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COEFFICIENT ESTIMATES FOR A NEW SUBCLASS OF m-FOLD SYMMETRIC ANALYTIC BI-UNIVALENT FUNCTIONS

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Abstract. Considering a new subclass of m-fold symmetric analytic bi-univalent functions, we determine estimates the coefficient bounds for the Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ of the functions in this class. In certain cases, our estimates improve some of those existing coeffcient bounds.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For two functions f and Θ, analytic in $\mathbb{U}$, we say that the function f is subordinate to Θ in $\mathbb{U}$, and write

$$f(z) \prec \Theta(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = \Theta(\omega(z)) \quad (z \in \mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

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and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f)\,;\; r_0(f) \geq \frac{1}{4}\right).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by (1.1).

For a brief history and interesting examples of functions in the class Σ, see [15] (see also [3]). In fact, the aforecited work of Srivastava *et al.* [15] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Bulut *et al.* [5], Frasin and Aouf [6], Ramachandran *et al.* [9], Srivastava *et al.* [10, 13], Xu *et al.* [18, 19] and the references cited in each of them.

Let $m \in \mathbb{N} = \{1, 2, 3, \ldots\}$. A domain D is said to be m-fold symmetric if a rotation of D about the origin through an angle $2\pi/m$ carries D on itself. It follows that, a function $f(z)$ analytic in $\mathbb{U}$ is said to be m-fold symmetric $(m \in \mathbb{N})$ if

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

In particular, every $f(z)$ is 1-fold symmetric and every odd $f(z)$ is 2-fold symmetric. We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

A simple argument shows that $f \in \mathcal{S}_m$ is characterized by having a power series of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U},\ m \in \mathbb{N}). \quad (1.3)$$

Srivastava *et al.* [14] defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. For normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as following:

$$\begin{aligned} g(w) &= f^{-1}(w) \\ &= w - a_{m+1} w^{m+1} + \left[(m+1)\, a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)\, a_{m+1}^3 - (3m+2)\, a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} \\ &\quad + \cdots \\ &= : w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1}. \end{aligned} \quad (1.4)$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$ given by (1.3). For $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. For some examples of m-fold symmetric bi-univalent functions, see [14].

We also denote by $\mathcal{P}$ the family of all functions p analytic in $\mathbb{U}$ for which

$$\Re(p(z)) > 0, \quad p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

Thus the m-fold symmetric function p in the class $\mathcal{P}$ is of the form (see [8]),

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + \cdots \qquad (z \in \mathbb{U}).$$

The coefficient problem for m-fold symmetric analytic bi-univalent functions is one of the favorite subjects of geometric function theory in these days (see [1, 4, 7, 11, 14, 16]). The object of the present paper is to introduce a new subclass of bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this new subclass.

2. The Class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$

Throughout this paper, we assume that φ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \quad (B_1 > 0,\ B_n \in \mathbb{R},\ n = 2, 3, \ldots). \tag{2.1}$$

With this assumption on φ, we now introduce the following new class of m-fold symmetric analytic bi-univalent functions.

Definition 1. For $\tau \in \mathbb{C}\setminus\{0\}$, $\lambda \geq 1$ and $0 \leq \delta \leq 1$, a function $f \in \Sigma_m$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\tau}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right\} \prec \varphi(z) \tag{2.2}$$

and

$$1 + \frac{1}{\tau}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) - 1\right\} \prec \varphi(w) \tag{2.3}$$

where $m \in \mathbb{N}$; $z, w \in \mathbb{U}$ and $g = f^{-1}$ is defined by (1.4).

Remark 1. ***(i)*** If we set $\tau = 1$ and $\delta = 0$, then we have the class

$$\mathcal{N}_{\Sigma,m}(1,\lambda,0,\varphi) = \mathcal{B}_{\Sigma,m}(\lambda,\varphi)$$

introduced and studied by Tang *et al.* [17].

(ii) There are many choices of the function $\varphi(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U}),$$

or

$$\varphi(z) = \frac{1 + (1-2\beta)z}{1-z} \qquad (0 \leq \beta < 1,\ z \in \mathbb{U}),$$

it is easy to verify that these functions are of the form (2.1). If $f \in \mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$, then $f \in \Sigma_m$ and

$$\left|\arg\left(1+\frac{1}{\tau}\left\{(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)-1\right\}\right)\right|<\frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(1+\frac{1}{\tau}\left\{(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right\}\right)\right|<\frac{\alpha\pi}{2},$$

or

$$\Re\left(1+\frac{1}{\tau}\left\{(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)-1\right\}\right)>\beta$$

and

$$\Re\left(1+\frac{1}{\tau}\left\{(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right\}\right)>\beta,$$

where the function $g=f^{-1}$ is defined by (1.4). This means that

$$f \in \mathcal{R}_{\Sigma_m}(\tau,\lambda,\delta;\alpha) \qquad \text{or} \qquad f \in \mathcal{R}_{\Sigma_m}(\tau,\lambda,\delta;\beta),$$

respectively. These classes are introduced and studied by Atshan and Al-Ziadi [2]. In these classes of m-fold symmetric bi-univalent functions, in particular we have

$$\mathcal{R}_{\Sigma_m}(\tau,1,\delta;\alpha)=\mathcal{H}_{\Sigma_m}(\tau,\delta;\alpha), \qquad \mathcal{R}_{\Sigma_m}(\tau,1,\delta;\beta)=\mathcal{H}_{\Sigma_m}(\tau,\delta;\beta) \qquad (\text{see [11]}),$$

$$\mathcal{R}_{\Sigma_m}(\tau,\lambda,0;\alpha)=\mathcal{B}_{\Sigma_m}(\tau,\lambda;\alpha), \qquad \mathcal{R}_{\Sigma_m}(\tau,\lambda,0;\beta)=\mathcal{B}^*_{\Sigma_m}(\tau,\lambda;\beta) \qquad (\text{see [12]}),$$

and

$$\mathcal{R}_{\Sigma_m}(1,\lambda,0;\alpha)=\mathcal{A}^{\alpha,\lambda}_{\Sigma,m}, \qquad \mathcal{R}_{\Sigma_m}(1,\lambda,0;\beta)=\mathcal{A}^{\lambda}_{\Sigma,m}(\beta) \qquad (\text{see [16]}).$$

Here we propose to investigate the m-fold symmetric bi-univalent function class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$ introduced in Definition 1 and derive coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for a function $f \in \mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$ given by (1.3). Our results for the m-fold symmetric bi-univalent function class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$ would generalize and improve the related works of Tang *et al.* [17], Srivastava *et al.* [11, 12] and Sümer Eker [16].

3. A Set of General Coefficient Estimates

In this section, we state and prove our general results involving the m-fold symmetric bi-univalent function class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$ given by Definition 1.

Theorem 1. *Let the function f given by (1.3) be in the class $\mathcal{N}_{\Sigma,m}(\tau,\lambda,\delta,\varphi)$, $\tau \in \mathbb{C}\backslash\{0\}$, $\lambda \geq 1$, $0 \leq \delta \leq 1$ and $m \in \mathbb{N}$. Then*

$$|a_{m+1}| \leq \frac{|\tau|\, B_1\sqrt{2B_1}}{\sqrt{\left|\tau(m+1)\psi_2 B_1^2-2\psi_1^2 B_2\right|+2\psi_1^2 B_1}} \tag{3.1}$$

and

$$|a_{2m+1}| \leq \begin{cases} \Phi & , \quad B_1 \geq \frac{2\psi_1^2}{|\tau|(m+1)\psi_2} \\ \frac{|\tau|^2(m+1)B_1^2}{2\psi_1^2} & , \quad B_1 < \frac{2\psi_1^2}{|\tau|(m+1)\psi_2} \end{cases}, \tag{3.2}$$

where

$$\Phi := \left(\frac{m+1}{2} - \frac{\psi_1^2}{|\tau|\,\psi_2 B_1}\right)\frac{2\,|\tau|^2\,B_1^3}{\left|\tau\,(m+1)\,\psi_2 B_1^2 - 2\psi_1^2 B_2\right| + 2\psi_1^2 B_1} + \frac{|\tau|\,B_1}{\psi_2}, \tag{3.3}$$

$$\psi_j := 1 + jm\lambda + jm\,(jm+1)\,\delta \qquad (j = 1, 2)\,. \tag{3.4}$$

Proof. Let $f \in \mathcal{N}_{\Sigma,m}(\tau, \lambda, \delta, \varphi)$ and $g = f^{-1}$ be defined by (1.4). Then there exist two Schwarz functions $u, v : \mathbb{U} \to \mathbb{U}$, with $u(0) = v(0) = 0$, such that

$$1 + \frac{1}{\tau}\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right\} = \varphi(u(z)) \tag{3.5}$$

and

$$1 + \frac{1}{\tau}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) - 1\right\} = \varphi(v(w)). \tag{3.6}$$

We suppose that

$$u(z) = p_m z^m + p_{2m} z^{2m} + \cdots \qquad (z \in \mathbb{U}) \tag{3.7}$$

and

$$v(w) = q_m w^m + q_{2m} w^{2m} + \cdots \qquad (w \in \mathbb{U})\,. \tag{3.8}$$

We note that

$$|p_m| \leq 1, \quad |p_{2m}| \leq 1 - |p_m|^2, \quad |q_m| \leq 1, \quad |q_{2m}| \leq 1 - |q_m|^2. \tag{3.9}$$

Using (3.7) and (3.8) together with (2.1), it is evident that

$$\varphi(u(z)) = 1 + B_1 p_m z^m + \left(B_1 p_{2m} + B_2 p_m^2\right) z^{2m} + \cdots \qquad (z \in \mathbb{U}) \tag{3.10}$$

and

$$\varphi(v(w)) = 1 + B_1 q_m w^m + \left(B_1 q_{2m} + B_2 q_m^2\right) w^{2m} + \cdots \qquad (w \in \mathbb{U}), \tag{3.11}$$

respectively. Since

$$\begin{aligned} 1 + \frac{1}{\tau}&\left\{(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right\} \\ &= 1 + \frac{1}{\tau}\left\{1 + m\lambda + m\,(m+1)\,\delta\right\} a_{m+1} z^m \\ &+ \frac{1}{\tau}\left\{1 + 2m\lambda + 2m\,(2m+1)\,\delta\right\} a_{2m+1} z^{2m} + \cdots \end{aligned}$$

and

$$1 + \frac{1}{\tau}\left\{(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) - 1\right\}$$

$$= 1 + \frac{1}{\tau}\left\{1 + m\lambda + m(m+1)\delta\right\} A_{m+1}w^m$$
$$+\frac{1}{\tau}\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\} A_{2m+1}w^{2m} + \cdots,$$

then (3.5), (3.6), (3.10) and (3.11) together with (1.4) yield

$$\frac{1}{\tau}\left\{1 + m\lambda + m(m+1)\delta\right\} a_{m+1} = B_1 p_m, \tag{3.12}$$

$$\frac{1}{\tau}\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\} a_{2m+1} = B_1 p_{2m} + B_2 p_m^2, \tag{3.13}$$

$$-\frac{1}{\tau}\left\{1 + m\lambda + m(m+1)\delta\right\} a_{m+1} = B_1 q_m \tag{3.14}$$

and

$$\frac{1}{\tau}\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\}\left\{(m+1)a_{m+1}^2 - a_{2m+1}\right\} = B_1 q_{2m} + B_2 q_m^2. \tag{3.15}$$

Now, considering (3.12) and (3.14), we get

$$p_m = -q_m \tag{3.16}$$

and

$$2\frac{1}{\tau^2}\left\{1 + m\lambda + m(m+1)\delta\right\}^2 a_{m+1}^2 = B_1^2\left(p_m^2 + q_m^2\right). \tag{3.17}$$

Now from (3.13), (3.15) and (3.17), we obtain

$$a_{m+1}^2 = \frac{\tau^2 B_1^3 (p_{2m} + q_{2m})}{\tau(m+1)\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\} B_1^2 - 2\left\{1 + m\lambda + m(m+1)\delta\right\}^2 B_2}. \tag{3.18}$$

Therefore, by $(3.9), (3.16)$ from (3.18), we get

$$|a_{m+1}|^2 \leq \frac{2|\tau|^2 B_1^3\left(1 - |p_m|^2\right)}{\left|\tau(m+1)\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\} B_1^2 - 2\left\{1 + m\lambda + m(m+1)\delta\right\}^2 B_2\right|}.$$

The equality (3.12) and the above inequality give

$$|a_{m+1}|^2 \leq \frac{2|\tau|^2 B_1^3}{\left|\tau(m+1)\psi_2 B_1^2 - 2\psi_1^2 B_2\right| + 2\psi_1^2 B_1}, \tag{3.19}$$

where ψ_j $(j = 1, 2)$ is defined by (3.4), which is the desired estimate on the coefficient $|a_{m+1}|$ as asserted in (3.1).

Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (3.15) from (3.13). Observing (3.16) we get

$$a_{2m+1} = \frac{m+1}{2}a_{m+1}^2 + \frac{\tau B_1}{2\left\{1 + 2m\lambda + 2m(2m+1)\delta\right\}}(p_{2m} - q_{2m}). \tag{3.20}$$

We obtain from the (3.9), (3.12) and the above equality

$$\begin{aligned} |a_{2m+1}| &\leq \frac{m+1}{2}|a_{m+1}|^2 + \frac{|\tau| B_1}{1+2m\lambda+2m(2m+1)\delta}\left(1-|p_m|^2\right) \\ &\leq \left(\frac{m+1}{2} - \frac{\{1+m\lambda+m(m+1)\delta\}^2}{|\tau|\{1+2m\lambda+2m(2m+1)\delta\}B_1}\right)|a_{m+1}|^2 \\ &\quad + \frac{|\tau| B_1}{1+2m\lambda+2m(2m+1)\delta}. \end{aligned} \tag{3.21}$$

Upon substituting the value of a_{m+1}^2 from (3.12) and (3.19) into (3.21), it follows that

$$|a_{2m+1}| \leq \frac{|\tau|^2 (m+1) B_1^2}{2\psi_1^2} \tag{3.22}$$

and

$$|a_{2m+1}| \leq \left(\frac{m+1}{2} - \frac{\psi_1^2}{|\tau|\psi_2 B_1}\right)\frac{2|\tau|^2 B_1^3}{\left|\tau(m+1)\psi_2 B_1^2 - 2\psi_1^2 B_2\right| + 2\psi_1^2 B_1} + \frac{|\tau| B_1}{\psi_2}, \tag{3.23}$$

where ψ_j $(j=1,2)$ is defined by (3.4), respectively. Therefore considering (3.22) and (3.23), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in (3.2). This completes the proof of the Theorem 1. □

Setting $\tau = 1$ and $\delta = 0$ in Theorem 1, we have the following corollary.

Corollary 1. *Let the function f given by* (1.3) *be in the class $\mathcal{B}_{\Sigma,m}(\lambda,\varphi)$, $\lambda \geq 1$ and $m \in \mathbb{N}$. Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|(m+1)(1+2m\lambda)B_1^2 - 2(1+m\lambda)^2 B_2\right| + 2(1+m\lambda)^2 B_1}}$$

and

$$|a_{2m+1}| \leq \begin{cases} \Phi & B_1 \geq \frac{2(1+m\lambda)^2}{(m+1)(1+2m\lambda)} \\ \frac{(m+1)B_1^2}{2(1+m\lambda)^2} & B_1 < \frac{2(1+m\lambda)^2}{(m+1)(1+2m\lambda)} \end{cases},$$

where

$$\Phi = \left(\tfrac{m+1}{2} - \tfrac{(1+m\lambda)^2}{(1+2m\lambda)B_1}\right)\frac{2B_1^3}{\left|(m+1)(1+2m\lambda)B_1^2 - 2(1+m\lambda)^2 B_2\right| + 2(1+m\lambda)^2 B_1} + \frac{B_1}{1+2m\lambda}.$$

Remark 2. Note that Corollary 1 is an improvement of the estimates obtained by Tang *et al.* [17, Theorem 7].

Taking the function

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leq 1,\ z \in \mathbb{U})$$

in Theorem 1, we have the following corollary.

Corollary 2. *Let the function f given by* (1.3) *be in the class* $\mathcal{R}_{\Sigma_m}(\tau,\lambda,\delta;\alpha)$, $\tau\in\mathbb{C}\backslash\{0\}$, $\lambda\geq 1$, $0\leq\delta\leq 1$, $0<\alpha\leq 1$ *and* $m\in\mathbb{N}$. *Then*

$$|a_{m+1}|\leq\frac{2|\tau|\alpha}{\sqrt{\left|\tau(m+1)\psi_2-\psi_1^2\right|\alpha+\psi_1^2}}$$

and

$$|a_{2m+1}|\leq\begin{cases}\Phi & , \quad \alpha\geq\frac{\psi_1^2}{|\tau|(m+1)\psi_2}\\ \frac{2|\tau|^2(m+1)\alpha^2}{\psi_1^2} & , \quad \alpha<\frac{\psi_1^2}{|\tau|(m+1)\psi_2}\end{cases},$$

where

$$\Phi=\left((m+1)-\frac{\psi_1^2}{|\tau|\psi_2\alpha}\right)\frac{2|\tau|^2\alpha^2}{\left|\tau(m+1)\psi_2-\psi_1^2\right|\alpha+\psi_1^2}+\frac{2|\tau|\alpha}{\psi_2},$$

and ψ_j *is defined by* (3.4).

Remark 3. Note that Corollary 2 is an improvement of the estimates obtained by Atshan and Al-Ziadi [2, Theorem 2.1], Srivastava *et al.* [11, Theorem 2] for $\lambda=1$, Srivatava *et al.* [12, Theorem 2.1] for $\delta=0$, and Sümer Eker [16, Theorem 1] for $\tau=1$ and $\delta=0$, respectively.

Taking the function

$$\varphi(z)=\frac{1+(1-2\beta)z}{1-z}\qquad(0\leq\beta<1,\ z\in\mathbb{U})$$

in Theorem 1, we have the following corollary.

Corollary 3. *Let the function f given by* (1.3) *be in the class* $\mathcal{R}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$, $\tau\in\mathbb{C}\backslash\{0\}$, $\lambda\geq 1$, $0\leq\delta\leq 1$, $0\leq\beta<1$ *and* $m\in\mathbb{N}$. *Then*

$$|a_{m+1}|\leq\frac{\sqrt{2}|\tau|(1-\beta)}{\sqrt{\left|\tau(m+1)\psi_2(1-\beta)-\psi_1^2\right|+\psi_1^2}}$$

and

$$|a_{2m+1}|\leq\begin{cases}\Phi & , \quad \beta\leq 1-\frac{\psi_1^2}{|\tau|(m+1)\psi_2}\\ \frac{2|\tau|^2(m+1)(1-\beta)^2}{\psi_1^2} & , \quad \beta>1-\frac{\psi_1^2}{|\tau|(m+1)\psi_2}\end{cases},$$

where

$$\Phi=\left((m+1)-\frac{\psi_1^2}{|\tau|\psi_2(1-\beta)}\right)\frac{2|\tau|^2(1-\beta)^2}{\left|\tau(m+1)\psi_2(1-\beta)-\psi_1^2\right|+\psi_1^2}+\frac{2|\tau|(1-\beta)}{\psi_2},$$

and ψ_1 *and* ψ_2 *are defined by* (3.4).

Remark 3. Note that Corollary 3 is an improvement of the estimates obtained by Atshan and Al-Ziadi [2, Theorem 3.1], Srivastava *et al.* [11, Theorem 3] for $\lambda = 1$, Srivatava *et al.* [12, Theorem 3.1] for $\delta = 0$, and Sümer Eker [16, Theorem 2] for $\tau = 1$ and $\delta = 0$, respectively.

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