lambda}+B)z}{1+Bz}. \tag{6}$$

Since the function from the left-hand side of the subordination is analytic in U, it follows $f(z)\neq 0, z\in U^*=U\backslash\{0\}$; that is, $\frac{1}{z}f(z)\neq 0$ and this is equivalent to the fact that (4) holds for $L=1$. From (6) according to the subordination of two analytic functions, we say that there exists a function w analytic in U with $w(0)=0, |w(z)|<1$ such that

$$\frac{R_q^{m+1}f(z)}{R_q^m f(z)}=\frac{1+((A-B)\cos\lambda e^{-i\lambda}+B)w(z)}{1+Bw(z)}, \tag{7}$$

which is equivalent to

$$\frac{R_q^{m+1}f(z)}{R_q^m f(z)} \neq \frac{1+((A-B)\cos\lambda e^{-i\lambda}+B)e^{i\theta}}{1+Be^{i\theta}} \tag{8}$$

or

$$\frac{1}{z}\left[(1+Be^{i\theta})R_q^{m+1}f(z)-(1+((A-B)\cos\lambda e^{-i\lambda}+B)e^{i\theta})R_q^m f(z)\right]\neq 0. \tag{9}$$

Since

$$R_q^m f(z) * \frac{z}{1-z} = R_q^m f(z),$$
$$R_q^m f(z) * \frac{z}{(1-z)(1-qz)} = R_q^{m+1} f(z),$$

we may write (9) as

$$\frac{1}{z}\left[R_q^m f(z) * \left(\frac{(1+Be^{i\theta})z}{(1-z)(1-qz)} - \frac{(1+((A-B)\cos\lambda e^{-i\lambda}+B)e^{i\theta})z}{(1-z)}\right)\right]\neq 0.$$

Therefore we obtain

$$\frac{((B-A)\cos\lambda e^{-i\lambda})e^{i\theta}}{z}\left[R_q^m f(z) * \frac{z-\frac{e^{-i\theta}+(A-B)\cos\lambda e^{-i\lambda}+B}{(A-B)\cos\lambda e^{-i\lambda}}qz^2}{(1-z)(1-qz)}\right]\neq 0, \tag{10}$$

which leads to (4) and the necessary part of Theorem 3.

Conversely, because assumption (4) holds for $L=1$, it follows that $\frac{1}{z}f(z)\neq 0$ for all $z\in U$; hence, the function $\varphi(z)=1+\frac{e^{i\lambda}}{\cos\lambda}(\frac{R_q^{m+1}f(z)}{R_q^m f(z)}-1)$ is analytic in U. Since it was shown in the first part of the proof that assumption (4) is equivalent to (8), we obtain that

$$\frac{R_q^{m+1}f(z)}{R_q^m f(z)} \neq \frac{1+((A-B)\cos\lambda e^{-i\lambda}+B)e^{i\theta}}{1+Be^{i\theta}} \tag{11}$$

and if we denote

$$\psi(z)=\frac{1+((A-B)\cos\lambda e^{-i\lambda}+B)z}{1+Bz}, \tag{12}$$

relation (11) shows that $\varphi(U)\cap\psi(U)=\emptyset$. Thus, the simply connected domain $\varphi(U)$ is included in a connected component of $C\backslash\psi(\partial U)$. From here, using the fact that $\varphi(0)=\psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, which represents in fact subordination (6); that is, $f\in\mathcal{S}_q^{m,\lambda}[A,B]$. This completes the proof of Theorem 3. □

Taking $q\to 1^-$ in Theorem 3, we obtain the following result.

Corollary 4. *The function f defined by (1) is in the class $\mathcal{S}^{m,\lambda}[A,B]$ if and only if*

$$\frac{1}{z}\left[R^m f(z) * \frac{z-Lz^2}{(1-z)^2}\right]\neq 0 \tag{13}$$

for all $L = \frac{e^{-i\theta}+(A-B)\cos\lambda e^{-i\lambda}+B}{(A-B)\cos\lambda e^{-i\lambda}}$, *where* $\theta \in [0, 2\pi]$, $|\lambda| < \frac{\pi}{2}$ *and also* $L = 1$.

Theorem 5. *A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{S}_q^{m,\lambda}[A,B]$ is that*

$$1 - \sum_{n=2}^{\infty} [n]_q^m \frac{[n]_q(e^{-i\theta}+B) - e^{-i\theta} + (B-A)\cos\lambda e^{-i\lambda} - B}{(A-B)\cos\lambda e^{-i\lambda}} a_n z^{n-1} \neq 0. \tag{14}$$

Proof. From Theorem 3, $f \in \mathcal{S}_q^{m,\lambda}[A,B]$ if and only if

$$\frac{1}{z}\left[R_q^m f(z) * \frac{z - Lqz^2}{(1-z)(1-qz)}\right] \neq 0 \tag{15}$$

for all $L = \frac{e^{-i\theta}+(A-B)\cos\lambda e^{-i\lambda}+B}{(A-B)\cos\lambda e^{-i\lambda}}$ and also $L = 1$. The left-hand side of (15) can be written as

$$\begin{aligned}
&\frac{1}{z}\left[R_q^m f(z) * \left(\frac{z}{(1-z)(1-qz)} - \frac{Lqz^2}{(1-z)(1-qz)}\right)\right] \\
&= \frac{1}{z}\{R_q^{m+1} f(z) - L[R_q^{m+1} f(z) - R_q^m f(z)]\} \\
&= 1 - \sum_{n=2}^{\infty} [n]_q^m ([n]_q(L-1) - L) a_n z^{n-1} \\
&= 1 - \sum_{n=2}^{\infty} [n]_q^m \frac{[n]_q(e^{-i\theta}+B) - e^{-i\theta} + (B-A)\cos\lambda e^{-i\lambda} - B}{(A-B)\cos\lambda e^{-i\lambda}} a_n z^{n-1}.
\end{aligned}$$

Thus, the proof is completed. □

Taking $q \to 1^-$ in Theorem 5, we get the following result.

Corollary 6. *A necessary and sufficient condition for the function f defined by (1) is in the class $\mathcal{S}^{m,\lambda}[A,B]$ is that*

$$1 - \sum_{n=2}^{\infty} n^m \frac{n(e^{-i\theta}+B) - e^{-i\theta} + (B-A)\cos\lambda e^{-i\lambda} - B}{(A-B)\cos\lambda e^{-i\lambda}} a_n z^{n-1} \neq 0. \tag{16}$$

We next determine coefficient estimate for a function of form (1) to be in the class $\mathcal{S}_q^{m,\lambda}[A,B]$.

Theorem 7. *If the function f defined by (1) satisfies the following inequality*

$$\sum_{n=2}^{\infty} [n]_q^m \big\{[n]_q(1-B) - 1 + (A-B)\cos\lambda + B\big\}|a_n| \leq (A-B)\cos\lambda, \tag{17}$$

then $f \in \mathcal{S}_q^{m,\lambda}[A,B]$.

Proof. From Theorem 5, we write

$$\begin{aligned}
&\left|1-\sum_{n=2}^{\infty}[n]_q^m\frac{[n]_q(e^{-i\theta}+B)-e^{-i\theta}+(B-A)\cos\lambda e^{-i\lambda}-B}{(A-B)\cos\lambda e^{-i\lambda}}a_nz^{n-1}\right|\\
&>1-\sum_{n=2}^{\infty}\left|[n]_q^m\frac{[n]_q(e^{-i\theta}+B)-e^{-i\theta}+(B-A)\cos\lambda e^{-i\lambda}-B}{(A-B)\cos\lambda e^{-i\lambda}}\right||a_n|\\
&\geq 1-\sum_{n=2}^{\infty}[n]_q^m\frac{[n]_q(1-B)-1+|(A-B)\cos\lambda e^{-i\lambda}|+B}{|(A-B)\cos\lambda e^{-i\lambda}|}|a_n|\\
&=1-\sum_{n=2}^{\infty}[n]_q^m\frac{[n]_q(1-B)-1+(A-B)\cos\lambda+B}{(A-B)\cos\lambda}|a_n|>0,
\end{aligned}$$

then $f\in\mathcal{S}_q^{m,\lambda}[A,B]$. □

Corollary 8. *Taking $q\to 1^-$ in Theorem 7, we obtain*

$$\sum_{n=2}^{\infty}n^m\big\{n(1-B)-1+(A-B)\cos\lambda+B\big\}|a_n|\leq(A-B)\cos\lambda, \tag{18}$$

then $f\in\mathcal{S}^{m,\lambda}[A,B]$.

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