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THE CUBIC EIGENPARAMETER DEPENDENT DISCRETE DIRAC EQUATIONS WITH PRINCIPAL FUNCTIONS

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Abstract. Let us consider the Boundary Value Problem (BVP) for the discrete Dirac Equations

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases}, \quad n \in \mathbb{N}, \tag{0.1}$$

$$(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3)y_1^{(2)} + (\beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \beta_3\lambda^3)y_0^{(1)} = 0, \tag{0.2}$$

where (a_n), (b_n), (p_n) and (q_n), $n \in \mathbb{N}$ are complex sequences, γ_i, $\beta_i \in \mathbb{C}$, $i = 0, 1, 2$ and λ is a eigenparameter. Discussing the eigenvalues and the spectral singularities, we prove that the BVP (0.1), (0.2) has a finite number of eigenvalues and spectral singularities with a finite multiplicities, if

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta}) \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|\right) < \infty,$$

holds, for some $\varepsilon > 0$ and $\frac{1}{2} \leq \delta \leq 1$.

1. Introduction

Difference equations are well suited to be solved with the computers since they become easily to an algorithmic form and they help to solve differential equations approximately with making discretizations. Also they arise as mathematical models of many practical problems arising in engineering, biology, economics and control theory. On the other hand, studies related on them lead to the rapid development of the theory of discrete difference equations. In the last decade, discrete boundary value problems have been intensively studied and the spectral analysis of the difference equations have been treated by various authors in connection with the classical moment problem ($[1-5]$). Moreover the spectral theory of the difference equations have been applied to the solution of classes of nonlinear discrete Korteveg-de Vriez equations and Toda lattices ($[6, 7]$).

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Let the discrete boundary value problem (BVP)

$$\begin{cases} y_{n+1}^{(2)} - y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ -y_n^{(1)} + y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases} \tag{1.1}$$

$$y_0^{(1)} = 0, \tag{1.2}$$

is considered where (p_n) and (q_n) are complex sequences for $n = 1, 2, ...$ and λ is a spectral parameter. The spectral analysis of the BVP (1.1)-(1.2) with spectrum and principal functions has been investigated in [8]. Moreover the authors in [8] found the integral representation for the Weyl function and the spectral expansion of (1.1)-(1.2) in terms of the principal functions. Some problems related to the spectral analysis of difference equations with spectral singularities have been studied in $[9-14]$. The spectral analysis of eigenparameter dependent non-selfadjoint BVP for the system of difference equations of first order have been studied in $[15-18]$.

Let us consider the discrete Dirac equations with cubic eigenparameter dependent boundary conditions such as

$$\begin{cases} a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ \\ a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \qquad , \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

$$(\gamma_0 + \gamma_1 \lambda + \gamma_2 \lambda^2 + \gamma_3 \lambda^3) y_1^{(2)} + (\beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \beta_3 \lambda^3) y_0^{(1)} = 0 \quad , \tag{1.4}$$

where $\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}$, $n \in \mathbb{N}$ are vector sequences, $a_n \neq 0$, $b_n \neq 0$ for all n. Also $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ and $(\beta_0, \beta_1, \beta_2, \beta_3)$ are linearly independent with $|\gamma_3| + |\beta_3| \neq 0$ and $\gamma_3 \neq \frac{\beta_2}{a_0}$ where γ_i, $\beta_i \in \mathbb{C}$, $i = 0, 1, 2$. If $a_n \equiv 1$ and $b_n \equiv -1$ for all $n \in \mathbb{N}$, then the system (1.3) reduces to

$$\begin{cases} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \quad , n \in \mathbb{N} \end{cases} \tag{1.5}$$

where Δ is a forward difference operator. The system (1.5) is the discrete analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

([19], Chap. 2). Therefore the system (1.5) (also (1.3)) is called the discrete Dirac system. In this article, we intend to investigate of spectrum and principal functions of the BVP (1.3)-(1.4) under the condition

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta}) \left(|1 - a_n| + |1 + b_n| + |p_n| + |q_n| \right) < \infty,$$

for some $\varepsilon > 0$ and $\frac{1}{2} \le \delta \le 1$.

2. Jost solution of (1.3)

Suppose that the condition

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta}) \left(|1-a_n| + |1+b_n| + |p_n| + |q_n|\right) < \infty \tag{2.1}$$

is satisfied for some $\varepsilon > 0$ and $\frac{1}{2} \le \delta \le 1$. It is well-known that [14], eq. (1.3) has the bounded solution

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n \left(I_2 + \sum_{m=1}^{\infty} A_{nm} e^{imz} \right) \begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix} e^{inz} \quad , \; n \in \mathbb{N}, \tag{2.2}$$

$$f_0^{(1)}(z) = \alpha_0^{11} \left\{ e^{i\frac{z}{2}} \left[1 + \sum_{m=1}^{\infty} A_{0m}^{11} e^{imz} \right] - i \sum_{m=1}^{\infty} A_{0m}^{12} e^{imz} \right\}, \tag{2.3}$$

under the condition (2.1) for $\lambda = 2\sin\frac{z}{2}$ and $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \; \operatorname{Im} z \ge 0\}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix} \quad , \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}$$

Note that α_n^{ij} and A_{nm}^{ij} $(i, j = 1, 2)$ are expressed in terms of (a_n), (b_n), (p_n) and (q_n), $n \in \mathbb{N}$. Also

$$\left|A_{nm}^{ij}\right| \le C \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} \left(|1-a_k| + |1+b_k| + |p_k| + |q_k|\right) \tag{2.4}$$

holds, where $C > 0$ is a constant and $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$. Therefore f_n is vector-valued analytic function with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \; \operatorname{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$ ([14]). The solution $f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix}$ is called Jost solution of (1.3).

Let $\widehat{\varphi}_n(\lambda) = \begin{pmatrix} \widehat{\varphi}_n^{(1)}(\lambda) \\ \widehat{\varphi}_n^{(2)}(\lambda) \end{pmatrix}$, $n \in \mathbb{N} \cup \{0\}$ be the another solution of (1.3) subject to the initial conditions

$$\widehat{\varphi}_0^{(1)}(\lambda) = -(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3) \quad , \quad \widehat{\varphi}_1^{(2)}(\lambda) = (\beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \beta_3\lambda^3).$$

If we characterize

$$\varphi_n(z) = \left\{ \widehat{\varphi}_n(2\sin\frac{z}{2}) \right\}, \; n \in \mathbb{N} \cup \{0\},$$

then φ_n is an entire function and is 4π periodic.

Let us take the semi-strips $T_0 := \{z : z \in \mathbb{C},\ z = x + iy,\ 0 \leq x \leq 4\pi,\ y > 0\}$ and $T := T_0 \cup [0, 4\pi]$. Then the Wronskian of the solutions $f_n(z)$ and $\varphi_n(z)$ is given by

$$\begin{aligned} W\left[f_n(z), \varphi_n(z)\right] &= a_n \left[\widehat{\varphi}_{n+1}^{(2)}(2\sin\frac{z}{2}) f_n^{(1)}(z) - f_{n+1}^{(2)}(z)\widehat{\varphi}_n^{(1)}(2\sin\frac{z}{2})\right] \\ &= a_0 \left[\widehat{\varphi}_1^{(2)}(2\sin\frac{z}{2}) f_0^{(1)}(z) - f_1^{(2)}(z)\widehat{\varphi}_0^{(1)}(2\sin\frac{z}{2})\right]. \end{aligned}$$

If we define

$$f(z) = \widehat{\varphi}_1^{(2)}(2\sin\frac{z}{2}) f_0^{(1)}(z) - f_1^{(2)}(z)\widehat{\varphi}_0^{(1)}(2\sin\frac{z}{2}),$$

then f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$ and $f(z) = f(z + 4\pi)$. When $f(z) \neq 0$ for all $z \in S$, $f_n(z)$ and $\varphi_n(z)$ are linearly independent. Here

$$\widehat{f}(z) = W\left[f_n(z), \varphi_n(z)\right] = a_0 f(z) \tag{2.5}$$

is called Jost function of the BVP (1.3)-(1.4). Moreover, if we define $g_n = (g_n^{(1)}, g_n^{(2)})$ then,

$$\begin{aligned} R_\lambda(L) g_n := -\frac{1}{\widehat{f}(z)} &\left\{ \sum_{k=1}^{n} (g_{k-1}^{(1)}, g_k^{(2)}) \begin{pmatrix} \frac{a_{k-1}}{a_k}\varphi_{k-1}^{(1)} \\ \varphi_k^{(2)} \end{pmatrix} \begin{pmatrix} f_n^{(1)} \\ f_n^{(2)} \end{pmatrix} \right. \\ &\left. + \sum_{k=n+1}^{\infty} (g_{k-1}^{(1)}, g_k^{(2)}) \begin{pmatrix} \frac{a_{k-1}}{a_k} f_{k-1}^{(1)} \\ f_k^{(2)} \end{pmatrix} \begin{pmatrix} \varphi_n^{(1)} \\ \varphi_n^{(2)} \end{pmatrix} \right\} \end{aligned}$$

is the resolvent of the BVP (1.3)-(1.4).

3. Eigenvalues and spectral singularities of (1.3)-(1.4)

From (2.5), we clearly obtain that the function

$$\begin{aligned} \widehat{f}(z) \quad = \quad & a_0 \left[f_1^{(2)}(z)(\gamma_0 + \gamma_1\lambda + \gamma_2\lambda^2 + \gamma_3\lambda^3) \right. \\ & \left. + f_0^{(1)}(z)(\beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \beta_3\lambda^3) \right] \end{aligned} \tag{3.1}$$

is analytic in $\mathbb{C}_+$, continuous up to the real axis and is 4π periodic. Also if we denote the set of all eigenvalues and spectral singularities of the BVP (1.3)-(1.4) by σ_d and σ_{ss} respectively, then it is clear that

$$\sigma_d = \left\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in T_0,\ \widehat{f}(z) = 0 \right\}, \tag{3.2}$$

$$\sigma_{ss} = \left\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in [0, 4\pi],\ \widehat{f}(z) = 0 \right\}. \tag{3.3}$$

From (2.2), (2.3) and (3.1) we obtain

$$\begin{aligned} \widehat{f}(z) = a_0 &\left\{ -i\alpha_0^{11}\beta_3 e^{-iz} - (\beta_2\alpha_0^{11} + \alpha_1^{22}\gamma_3) e^{-i\frac{z}{2}} \right. \\ &+ i\left[(\beta_1 + 3\beta_3)\alpha_0^{11} - \gamma_3\alpha_1^{21} + \gamma_2\alpha_1^{22} \right] \\ &- \left[-(\beta_0 + 2\beta_2)\alpha_0^{11} + \gamma_2\alpha_1^{21} - (\gamma_1 + 3\gamma_3)\alpha_1^{22} \right] e^{i\frac{z}{2}} \\ &+ i\left[-(\beta_1 + 3\beta_3)\alpha_0^{11} + (\gamma_1 + 3\gamma_3)\alpha_1^{21} - (\gamma_0 + 2\gamma_2)\alpha_1^{22} \right] e^{iz} \end{aligned}$$

$$
\begin{aligned}
&-\left[\beta_2\alpha_0^{11}-(\gamma_0+2\gamma_2)\,\alpha_1^{21}+(\gamma_1+3\gamma_3)\,\alpha_1^{22}\right]e^{i\frac{3z}{2}}\\
&+i\left[\beta_3\alpha_0^{11}-(\gamma_1+3\gamma_3)\,\alpha_1^{21}+\gamma_2\alpha_1^{22}\right]e^{i2z}\\
&-\left[\gamma_2\alpha_1^{21}-\gamma_3\alpha_1^{22}\right]e^{i\frac{5z}{2}}+i\gamma_3\alpha_1^{21}e^{i3z}-\sum_{m=1}^{\infty}\beta_3A_{0m}^{12}\alpha_0^{11}e^{i\left(m-\frac{3}{2}\right)z}\\
&+i\sum_{m=1}^{\infty}\left(\beta_2A_{0m}^{12}-\beta_3A_{0m}^{11}\right)\alpha_0^{11}e^{i(m-1)z}\\
&-\sum_{m=1}^{\infty}\left\{\left[\beta_2A_{0m}^{11}-(\beta_1+3\beta_3)\,A_{0m}^{12}\right]\alpha_0^{11}+\gamma_3A_{1m}^{12}\alpha_1^{21}+\gamma_3A_{1m}^{22}\alpha_1^{22}\right\}e^{i\left(m-\frac{1}{2}\right)z}\\
&+i\sum_{m=1}^{\infty}\left\{\left[(\beta_1+3\beta_3)A_{0m}^{11}-(\beta_0+2\beta_2)\,A_{0m}^{12}\right]\alpha_0^{11}\right.\\
&\qquad\left.+\left(\gamma_2A_{1m}^{12}-\gamma_3A_{1m}^{11}\right)\alpha_1^{21}+\left(\gamma_2A_{1m}^{22}-\gamma_3A_{1m}^{21}\right)\alpha_1^{22}\right\}e^{imz}\\
&-\sum_{m=1}^{\infty}\left\{\left[(\beta_1+3\beta_3)\,A_{0m}^{12}-(\beta_0+2\beta_2)\,A_{0m}^{11}\right]\alpha_0^{11}+\left[\gamma_2A_{1m}^{11}-(\gamma_1+3\gamma_3)A_{1m}^{12}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[\gamma_2A_{1m}^{21}-(\gamma_1+3\gamma_3)A_{1m}^{22}\right]\alpha_1^{22}\right\}e^{i\left(m+\frac{1}{2}\right)z}\\
&+i\sum_{m=1}^{\infty}\left\{\left[\beta_2A_{0m}^{12}-(\beta_1+3\beta_3)\,A_{0m}^{11}\right]\alpha_0^{11}+\left[(\gamma_1+3\gamma_3)A_{1m}^{11}-(\gamma_0+2\gamma_2)\,A_{1m}^{12}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[(\gamma_1+3\gamma_3)A_{1m}^{21}-(\gamma_0+2\gamma_2)A_{1m}^{22}\right]\alpha_1^{22}\right\}e^{i(m+1)z}\\
&-\sum_{m=1}^{\infty}\left\{\left(\beta_2A_{0m}^{11}-\beta_3A_{0m}^{12}\right)\alpha_0^{11}+\left[(\gamma_1+3\gamma_3)A_{1m}^{12}-(\gamma_0+2\gamma_2)\,A_{1m}^{11}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[(\gamma_1+3\gamma_3)A_{1m}^{22}-(\gamma_0+2\gamma_2)A_{1m}^{21}\right]\alpha_1^{22}\right\}e^{i\left(m+\frac{3}{2}\right)z}\\
&+i\sum_{m=1}^{\infty}\left\{\beta_3A_{0m}^{11}\alpha_0^{11}+\left[\gamma_2A_{1m}^{12}-(\gamma_1+3\gamma_3)\,A_{1m}^{11}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[\gamma_2A_{1m}^{22}-(\gamma_1+3\gamma_3)A_{1m}^{21}\right]\alpha_1^{22}\right\}e^{i(m+2)z}\\
&-\sum_{m=1}^{\infty}\left[\left(\gamma_2A_{1m}^{11}-\gamma_3A_{1m}^{12}\right)\alpha_1^{21}+\left(\gamma_2A_{1m}^{21}-\gamma_3A_{1m}^{22}\right)\alpha_1^{22}\right]e^{i\left(m+\frac{5}{2}\right)z}\\
&+i\sum_{m=1}^{\infty}\left(\gamma_3A_{1m}^{11}\alpha_1^{21}+\gamma_3A_{1m}^{21}\alpha_1^{22}\right)e^{i(m+3)z}. \qquad (3.4)
\end{aligned}
$$

Let

$$F(z):=\widehat{f}(z)e^{iz}, \qquad (3.5)$$

then, the function F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$,

$F(z)=a_0\left\{-i\alpha_0^{11}\beta_3-(\beta_2\alpha_0^{11}+\alpha_1^{22}\gamma_3)e^{i\frac{z}{2}}\right.$

$$
\begin{aligned}
&+ i\left[(\beta_1+3\beta_3)\,\alpha_0^{11}-\gamma_3\alpha_1^{21}+\gamma_2\alpha_1^{22}\right]e^{iz}\\
&-\left[-(\beta_0+2\beta_2)\,\alpha_0^{11}+\gamma_2\alpha_1^{21}-(\gamma_1+3\gamma_3)\,\alpha_1^{22}\right]e^{i\frac{3z}{2}}\\
&+ i\left[-(\beta_1+3\beta_3)\,\alpha_0^{11}+(\gamma_1+3\gamma_3)\,\alpha_1^{21}-(\gamma_0+2\gamma_2)\,\alpha_1^{22}\right]e^{i2z}\\
&-\left[\beta_2\alpha_0^{11}-(\gamma_0+2\gamma_2)\,\alpha_1^{21}+(\gamma_1+3\gamma_3)\,\alpha_1^{22}\right]e^{i\frac{5z}{2}}\\
&+ i\left[\beta_3\alpha_0^{11}-(\gamma_1+3\gamma_3)\,\alpha_1^{21}+\gamma_2\alpha_1^{22}\right]e^{i3z}\\
&-\left[\gamma_2\alpha_1^{21}-\gamma_3\alpha_1^{22}\right]e^{i\frac{7z}{2}}+i\gamma_3\alpha_1^{21}e^{i4z}-\sum_{m=1}^{\infty}\beta_3A_{0m}^{12}\alpha_0^{11}e^{i\left(m-\frac{1}{2}\right)z}\\
&+ i\sum_{m=1}^{\infty}\left(\beta_2A_{0m}^{12}-\beta_3A_{0m}^{11}\right)\alpha_0^{11}e^{imz}\\
&-\sum_{m=1}^{\infty}\left\{\left[\beta_2A_{0m}^{11}-(\beta_1+3\beta_3)\,A_{0m}^{12}\right]\alpha_0^{11}+\gamma_3A_{1m}^{12}\alpha_1^{21}+\gamma_3A_{1m}^{22}\alpha_1^{22}\right\}e^{i\left(m+\frac{1}{2}\right)z}\\
&+ i\sum_{m=1}^{\infty}\left\{\left[(\beta_1+3\beta_3)A_{0m}^{11}-(\beta_0+2\beta_2)\,A_{0m}^{12}\right]\alpha_0^{11}\right.\\
&\qquad\left.+\left(\gamma_2A_{1m}^{12}-\gamma_3A_{1m}^{11}\right)\alpha_1^{21}+\left(\gamma_2A_{1m}^{22}-\gamma_3A_{1m}^{21}\right)\alpha_1^{22}\right\}e^{i(m+1)z}\\
&-\sum_{m=1}^{\infty}\left\{\left[(\beta_1+3\beta_3)\,A_{0m}^{12}-(\beta_0+2\beta_2)\,A_{0m}^{11}\right]\alpha_0^{11}+\left[\gamma_2A_{1m}^{11}-(\gamma_1+3\gamma_3)A_{1m}^{12}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[\gamma_2A_{1m}^{21}-(\gamma_1+3\gamma_3)A_{1m}^{22}\right]\alpha_1^{22}\right\}e^{i\left(m+\frac{3}{2}\right)z}\\
&+ i\sum_{m=1}^{\infty}\left\{\left[\beta_2A_{0m}^{12}-(\beta_1+3\beta_3)\,A_{0m}^{11}\right]\alpha_0^{11}+\left[(\gamma_1+3\gamma_3)A_{1m}^{11}-(\gamma_0+2\gamma_2)\,A_{1m}^{12}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[(\gamma_1+3\gamma_3)A_{1m}^{21}-(\gamma_0+2\gamma_2)A_{1m}^{22}\right]\alpha_1^{22}\right\}e^{i(m+2)z}\\
&-\sum_{m=1}^{\infty}\left\{\left(\beta_2A_{0m}^{11}-\beta_3A_{0m}^{12}\right)\alpha_0^{11}+\left[(\gamma_1+3\gamma_3)A_{1m}^{12}-(\gamma_0+2\gamma_2)\,A_{1m}^{11}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[(\gamma_1+3\gamma_3)A_{1m}^{22}-(\gamma_0+2\gamma_2)A_{1m}^{21}\right]\alpha_1^{22}\right\}e^{i\left(m+\frac{5}{2}\right)z}\\
&+ i\sum_{m=1}^{\infty}\left\{\beta_3A_{0m}^{11}\alpha_0^{11}+\left[\gamma_2A_{1m}^{12}-(\gamma_1+3\gamma_3)\,A_{1m}^{11}\right]\alpha_1^{21}\right.\\
&\qquad\left.+\left[\gamma_2A_{1m}^{22}-(\gamma_1+3\gamma_3)A_{1m}^{21}\right]\alpha_1^{22}\right\}e^{i(m+3)z}\\
&-\sum_{m=1}^{\infty}\left[\left(\gamma_2A_{1m}^{11}-\gamma_3A_{1m}^{12}\right)\alpha_1^{21}+\left(\gamma_2A_{1m}^{21}-\gamma_3A_{1m}^{22}\right)\alpha_1^{22}\right]e^{i\left(m+\frac{7}{2}\right)z}\\
&+ i\sum_{m=1}^{\infty}\left(\gamma_3A_{1m}^{11}\alpha_1^{21}+\gamma_3A_{1m}^{21}\alpha_1^{22}\right)e^{i(m+4)z} \qquad (3.6)
\end{aligned}
$$

and

$$F(z+4\pi) = F(z).$$

Using (3.2)-(3.5),

$$\sigma_d = \left\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in T_0,\ F(z) = 0 \right\}, \tag{3.7}$$

$$\sigma_{ss} = \left\{ \lambda : \lambda = 2\sin\frac{z}{2},\ z \in [0, 4\pi],\ F(z) = 0 \right\}. \tag{3.8}$$

Definition 3.1. *The multiplicity of a zero of F in T is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.3)-(1.4).*

It follows from (3.2) and (3.3) that, in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of the BVP (1.3)-(1.4), we need to discuss the quantitative properties of the zeros of F in T.

Let

$$\begin{aligned} M_1 &:= \{ z : z \in T_0,\ F(z) = 0 \}, \\ M_2 &:= \{ z : z \in [0, 4\pi],\ F(z) = 0 \}. \end{aligned} \tag{3.9}$$

We also denote the set of all limit points of M_1 by M_3 and the set of all zeros of F with infinite multiplicity by M_4.

From (3.2), (3.3) and (3.9) we get that

$$\begin{aligned} \sigma_d &= \left\{ \lambda : \lambda = 2\sin\tfrac{z}{2},\ z \in M_1 \right\}, \\ \sigma_{ss} &= \left\{ \lambda : \lambda = 2\sin\tfrac{z}{2},\ z \in M_2 \right\}. \end{aligned} \tag{3.10}$$

Theorem 3.1. *If (2.1) holds, then*

(i) The set M_1 is bounded and countable.

(ii) $M_1 \cap M_3 = \varnothing$, $M_1 \cap M_4 = \varnothing$.

(iii) The set M_2 is compact and $\mu(M_2) = 0$, where μ denotes the Lebesque measure in the real axis.

(iv) $M_3 \subset M_2$, $M_4 \subset M_2$; $\mu(M_3) = \mu(M_4) = 0$.

(v) $M_3 \subset M_4$.

Proof. Using (1.4), (2.4) and (3.6), we have

$$F(z) = \begin{cases} -i\alpha_0^{11}\beta_3 + o(e^{-y}) & ,\ \beta_3 \neq 0,\ z \in T,\ y \to \infty \\ -(\beta_2\alpha_0^{11} + \alpha_1^{22}\gamma_3)e^{i\frac{z}{2}} + o(e^{-y}) & ,\ \beta_3 = 0,\ z \in T,\ y \to \infty. \end{cases} \tag{3.11}$$

Eq. (3.11) shows that M_1 is bounded. Since F is analytic in $\mathbb{C}_+$ and is a 4π periodic function we get that M_1 has at most a countable number of elements. This proves (i).

From the uniqueness theorems of analytic functions we obtain (ii)-(iv) [20].

Using the continuity of all derivatives of F on $[0, 4\pi]$ we get (v). □

From (3.10) and Theorem 3.1, we have the following.

Theorem 3.2. *Under the condition (2.1)*

(i) the set of eigenvalues of the BVP (1.3)-(1.4) is bounded and countable and its limit points can lie only in [-2,2].

(ii) $\sigma_{ss} \subset [-2,2]$, $\sigma_{ss} = \overline{\sigma}_{ss}$ *and* $\mu(\sigma_{ss}) = 0$.

For $\delta = 1$ condition (2.1) reduces to

$$\sum_{n=1}^{\infty} \exp(\varepsilon n) \left(|1-a_n| + |1+b_n| + |p_n| + |q_n|\right) < \infty. \tag{3.12}$$

Theorem 3.3. *Under the condition (3.12) the BVP (1.3)-(1.4) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Proof. Using (2.4) we find that

$$\left|A_{nm}^{ij}\right| \le C \exp[-\frac{\varepsilon}{5}(n+m)] \ , \ i,j = 1,2 \ , \ n,m \in \mathbb{N}, \tag{3.13}$$

where $C > 0$ is a constant. From (3.6) and (3.13) we observe that the function F has an analytic continuation to the half-plane $\operatorname{Im} z > -\frac{\varepsilon}{5}$. Since F is a 4π periodic function, the limit points its zeros in T cannot lie in $[0, 4\pi]$. Using Theorem 3.1 we have the bounded sets M_1 and M_2 have a finite number of elements. From analyticity of F in $\operatorname{Im} z > -\frac{\varepsilon}{5}$, we get that all zeros of F in T a finite multiplicity. Therefore using (3.10), we obtain the finiteness of eigenvalues and spectral singularities of the BVP (1.3)-(1.4). □

It is seen that the condition (3.12) guaranties of the analytic continuation of F from the real axis to lower half-plane. So the finiteness of eigenvalues and spectral singularities of the BVP (1.3)-(1.4) are obtained as a result of this analytic continuation.

Now let us suppose that

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta}) \left(|1-a_n| + |1+b_n| + |p_n| + |q_n|\right) < \infty \ , \quad \varepsilon > 0, \quad \frac{1}{2} \le \delta < 1 \tag{3.14}$$

which is weaker than (3.12). It is evident that under the condition (3.14) the function F is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But F does not have an analytic continuation from the real axis to lower half-plane. Therefore under the condition (3.14) the finiteness of eigenvalues and spectral singularities of the BVP (1.3)-(1.4) cannot be shown in a way similar to Theorem 3.3.

Under the condition (3.14), to prove that the eigenvalues and the spectral singularities of the BVP (1.3)-(1.4) are of finite number we will use the following.

Theorem 3.4. ([**8**]) *Let us assume that the* 4π *periodic function* g *is analytic in* $\mathbb{C}_+$, *all of its derivatives are continuous in* $\overline{\mathbb{C}}_+$ *and*

$$\sup_{z \in T} \left| g^{(k)}(z) \right| \le A_k \quad , \quad k \in \mathbb{N} \cup \{0\} \, .$$

If the set $G \subset [0, 4\pi]$ with Lebesque measure zero is the set of all zeros the function g with infinite multiplicity in T, if

$$\int_0^{\omega} \ln K(s) d\mu(G_s) = -\infty, \tag{3.15}$$

where $K(s) = \inf_k \frac{A_k s^k}{k!}$ and $\mu(G_s)$ is the Lebesque measure of s-neighborhood of G and $\omega \in (0, 4\pi)$ is an arbitrary constant, then $g \equiv 0$ in $\overline{\mathbb{C}}_+$.

Under the condition (3.14) from (2.4) and (3.6) we find

$$\left| F^{(k)}(z) \right| \leq A_k \quad , \quad k \in \mathbb{N} \cup \{0\}$$

where

$$A_k = 5^k C \sum_{m=1}^{\infty} m^k \exp(-\frac{\varepsilon}{5} m^{\delta})$$

and $C > 0$ is a constant. We can obtain the following estimate,

$$A_k \leq 5^k C \int_0^{\infty} x^k \exp(-\frac{\varepsilon}{5} x^{\delta}) dx \leq D d^k k! k^{k\frac{1-\delta}{\delta}}, \tag{3.16}$$

where D and d are constants depending C, ε and δ.

Theorem 3.5. *If (3.14) holds, then $M_4 = \varnothing$.*

Proof. The function F satisfies all conditions of Theorem 3.4 except (3.15). But F is not identically equal to zero. In this case the function F satisfies the condition

$$\int_0^{\omega} \ln K(s) d\mu(M_{4,s}) > -\infty \tag{3.17}$$

instead of (3.15), where $K(s) = \inf_k \frac{A_k s^k}{k!}$, $k \in \mathbb{N} \cup \{0\}$ and $\mu(M_{4,s})$ is the Lebesque measure of s-neighborhood of M_4 and A_k is defined by (3.16). Substituting (3.16) in the definition of $K(s)$, we get

$$K(s) = D \exp\left\{ -\frac{1-\delta}{\delta} e^{-1} d^{-\frac{\delta}{1-\delta}} s^{-\frac{\delta}{1-\delta}} \right\}. \tag{3.18}$$

It follows from (3.17) and (3.18) that

$$\int_0^{\omega} s^{-\frac{\delta}{1-\delta}} d\mu(M_{4,s}) < \infty. \tag{3.19}$$

Since $\frac{\delta}{1-\delta} \geq 1$, consequently (3.19) holds for arbitrary s if and only if $\mu(M_{4,s}) = 0$ or $M_4 = \varnothing$. $\square$

Theorem 3.6. *Under the condition (3.14) the BVP (1.3)-(1.4) has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the function F has a finite number of zeros with finite multiplicities in T.

From Theorem 3.1 and Theorem 3.5 we get that $M_3 = \varnothing$. So the bounded sets M_1 and M_2 have no limit points, i.e., the function F has only a finite number of zeros in T. Since $M_4 = \varnothing$, these zeros are of finite multiplicity. □

4. Principal functions of (1.3)-(1.4)

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ and $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_\nu$ denote the zeros of F in T_0 and $[0, 4\pi]$ with multiplicities $m_1, m_2, \ldots, m_k$ and $m_{k+1}, m_{k+2}, \ldots, m_\nu$, respectively.

Let us define $\ell := \begin{pmatrix} \ell^{(1)} \\ \ell^{(2)} \end{pmatrix}$ where

$$\left(\ell^{(1)} y\right)_n = a_{n+1} y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)}, \quad n \in \mathbb{N}$$

and

$$\left(\ell^{(2)} y\right)_n = a_{n-1} y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)}, \quad n \in \mathbb{N}.$$

Definition 4.1. *Let $\lambda = \lambda_0$ be an eigenvalue of the BVP (1.3)-(1.4). If the vectors y_n, $\frac{d}{d\lambda} y_n$, $\frac{d^2}{d\lambda^2} y_n, \ldots,$ $\frac{d^\nu}{d\lambda^\nu} y_n$;*

$$\frac{d^j}{d\lambda^j} y := \left\{ \frac{d^j}{d\lambda^j} y_n \right\}_{n \in \mathbb{N}}, \ j = 0, 1, \ldots, \nu \ ; \ n \in \mathbb{N}$$

satisfy the conditions

$$(\ell y)_n - \lambda_0 y_n = 0,$$

$$\left(\ell \left(\frac{d^j}{d\lambda^j} y\right)\right)_n - \lambda_0 \frac{d^j}{d\lambda^j} y_n - \frac{d^{j-1}}{d\lambda^{j-1}} y_n = 0, \quad j = 1, 2, \ldots, \nu \quad ; \quad n \in \mathbb{N}$$

then the vector y_n is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of the BVP (1.3)-(1.4). The vectors $\frac{d}{d\lambda} y_n$, $\frac{d^2}{d\lambda^2} y_n, \ldots,$ $\frac{d^\nu}{d\lambda^\nu} y_n$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$. The principal vectors of the spectral singularities of the BVP (1.3)-(1.4) are defined analogously.

We define the vectors

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) = \begin{pmatrix} \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda = \lambda_i} \\ \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda = \lambda_i} \end{pmatrix}, \ n \in \mathbb{N}$$
$$j = 0, 1, \ldots, m_i - 1 \ ; \ i = 1, 2, \ldots, k, k+1, \ldots, \nu \tag{4.1}$$

where $\lambda = 2\sin\frac{z}{2}$ and

$$E_n(\lambda) = \begin{pmatrix} E_n^{(1)}(\lambda) \\ E_n^{(2)}(\lambda) \end{pmatrix} := f_n(2\arcsin\lambda/2) = \begin{pmatrix} f_n^{(1)}(2\arcsin\lambda/2) \\ f_n^{(2)}(2\arcsin\lambda/2) \end{pmatrix}. \tag{4.2}$$

If

$$y(\lambda) = \{y_n(\lambda)\} := \begin{pmatrix} y_n^{(1)}(\lambda) \\ y_n^{(2)}(\lambda) \end{pmatrix}_{n\in\mathbb{N}}$$

is a solution of (1.3), then

$$\frac{d^j}{d\lambda^j} y(\lambda) = \left\{ \left(\frac{d^j}{d\lambda^j} \right) y_n(\lambda) \right\}_{n\in\mathbb{N}} := \left\{ \begin{array}{c} \left(\frac{d^j}{d\lambda^j}\right) y_n^{(1)}(\lambda) \\ \left(\frac{d^j}{d\lambda^j}\right) y_n^{(2)}(\lambda) \end{array} \right\}$$

satisfies

$$\begin{pmatrix} a_{n+1}\frac{d^j}{d\lambda^j} y_{n+1}^{(2)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + p_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) \\ a_{n-1}\frac{d^j}{d\lambda^j} y_{n-1}^{(1)}(\lambda) + b_n \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + q_n \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda \frac{d^j}{d\lambda^j} y_n^{(1)}(\lambda) + j\frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(1)}(\lambda) \\ \lambda \frac{d^j}{d\lambda^j} y_n^{(2)}(\lambda) + j\frac{d^{j-1}}{d\lambda^{j-1}} y_n^{(2)}(\lambda) \end{pmatrix}. \tag{4.3}$$

From (4.1)-(4.3) we get that

$$\begin{gathered} (\ell V(\lambda_i))_n - \lambda_0 V_n(\lambda_i) = 0, \\ \left(\ell \left(\frac{d^j}{d\lambda^j} V(\lambda_i) \right) \right)_n - \lambda_0 \frac{d^j}{d\lambda^j} V_n(\lambda_i) - \frac{d^{j-1}}{d\lambda^{j-1}} V_n(\lambda_i) = 0, \quad n \in \mathbb{N} \\ j = 1, 2, \ldots, m_i - 1 \; ; \; i = 1, 2, \ldots, \nu. \end{gathered}$$

The vectors $\frac{d^j}{d\lambda^j} V_n(\lambda_i)$ for $j = 0, 1, 2, \ldots, m_i - 1$; $i = 1, 2, \ldots, k$ and $\frac{d^j}{d\lambda^j} V_n(\lambda_i)$ for $j = 0, 1, 2, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$ are the principal vectors of eigenvalues and spectral singularities of the BVP (1.3)-(1.4), respectively.

Theorem 4.1.

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) \in \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1 \; ; \; i = 1, 2, \ldots, k$$

and

$$\frac{d^j}{d\lambda^j} V_n(\lambda_i) \notin \ell_2\left(\mathbb{N}, \mathbb{C}^2\right), \quad j = 0, 1, 2, \ldots, m_i - 1 \; ; \; i = k+1, k+2, \ldots, \nu.$$

Proof. Using (4.2) we get that

$$\left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} = \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z) \right\}_{z=z_i}, \quad n \in \mathbb{N}$$

and

$$\left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} = \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z) \right\}_{z=z_i}, \quad n \in \mathbb{N}$$

where $\lambda_i = 2\sin z_i/2,\ z_i \in T$ for $i = 1, 2, \ldots, k$ and $C_t,\ D_t$ are constants depending on λ. From (2.2) we obtain that

$$\begin{aligned} \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z) \right\}_{z=z_i} &= \alpha_n^{11} i^t (n+1/2)^t e^{iz_i(n+1/2)} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{11} \Big\{ A_{nm}^{11} i^t (m+n+1/2)^t e^{i(m+n+1/2)z_i} \\ &- A_{nm}^{12} i^{t+1} (m+n)^t e^{i(m+n)z_i} \Big\} \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z) \right\}_{z=z_i} &= \alpha_n^{21} i^t (n+1/2)^t e^{iz_i(n+1/2)} - i(in)^t \alpha_n^{22} e^{inz_i} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{21} \Big\{ A_{nm}^{11} i^t (m+n+1/2)^t e^{i(m+n+1/2)z_i} \\ &- A_{nm}^{12} i^{t+1} (m+n)^t e^{i(m+n)z_i} \Big\} \\ &+ \sum_{m=1}^{\infty} \alpha_n^{22} \Big\{ A_{nm}^{21} i^t (m+n+1/2)^t e^{i(m+n+1/2)z_i} \\ &- A_{nm}^{22} i^{t+1} (m+n)^t e^{i(m+n)z_i} \Big\}. \end{aligned} \tag{4.5}$$

For the principal vectors $\frac{d^j}{d\lambda^j} V_n(\lambda_i) = \left\{ \frac{d^j}{d\lambda^j} V_n(\lambda_i) \right\}_{n\in\mathbb{N}}$ for $j = 0, 1, \ldots, m_i - 1$; $i = 1, 2, \ldots, k$ corresponding to the eigenvalues of the BVP (1.3)-(1.4) we get

$$\frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} = \frac{1}{j!} \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\}$$

$$j = 0, 1, \ldots, m_i - 1\ ; i = 1, 2, \ldots, k \tag{4.6}$$

and

$$\frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} = \frac{1}{j!} \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\}$$

$$j=0,1,\ldots,m_i-1\ ;i=1,2,\ldots,k. \tag{4.7}$$

Since $\operatorname{Im}\lambda_i>0$ for $i=1,2,\ldots,k$ from (4.6) and (4.7) we obtain that

$$\begin{aligned}\left\|\frac{d^j}{d\lambda^j}V_n\right\|^2 &= \sum_{n=1}^{\infty}\left(\left|\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\right\}_{\lambda=\lambda_i}\right|^2\right.\\ &\quad\left.+\left|\frac{1}{j!}\left\{\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\right\}_{\lambda=\lambda_i}\right|^2\right)\\ &\le\left(\frac{1}{j!}\right)^2\left[\sum_{n=1}^{\infty}\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\right.\\ &\quad\left.\times\left(\left|\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\}\right|+\left|\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\}\right|\right)\right]^2\end{aligned} \tag{4.8}$$

or

$$\begin{aligned}\left\|\frac{d^j}{d\lambda^j}V_n\right\|^2 \le\left(\frac{1}{j!}\right)^2&\left\{\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\left\{\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\right.\right.\right.\\ &\times\left.\left(|n+1/2|^t e^{-(n+1/2)\operatorname{Im}z_i}\right)+\left|\alpha_n^{22}\right||n|^t e^{-n\operatorname{Im}z_i}\right\}\\ &+\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}\left\{\sum_{m=1}^{\infty}\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\right.\\ &\times\left(\left|A_{nm}^{11}\right||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im}z_i}\right)\\ &\left.+\left|A_{nm}^{12}\right||m+n|^t e^{-(m+n)\operatorname{Im}z_i}\right\}\\ +\sum_{t=0}^{j}\max\{|C_t|,|D_t|\}&\left\{\sum_{m=1}^{\infty}\left|\alpha_n^{22}\right|\left(\left|A_{nm}^{21}\right||m+n+1/2|^t e^{-(m+n+1/2)\operatorname{Im}z_i}\right.\right.\\ &\left.\left.\left.\left.+\left|A_{nm}^{22}\right||m+n|^t e^{-(m+n)\operatorname{Im}z_i}\right)\right\}\right]\right\}^2.\end{aligned} \tag{4.9}$$

From (4.9), if we say

$$\begin{aligned}Y=\frac{1}{j!}\sum_{n=1}^{\infty}\sum_{t=0}^{j}&\max\{|C_t|,|D_t|\}\left\{\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\right.\\ &\times\left.\left(|n+1/2|^t e^{-(n+1/2)\operatorname{Im}z_i}\right)+\left|\alpha_n^{22}\right||n|^t e^{-n\operatorname{Im}z_i}\right\}\end{aligned}$$

then

$$Y \leq \frac{A\,(j+1)}{j!} \sum_{n=1}^{\infty} \left[(n+1/2)^j \, e^{-(n+1/2)\,\mathrm{Im}\, z_i} + n^j e^{-n\,\mathrm{Im}\, z_i} \right] < \infty \tag{4.10}$$

holds where

$$A = \max\left\{ |C_t|\,, |D_t| \right\} \max\left\{ \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right), \left|\alpha_n^{22}\right| \right\}.$$

Now we define the function

$$\begin{aligned} g_n\,(z) = {} & \sum_{t=0}^{j} \max\left\{ |C_t|\,, |D_t| \right\} \left\{ \sum_{m=1}^{\infty} \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \right. \\ & \times \left(\left|A_{nm}^{11}\right| |m+n+1/2|^t \, e^{-(m+n+1/2)\,\mathrm{Im}\, z_i} \right. \\ & \left. \left. + \left|A_{nm}^{12}\right| |m+n|^t \, e^{-(m+n)\,\mathrm{Im}\, z_i} \right) \right\} \\ & + \sum_{t=0}^{j} \max\left\{ |C_t|\,, |D_t| \right\} \left\{ \sum_{m=1}^{\infty} \left|\alpha_n^{22}\right| \left(\left|A_{nm}^{21}\right| |m+n+1/2|^t \, e^{-(m+n+1/2)\,\mathrm{Im}\, z_i} \right. \right. \\ & \left. \left. + \left|A_{nm}^{22}\right| |m+n|^t \, e^{-(m+n)\,\mathrm{Im}\, z_i} \right) \right\}. \end{aligned} \tag{4.11}$$

So we get,

$$\begin{aligned} & \frac{1}{j!} \sum_{n=1}^{\infty} \left[\sum_{t=0}^{j} \max\left\{ |C_t|\,, |D_t| \right\} \left\{ \sum_{m=1}^{\infty} \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \right. \right. \\ & \times \left. \left(\left|A_{nm}^{11}\right| |m+n+1/2|^t \, e^{-(m+n+1/2)\,\mathrm{Im}\, z_i} + \left|A_{nm}^{12}\right| |m+n|^t \, e^{-(m+n)\,\mathrm{Im}\, z_i} \right) \right\} \\ & + \sum_{t=0}^{j} \max\left\{ |C_t|\,, |D_t| \right\} \left\{ \sum_{m=1}^{\infty} \left|\alpha_n^{22}\right| \left(\left|A_{nm}^{21}\right| |m+n+1/2|^t \, e^{-(m+n+1/2)\,\mathrm{Im}\, z_i} \right. \right. \\ & \left. \left. \left. + \left|A_{nm}^{22}\right| |m+n|^t \, e^{-(m+n)\,\mathrm{Im}\, z_i} \right) \right\} \right] \\ & = \frac{1}{j!} \sum_{n=1}^{\infty} g_n\,(z)\,. \end{aligned}$$

Using the boundedness of A_{nm}^{ij} and α_n^{ij} for $i, j = 1, 2$, we obtain that

$$\begin{aligned} g_n\,(z) \leq {} & \max\left\{ |C_t|\,, |D_t| \right\} M \sum_{t=0}^{j} \sum_{m=1}^{\infty} \left\{ |m+n+1/2|^t \, e^{-(m+n+1/2)\,\mathrm{Im}\, z_i} \right. \\ & \left. + |m+n|^t \, e^{-(m+n)\,\mathrm{Im}\, z_i} \right\} \end{aligned}$$

where

$$M = \max\left\{ \left(\left|\alpha_n^{11}\right| + \left|\alpha_n^{21}\right| \right) \left|A_{nm}^{11}\right|, \left|\alpha_n^{22}\right| \left|A_{nm}^{21}\right|, \right.$$

$$\left(\left|\alpha_n^{11}\right|+\left|\alpha_n^{21}\right|\right)\left|A_{nm}^{12}\right|,\left|\alpha_n^{22}\right|\left|A_{nm}^{22}\right|\right\}.$$

If we take $\max\{|C_t|,|D_t|\}\,M=N$, we can write

$$\begin{aligned}
g_n(z) &\le N\sum_{t=0}^{j}e^{-n\operatorname{Im}z_i}\sum_{m=1}^{\infty}\left\{(m+n+1/2)^t e^{-m\operatorname{Im}z_i}+(m+n)^t e^{-m\operatorname{Im}z_i}\right\}\\
&= Ne^{-n\operatorname{Im}z_i}\left\{\sum_{m=1}^{\infty}2e^{-m\operatorname{Im}z_i}+\sum_{m=1}^{\infty}e^{-m\operatorname{Im}z_i}\left((m+n+1/2)+(m+n)\right)\right.\\
&\left.+\ldots+\sum_{m=1}^{\infty}e^{-m\operatorname{Im}z_i}\left((m+n+1/2)^j+(m+n)^j\right)\right\}\\
&\le Ne^{-n\operatorname{Im}z_i}\sum_{m=1}^{\infty}\sum_{t=0}^{j}e^{-m\operatorname{Im}z_i}\left((m+n+1/2)^t+(m+n)^t\right)\\
&\le Be^{-n\operatorname{Im}z_i}
\end{aligned}$$

where

$$B=N\sum_{t=0}^{j}e^{-m\operatorname{Im}z_i}\left((m+n+1/2)^t+(m+n)^t\right).$$

Therefore, we have,

$$\left(\frac{1}{j!}\sum_{n=1}^{\infty}g_n(z)\right)^2\le\left(\frac{1}{j!}\sum_{n=1}^{\infty}Be^{-n\operatorname{Im}z_i}\right)^2<\infty. \tag{4.12}$$

From (4.10) and (4.12), $\frac{d^j}{d\lambda^j}V_n(\lambda_i)\in\ell_2(\mathbb{N},\mathbb{C}^2)$ for $j=0,1,\ldots,m_i-1$; $i=1,2,\ldots,k$.

On the other hand; since $\operatorname{Im}z_i=0$ for $j=0,1,\ldots,m_i-1$; $i=k+1,k+2,\ldots,\nu$ using (4.4), we find that

$$\sum_{n=1}^{\infty}\left|\alpha_n^{11}i^t(n+1/2)^t e^{iz_i(n+1/2)}\right|^2=\infty,$$

but the other terms in (4.4) belong to $\ell_2(\mathbb{N},\mathbb{C}^2)$, so $\frac{d^j}{d\lambda^j}E_n^{(1)}(\lambda)\notin\ell_2(\mathbb{N},\mathbb{C}^2)$ .Similarly, from (4.5), we get $\frac{d^j}{d\lambda^j}E_n^{(2)}(\lambda)\notin\ell_2(\mathbb{N},\mathbb{C}^2)$, then we obtain that $\frac{d^j}{d\lambda^j}V_n(\lambda_i)\notin\ell_2(\mathbb{N},\mathbb{C}^2)$ for $j=0,1,\ldots,m_i-1$; $i=k+1,k+2,\ldots,\nu$. □

Let us introduce Hilbert space $H_{-j}(\mathbb{N})$, $j=0,1,2,\ldots,$

$$H_{-j}(\mathbb{N})=\left\{y=\begin{pmatrix}y_n^{(1)}\\ y_n^{(2)}\end{pmatrix}:\sum_{n\in\mathbb{N}}(1+|n|)^{-2j}\left(\left|y_n^{(1)}\right|^2+\left|y_n^{(2)}\right|^2\right)<\infty\right\}$$

with

$$\|y\|_{-j}^2=\sum_{n\in\mathbb{N}}(1+|n|)^{-2j}\left(\left|y_n^{(1)}\right|^2+\left|y_n^{(2)}\right|^2\right).$$

Now we have the following result:

Theorem 4.2. $\frac{d^j}{d\lambda^j}V_n(\lambda_i) \in H_{-(j+1)}(\mathbb{N})$ *for* $j = 0, 1, 2, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$.

Proof. Using (2.1), (2.6) and (2.7) we have

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} (1+|n|)^{-2(j+1)} \left(\left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(1)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right. \\
&\qquad \left. + \left| \frac{1}{j!} \left\{ \frac{d^j}{d\lambda^j} E_n^{(2)}(\lambda) \right\}_{\lambda=\lambda_i} \right|^2 \right) \\
&= \sum_{n\in\mathbb{N}} \frac{(1+|n|)^{-2(j+1)}}{(j!)^2} \left\{ \left| \sum_{t=0}^{j} C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right|^2 \right. \\
&\qquad \left. + \left| \sum_{t=0}^{j} D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right|^2 \right\} \\
&\leq \frac{1}{(j!)^2} \sum_{n=1}^{\infty} (1+|n|)^{-2(j+1)} \left\{ \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 \right. \\
&\qquad \left. + \left(\sum_{t=0}^{j} \left| D_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(2)}(z_i) \right\} \right| \right)^2 \right\}
\end{aligned} \tag{4.13}$$

for $j = 0, 1, 2, \ldots, m_i - 1$; $i = k+1, k+2, \ldots, \nu$. Since $\operatorname{Im} z_i = 0$, using (4.13) we get

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} (1+|n|)^{-2(j+1)} \frac{1}{(j!)^2} \left(\sum_{t=0}^{j} \left| C_t \left\{ \frac{d^t}{d\lambda^t} f_n^{(1)}(z_i) \right\} \right| \right)^2 \\
&\leq \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \sum_{t=0}^{j} (1+|n|)^{-(j+1)} (n+1/2)^t \left| \alpha_n^{11} \right| |C_t| \right. \\
&\quad + \sum_{t=0}^{j} |C_t| \left| \alpha_n^{11} \right| (1+|n|)^{-(j+1)} \sum_{m=1}^{\infty} \left| A_{nm}^{11} \right| (m+n+1/2)^t \\
&\qquad \left. + \left| A_{nm}^{12} \right| (m+n)^t \right\}^2 \\
&= \frac{1}{(j!)^2} \sum_{n=1}^{\infty} \left\{ \left(\sum_{t=0}^{j} (1+|n|)^{-(j+1)} (n+1/2)^t \left| \alpha_n^{11} \right| |C_t| \right)^2 \right.
\end{aligned}$$

$$+ 2\left(1+|n|\right)^{-2(j+1)}\left|\alpha_n^{11}\right|^2\left[\sum_{t=0}^{j}\left(n+1/2\right)^t|C_t|\right]$$
$$\times\left[\sum_{t=0}^{j}|C_t|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\left(m+n+1/2\right)^t+\left|A_{nm}^{12}\right|\left(m+n\right)^t\right]$$
$$+\left(\sum_{t=0}^{j}|C_t|\left(1+|n|\right)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\right.$$
$$\left.\times\left(m+n+1/2\right)^t+\left|A_{nm}^{12}\right|\left(m+n\right)^t\right)^2\Bigg\}. \qquad (4.14)$$

Using (4.14), (2.1) and (2.4) we first obtain that

$$\left(\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|\left(1+|n|\right)^{-(j+1)}\sum_{m=1}^{\infty}\left(\left|A_{nm}^{11}\right|\left(m+n+1/2\right)^t\right.\right.$$
$$\left.\left.+\left|A_{nm}^{12}\right|\left(m+n\right)^t\right)\right)^2$$
$$\leq 4\left\{\sum_{t=0}^{j}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left(1+|n|\right)^{-(j+1)}\left(m+n+1/2\right)^t C\exp\left(-\varepsilon\left(\left(n+m\right)/4\right)^{\delta}\right)\right.$$
$$\left.\times\sum_{j=n+\lceil m/2\rceil}^{\infty}e^{\varepsilon j^{\delta}}\left(|1-a_j|+|1+b_j|+|p_j|+|q_j|\right)\right\}^2$$
$$\leq C_1\left(\sum_{t=0}^{j}\left(1+|n|\right)^{-(j+1)}\sum_{m=1}^{\infty}\left(m+n+1/2\right)^t\exp\left(-\varepsilon\sqrt{2}\left(n^{1/2}+m^{1/2}\right)/4\right)\right)^2$$
$$= C_1\left(1+|n|\right)^{-2(j+1)}\exp\left(-\varepsilon\sqrt{2}n^{1/2}/2\right)$$
$$\times\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+1/2\right)^t\exp\left(-\varepsilon\sqrt{2}m^{1/2}/4\right)\right)^2$$
$$= G\exp\left(-\varepsilon\sqrt{2}n^{1/2}/2\right)\left(1+|n|\right)^{-2(j+1)} \qquad (4.15)$$

where

$$C_1=\left(2C\left|\alpha_n^{11}\right|\sum_{j=n+\lceil m/2\rceil}^{\infty}e^{\varepsilon j^{\delta}}\left(|1-a_j|+|1+b_j|+|p_j|+|q_j|\right)\right)^2$$
$$G=C_1\left(\sum_{t=0}^{j}\sum_{m=1}^{\infty}\left(m+n+1/2\right)^t\exp\left(-\varepsilon\sqrt{2}m^{1/2}/4\right)\right)^2.$$

Hence we get from (4.15)

$$\sum_{n=1}^{\infty}\left(\sum_{t=0}^{j}|C_t|\,(1+|n|)^{-(j+1)}\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}\left|A_{nm}^{11}\right|\right.$$
$$\left.\times(m+n+1/2)^t+\left|A_{nm}^{12}\right|(m+n)^t\right)^2$$
$$\leq G\sum_{n=1}^{\infty}\exp\left(-\varepsilon\sqrt{2}n^{1/2}/2\right)(1+|n|)^{-2(j+1)}$$
$$<\infty. \tag{4.16}$$

Secondly, using (4.14) and (4.15) we obtain that

$$\sum_{n=1}^{\infty}2\left\{\left[\sum_{t=0}^{j}\left|\alpha_n^{11}\right||C_t|\,(1+|n|)^{-(j+1)}(n+1/2)^t\right]\right.$$
$$\times\left[\sum_{t=0}^{j}|C_t|\left|\alpha_n^{11}\right|\sum_{m=1}^{\infty}(1+|n|)^{-(j+1)}\left((m+n+1/2)^t\left|A_{nm}^{11}\right|\right)\right.$$
$$\left.\left.+(m+n)^t\left|A_{nm}^{12}\right|\right]\right\}$$
$$\leq T\sum_{n=1}^{\infty}\left[\sum_{t=0}^{j}(1+|n|)^{-2(j+1)}(n+1/2)^t\exp\left(-\varepsilon\sqrt{2}n^{1/2}/4\right)\right]$$
$$<\infty \tag{4.17}$$

where

$$T=\left|\alpha_n^{11}\right|G^{1/2}\max|C_t|$$

and also expression of the left side of (4.15) is obviously convergent. So, we get from (4.16) and (4.17)

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|C_t\left\{\frac{d^t}{d\lambda^t}f_n^{(1)}(z_i)\right\}\right|\right)^2<\infty$$

and similarly

$$\sum_{n\in\mathbb{N}}(1+|n|)^{-2(j+1)}\frac{1}{(j!)^2}\left(\sum_{t=0}^{j}\left|D_t\left\{\frac{d^t}{d\lambda^t}f_n^{(2)}(z_i)\right\}\right|\right)^2<\infty.$$

Finally $\frac{d^j}{d\lambda^j}V_n(\lambda_i)\in H_{-(j+1)}(\mathbb{N})$ for $j=0,1,2,\ldots,m_i-1$; $i=k+1,k+2,\ldots,\nu$. □

References

[1] Agarwal, R. P., Difference equation and inequalities, Theory, Methods and Applications. Marcel Dekkar Inc., New York, Basel, 2000.

[2] Agarwal, R. P. and Wong, P. J. Y., Advanced Topics in Difference Equations. Kluwer, Dordrecht, 1997.

[3] Kelley, W. G. and Peterson, A. C., Difference Equations, An Introduction with Applications, Harcourt Academic Press, 2001.

[4] Agarwal, R. P., Perera, K. and O'Regan, D., Multiple positive solutions of singular and nonsingular discrete problems via variational methods, *Nonlinear Analysis*, 58, (*2004*), 69-73.

[5] Agarwal, R. P., Perera, K. and O'Regan, D., Multiple positive solutions of singular discrete p-Laplasian problems via variational methods, *Advances in Difference Equations*, 2005:2, (2005), 93-99.

[6] Berezanski, Y. M., Integration of nonlinear difference equations by the inverse spectral problem method, *Soviet Math. Dokl.* 31, (1985), 264-267.

[7] Toda, M., Theory of Nonlinear Lattices. Springer-Verlag, Berlin, 1981.

[8] Bairamov, E. and Celebi, A. O., Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators, *Quart. J. Math. Oxford*, 50 (2), (1999), 371-384.

[9] Bairamov, E., Cakar, O. and Krall, A. M., Non-selfadjoint difference operators and Jacobi matrices with spectral singularities, *Math. Nachr.*, 229, (2001), 5-14.

[10] Adivar, M. and Bairamov, E., Spectral properties of non-selfadjoint difference operators, *J. Math. Anal. Appl.,* 261, (2001), 461-478.

[11] Adivar, M. and Bairamov, E., Difference equations of second order with spectral singularities, *J. Math. Anal. Appl.*, 277, (2003), 714-721.

[12] Adivar, M. and Bohner, M., Spectral analysis of q-difference equations with spectral singularities, *Math. Comput. Modelling*, 43 (7-9), (2006),695-703.

[13] Adivar, M. and Bohner, M., Spectrum and principal vectors of second order q-difference equations, *Indian J. Math.*, 48 (1), (2006) , 17-33.

[14] Bairamov, E. and Coskun, C., Jost solutions and the spectrum of the system of difference equations, *Appl. Math. Lett.,* 17, (2004) , 1039-1045.

[15] Bairamov, E. and Koprubasi, T., Eigenparameter dependent discrete Dirac equations with spectral singularities, *Appl. Math. and Comp.*, 215, *2010*, 4216-4220.

[16] Aygar, Y., Olgun, M. and Koprubasi, T., Principal Functions of Nonselfadjoint Discrete Dirac Equations with Spectral Parameter in Boundary Conditions, *Abstract and Applied Analysis*, vol. 2012, (2012) , ID 924628.

[17] Koprubasi, T., Spectrum of the quadratic eigenparameter dependent discrete Dirac equations, Advances in Difference Equations, (2014), 2014:148.

[18] Koprubasi, T. and Yokus, N., Quadratic eigenparameter dependent discrete Sturm Liouville equations with spectral singularities, *Appl. Math. and Comp.*, 244, (2014), 57-62.

[19] Levitan, B. M. and Sargsjan, I. S., Introduction to Spectral Theory. Translations of Mathematical Monographs 39, 1975.

[20] Dolzhenko, E. P., Boundary value uniqueness theorems for analytic functions *Math. Notes*, 26 (6), (1979), 437-442.

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