TITLE: Remarks on almost ?-Ricci solitons in (?)-para Sasakian manifolds

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REMARKS ON ALMOST η-RICCI SOLITONS IN (ε)-PARA SASAKIAN MANIFOLDS

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ABSTRACT. We consider almost η-Ricci solitons in (ε)-para Sasakian manifolds satisfying certain curvature conditions. In the gradient case we give an estimation for the norm of the Ricci curvature tensor and express the scalar curvature of the manifold in terms of the two functions that define the soliton. We also prove that if the Ricci operator is of Codazzi type, then the gradient η-Ricci soliton is expanding if M is spacelike or shrinking if M is timelike.

1. INTRODUCTION

A pseudo-Riemannian manifold (M, g) admits a *Ricci soliton* [7] if there exists a smooth vector field V (called the potential vector field) such that

$$\frac{1}{2}\mathcal{L}_V\, g + S + \lambda g = 0, \tag{1}$$

where $\mathcal{L}_V$ denotes the Lie derivative along the vector field V and λ is a real constant. By perturbing (1) with a term $\mu\eta \otimes \eta$, for μ a real constant and η a 1-form, we obtain the *η-Ricci soliton*, introduced by Cho and Kimura [5] and more general, if we replace the two constants λ and μ by smooth functions, we get *almost η-Ricci solitons* [2].

In the present paper we consider almost η-Ricci solitons in (ε)-para Sasakian manifolds which satisfy certain curvature properties. (ε)-*para Sasakian manifolds* were defined by Tripathi, Kılıç, Yüksel Perktaş, and Keleş [10] as a counterpart of almost paracontact metric geometry [9]. Also, Ricci solitons in (ε)-para Sasakian manifolds satisfying certain curvature conditions were treated in [1]. Our interest is to characterize the geometry of an almost η-Ricci soliton on (ε)-para Sasakian manifolds in the case when the potential vector field is of gradient type and explicitly compute the scalar curvature for the case of gradient η-Ricci solitons. Remark that

some properties of η-Ricci solitons on (ε)-almost paracontact metric manifolds were studied in [4].

2. (ε)-para Sasakian structures

Recall that an *almost paracontact structure* on an n-dimensional manifold M is a triple (φ, ξ, η) [9] consisting of a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying:

$$\varphi^2 = I - \eta \otimes \xi, \tag{2}$$

$$\eta(\xi) = 1, \tag{3}$$

$$\varphi\xi = 0, \tag{4}$$

$$\eta \circ \varphi = 0. \tag{5}$$

One can easily check that (2) and one of (3), (4) and (5) imply the other two relations. Moreover, if g is a pseudo-Riemannian metric such that

$$g(\varphi\cdot, \varphi\cdot) = g(\cdot, \cdot) - \varepsilon\eta \otimes \eta, \tag{6}$$

where $\varepsilon = \pm 1$, then M is called (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$ [10]. In particular, if $\text{index}(g) = 1$ (that is when g is a Lorentzian metric), then the (ε)-almost paracontact metric manifold is called *Lorentzian almost paracontact manifold*. From (6) we obtain:

$$i_\xi g = \varepsilon\eta, \tag{7}$$

$$g(X, \varphi Y) = g(\varphi X, Y), \tag{8}$$

and from (7) it follows that

$$g(\xi, \xi) = \varepsilon, \tag{9}$$

that is, the structure vector field ξ is never lightlike.

Let $(M, \varphi, \xi, \eta, g, \varepsilon)$ be an (ε)-almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If $\varepsilon = 1$, then M is said to be a spacelike (ε)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold) and if $\varepsilon = -1$, then M is said to be a timelike (ε)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) [10].

An (ε)-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ is called (ε)-*para Sasakian structure* if

$$(\nabla_X\varphi)Y = -g(\varphi X, \varphi Y)\xi - \varepsilon\eta(Y)\varphi^2 X, \tag{10}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called (ε)-*para Sasakian manifold* [10]. In an (ε)-para Sasakian manifold, we have

$$\nabla\xi = \varepsilon\varphi \tag{11}$$

and the Riemann curvature tensor R and the Ricci tensor S satisfy the following equations [10]:

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{12}$$

$$R(\xi,X)Y = -\varepsilon g(X,Y)\xi + \eta(Y)X, \tag{13}$$

$$\eta(R(X,Y)Z) = -\varepsilon\eta(X)g(Y,Z) + \varepsilon\eta(Y)g(X,Z), \tag{14}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$.

Also remark that if $(\varphi, \xi, \eta, g, \varepsilon)$ is an (ε)-para Sasakian structure on the manifold (M, g) of constant curvature k, if M is spacelike (resp. timelike), then M is hyperbolic (resp. elliptic) manifold. Indeed, if $R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y]$, for any X, Y, $Z \in \Gamma(TM)$, applying η to this relation and using (14) we obtain $k = -\varepsilon$.

Example 1. [10] *Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining*

$$\eta = ds - ydx - tdz, \qquad \xi = \frac{\partial}{\partial s}$$

$$\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y}$$

$$\varphi\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial z} - t\frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial t}\right) = -\frac{\partial}{\partial t}, \qquad \varphi\left(\frac{\partial}{\partial s}\right) = 0$$

$$g_1 = (dx)^2 + (dy)^2 + (dz)^2 + (dt)^2 - \eta \otimes \eta$$

$$\begin{aligned} g_2 = &\ -(dx)^2 - (dy)^2 + (dz)^2 + (dt)^2 + (ds)^2 \\ &\ -t(dz \otimes ds + ds \otimes dz) - y(dx \otimes ds + ds \otimes dx), \end{aligned}$$

then $(\varphi, \xi, \eta, g_1)$ is a timelike Lorentzian almost paracontact structure in $\mathbb{R}^5$, while $(\varphi, \xi, \eta, g_2)$ is a spacelike (ε)-almost paracontact structure.

3. Almost η-Ricci solitons in $(M, \varphi, \xi, \eta, g, \varepsilon)$

Let (M, g) be an n-dimensional Riemannian manifold $(n > 2)$. Consider the equation:

$$\mathcal{L}_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{16}$$

where $\mathcal{L}_\xi$ is the Lie derivative operator along the vector field ξ, S is the Ricci curvature tensor field of the metric g, η is a 1-form and λ and μ are smooth functions on M.

The data (ξ, λ, μ) which satisfy the equation (16) is said to be an *almost η-Ricci soliton* on (M, g) called *steady* if $\lambda = 0$, *shrinking* if $\lambda < 0$ or *expanding* if $\lambda > 0$; in particular, if $\mu = 0$, (ξ, λ) is an *almost Ricci soliton* [8].

Example 2. *Let $M = \mathbb{R}^3$, (x, y, z) be the standard coordinates in $\mathbb{R}^3$ and g be the Lorentzian metric:*

$$g := e^{-2z} dx \otimes dx + e^{2x-2z} dy \otimes dy - dz \otimes dz.$$

Consider the $(1,1)$-tensor field φ, the vector field ξ and the 1-form η:

$$\varphi := -\frac{\partial}{\partial x} \otimes dx - \frac{\partial}{\partial y} \otimes dy, \quad \xi := \frac{\partial}{\partial z}, \quad \eta := dz.$$

In this case, $(M, \varphi, \xi, \eta, g)$ is a timelike Lorentzian almost paracontact manifold. Moreover, for the orthonormal vector fields:

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-x} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

we get:

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1, \quad \nabla_{E_2} E_1 = e^z E_2,$$

$$\nabla_{E_2} E_2 = -e^z E_1 - E_3, \quad \nabla_{E_2} E_3 = -E_2, \quad \nabla_{E_3} E_1 = 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0$$

and the Riemann tensor field and the Ricci curvature tensor field are given by:

$$R(E_1, E_2) E_2 = (1 - e^{2z}) E_1, \quad R(E_1, E_3) E_3 = -E_1, \quad R(E_2, E_1) E_1 = (1 - e^{2z}) E_2,$$

$$R(E_2, E_3) E_3 = -E_2, \quad R(E_3, E_1) E_1 = E_3, \quad R(E_3, E_2) E_2 = E_3,$$

$$S(E_1, E_1) = S(E_2, E_2) = 2 - e^{2z}, \quad S(E_3, E_3) = -2.$$

Therefore, the data (ξ, λ, μ) for $\lambda = e^{2z} - 1$ and $\mu = e^{2z} + 1$ defines an almost η-Ricci soliton on (M, g).

Writing $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection ∇, we obtain:

$$S(X, Y) = -\frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] - \lambda g(X, Y) - \mu \eta(X) \eta(Y), \qquad (17)$$

for any $X, Y \in \Gamma(TM)$.

If $(M, \varphi, \xi, \eta, g, \varepsilon)$ is an (ε)-para Sasakian manifold, then (17) becomes:

$$S(X, Y) = -\varepsilon g(\varphi X, Y) - \lambda g(X, Y) - \mu \eta(X) \eta(Y), \qquad (18)$$

for any $X, Y \in \Gamma(TM)$.

Also remark that on an (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$, since the Ricci curvature tensor field S satisfies (15), using (18) we obtain:

$$\varepsilon \lambda + \mu = n - 1 \qquad (19)$$

and we can state:

Proposition 3. *The scalar curvature of an (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ admitting an almost η-Ricci soliton (ξ, λ, μ) is:*

$$scal = -div(\xi) - n\lambda - \varepsilon\mu. \qquad (20)$$

From (19) we also deduce that:

Proposition 4. *An almost Ricci soliton (ξ, λ) on a spacelike (resp. timelike) (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ is expanding (resp. shrinking).*

Remark 5. *If the almost η-Ricci soliton is steady, then $\mu = n - 1$ and the scalar curvature of $(M, \varphi, \xi, \eta, g, \varepsilon)$ is $scal = -div(\xi) - \varepsilon(n-1)$.*

Similar like in the case of η-Ricci solitons on Lorentzian para-Sasakian manifolds [3], the next theorems formulate some results in case of (ε)-para Sasakian manifold which is Ricci symmetric, has η-parallel, of Codazzi type or cyclic η-recurrent Ricci curvature tensor.

Proposition 6. *Let $(\varphi, \xi, \eta, g, \varepsilon)$ be an (ε)-para Sasakian structure on the manifold M and let (ξ, λ, μ) be an almost η-Ricci soliton on (M, g).*

(1) *If the manifold (M, g) is Ricci symmetric (i.e. $\nabla S = 0$), then $\xi(\mu) = -\varepsilon\xi(\lambda)$.*

(2) *If the Ricci tensor is η-recurrent (i.e. $\nabla S = \eta \otimes S$), then $\xi(\varepsilon\lambda + \mu) = n - 1$.*

(3) *If the Ricci tensor is of Codazzi type (i.e. $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$, for any X, Y, $Z \in \Gamma(TM)$), then $d(\varepsilon\lambda + \mu) = \xi(\varepsilon\lambda + \mu)\eta$.*

(4) *If the Ricci tensor is η-parallel (i.e.$(\nabla_X S)(\varphi Y, \varphi Z) = 0$), then the scalar function λ is locally constant.*

Proof. Replacing the expression of S from (18) in $(\nabla_X S)(Y, Z) := X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$ we obtain:

$$(\nabla_X S)(Y, Z) = \eta(Y)g(X, Z) + \eta(Z)g(X, Y) - 2\varepsilon\eta(X)\eta(Y)\eta(Z) - \quad (21)$$

$$-X(\lambda)g(Y, Z) - X(\mu)\eta(Y)\eta(Z) - \mu[\eta(Y)g(\varphi X, Z) + \eta(Z)g(\varphi X, Y)].$$

For the first two assertions, just take $X = Y = Z := \xi$ in the expression of ∇S from (21) and we obtain the required results. Concerning the case when S is of Codazzi type, taking $Y = Z := \xi$ in $(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$ we obtain $\varepsilon X(\lambda) + X(\mu) = [\varepsilon\xi(\lambda) + \xi(\mu)]\eta(X)$, for any $X \in \Gamma(TM)$, which is equivalent to the stated relation. If S is η-parallel, then for any X, Y, $Z \in \Gamma(TM)$, $0 = (\nabla_X S)(\varphi Y, \varphi Z) = -X(\lambda)g(\varphi Y, \varphi Z)$ which implies $X(\lambda) = 0$, for any $X \in \Gamma(TM)$. $\square$

Remark 7. *If on the (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ we consider the almost η-Ricci soliton (V, λ, μ) with the potential vector field V conformal Killing (i.e. $\frac{1}{2}\pounds_\xi g = fg$), for f a smooth function on M, then*

$$S = -(f + \lambda)g - [n - 1 - \varepsilon(f + \lambda)]\eta \otimes \eta$$

and the manifold is Einstein if and only if $\lambda = \varepsilon(n - 1) - f$; in this case, we have an almost Ricci soliton.

The same conclusion is reached if the potential vector field $V := \xi$ is torse-forming (i.e. $\nabla\xi = f\varphi^2$ according to [4]).

When the potential vector field of (16) is of gradient type, i.e. $\xi = grad(f)$, then (ξ, λ, μ) is said to be a *gradient almost η-Ricci soliton* and the equation satisfied by it becomes:

$$Hess(f) + S + \lambda g + \mu \eta \otimes \eta = 0, \tag{22}$$

where $Hess(f)$ is the Hessian of f defined by $Hess(f)(X, Y) := g(\nabla_X \xi, Y)$.

Proposition 8. *Let $(M, \varphi, \xi, \eta, g, \varepsilon)$ be an (ε)-para Sasakian manifold. If (22) defines a gradient almost η-Ricci soliton on (M, g) with the potential vector field $\xi := grad(f)$ and $\eta = df$ is the g-dual of ξ, then:*

$$(\nabla_X Q)Y - (\nabla_Y Q)X = (d(f - \lambda) \otimes I - I \otimes d(f - \lambda) + (df \otimes d\mu - d\mu \otimes df) \otimes \xi + \tag{23}$$
$$+ \varepsilon\mu(df \otimes \varphi - \varphi \otimes df))(X, Y),$$

for any $X, Y \in \Gamma(TM)$, where Q stands for the Ricci operator defined by $g(QX, Y) := S(X, Y)$.

Proof. Notice that (22) can be written:

$$\nabla\xi + Q + \lambda I + \mu df \otimes \xi = 0. \tag{24}$$

Then:

$$\begin{aligned}(\nabla_X Q)Y &= -(\nabla_X \nabla_Y \xi - \nabla_{\nabla_X Y} \xi) - X(\lambda)Y - X(\mu)df(Y)\xi \\ &\quad - \mu[g(Y, \nabla_X \xi)\xi + df(Y)\nabla_X \xi].\end{aligned} \tag{25}$$

Replacing now $\nabla\xi = \varepsilon\varphi$ in the previous relation, after a long computation, we get the required relation. $\square$

Remark 9. *i) Remark that since ξ is geodesic vector field, from (24) follows that ξ is an eigenvector of Q corresponding to the eigenvalue $-(\lambda + \mu)$. In particular, if $\lambda = -\mu$, then $\xi \in \ker Q$.*

ii) The Ricci operator is φ-invariant (i.e. $Q \circ \varphi = \varphi \circ Q$).

iii) If Q is of Codazzi type (i.e. $(\nabla_X Q)Y = (\nabla_Y Q)X$, for any $X, Y \in \Gamma(TM)$), then for the gradient η-Ricci soliton case, $\mu\varphi X = \varepsilon\varphi^2 X$, for any $X \in \Gamma(TM)$ which implies $\mu^2 = 1$. Therefore, the soliton is expanding if M is spacelike or shrinking if M is timelike and it is given by $(\lambda, \mu) \in \{(\varepsilon n, -1), (\varepsilon(n - 2), 1)\}$.

A lower and an upper bound of the Ricci curvature tensor's norm [6] for a gradient almost η-Ricci soliton on an (ε)-para Sasakian manifold will be given [2].

Theorem 10. *If (22) defines a gradient almost η-Ricci soliton on the n-dimensional (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ and the 1-form $\eta = df$ is the g-dual of the gradient vector field $\xi := grad(f)$, then:*

$$n - 1 + \mu^2 - \frac{(\Delta(f) + \varepsilon\mu)^2}{n} \le |S|^2 \le n - 1 + \mu^2 + \frac{(scal)^2}{n}. \tag{26}$$

Remark 11. *The simultaneous equalities hold for $(scal)^2 = -(\Delta(f) + \varepsilon\mu)^2 \ (= 0)$ i.e. for steady gradient almost η-Ricci soliton $(\lambda = 0)$ with $scal = 0$ and $\Delta(f) = -\varepsilon\mu$. In this case, $|S|^2 = n - 1 + \mu^2$.*

Using a Bochner-type formula for the gradient almost η-Ricci solitons [2], in the case of (ε)-para Sasakian manifold we obtain the condition satisfied by the potential function:

Theorem 12. *If (22) defines a gradient almost η-Ricci soliton on the n-dimensional (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ and the 1-form $\eta = df$ is the g-dual of the gradient vector field $\xi := grad(f)$, then:*

$$\Delta(f) = \frac{1}{2}[\varepsilon(n-1) + \lambda + \varepsilon\mu + \varepsilon(n-2)\xi(\lambda) - \xi(\mu)]. \tag{27}$$

Considering (20) and (27) we obtain:

$$scal = -\frac{1}{2}[\varepsilon(n-1) + (2n+1)\lambda + 3\varepsilon\mu + \varepsilon(n-2)\xi(\lambda) - \xi(\mu)]. \tag{28}$$

Remark 13. *i) If (22) defines a gradient η-Ricci soliton on the n-dimensional (ε)-para Sasakian manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$, then:*

$$scal = -\frac{1}{2}[\varepsilon(n-1) + (2n+1)\lambda + 3\varepsilon\mu].$$

ii) In this case, if M is connected and has constant scalar curvature, then:

$$\mu = -\frac{2n+1}{3\varepsilon}\lambda + C, \quad C \in \mathbb{R}.$$

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