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MEAN ERGODIC TYPE THEOREMS

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Abstract. Let T be a bounded linear operator on a Banach space X. Replacing the Cesàro matrix by a regular matrix $A = (a_{nj})$ Cohen studied a mean ergodic theorem. In the present paper we extend his result by taking a sequence of infinite matrices $\mathcal{A} = (A^{(i)})$ that contains both convergence and almost convergence. This result also yields an $\mathcal{A}$-ergodic decomposition. When T is power bounded we give a characterization for T to be $\mathcal{A}$-ergodic.

1. Introduction

Let X be a Banach space and T be a bounded linear operator on X into itself. By $M_n(T)$ we denote the Cesàro averages of T given by $M_n(T) := \frac{1}{n+1}\sum_{j=0}^{n} T^j$. An operator $T \in B(X)$ is called mean ergodic, respectively uniformly ergodic, if $\{M_n(T)\}$ is strongly, respectively uniformly, convergent in $B(X)$. Cohen [3] considered the problem of determining a class of regular matrices $A = (a_{nj})$ for which

$$L_n := \sum_{j=1}^{\infty} a_{nj} T^j$$

converges strongly to an element invariant under T. It is the case when $\{L_n x : n \in \mathbb{N}\}$ is weakly compact and $\lim_k \sum_{j=k}^{\infty} |a_{n,j+1} - a_{nj}| = 0$ uniformly in n (see also [11]). Observe that Cohen's result is an extension of the mean ergodic theorems due to von Nuemann [10], F. Riesz [8] and K. Yosida [12].

In the present paper, replacing the matrix $A = (a_{nj})$ by a sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$ we study results in an analogy of Cohen.

Now, we give some basic notations concerning the sequence of infinite matrices. Let $\mathcal{A}$ be a sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$. Given a sequence $x = (x_j)$

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we write

$$A_n^{(i)}x = \sum_{j=1}^{\infty} a_{nj}^{(i)} x_j$$

if it exists for each n and $i \geq 0$. The sequence (x_j) is said to be summable to the value s by the method $\mathcal{A}$ if

$$A_n^{(i)}x \to s \quad (n \to \infty, \text{ uniformly in } i). \tag{1}$$

If (1) holds, we write $x \to s(\mathcal{A})$.

The method $\mathcal{A}$ is called conservative if $x \to s$ implies $x \to s'(\mathcal{A})$. If $\mathcal{A}$ is conservative and $s = s'$, we say that $\mathcal{A}$ is regular. We now recall a theorem which characterizes the regularity of the sequences of infinite matrices.

Theorem 1 ([2, 9])**.** *Let $\mathcal{A}$ be the sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$. Then, $\mathcal{A}$ is regular if and only if the following conditions hold:*

(1) $\sum_j |a_{nj}^{(i)}| < \infty$, *(for all n, for all i),*
(2) *There exists an integer m such that* $\sup_{i \geq 0,\, n \geq m} \sum_j |a_{nj}^{(i)}| < \infty$,
(3) *for all j,* $\lim_n a_{nj}^{(i)} = 0$, *(uniformly in i),*
(4) $\lim_n \sum_j a_{nj}^{(i)} = 1$, *(uniformly in i).*

In addition, we write

$$\|\mathcal{A}\| := \sup_{n,i} \sum_j |a_{nj}^{(i)}|, \tag{2}$$

and $\|\mathcal{A}\| < \infty$ to mean that, there exists a constant M such that $\sum_j |a_{nj}^{(i)}| \leq M$, (for all n, for all i) and the series $\sum_j a_{nj}^{(i)}$ converges uniformly in i for each n.

Throughout the paper we assume that the sequence of matrices $(A^{(i)}) = (a_{nj}^{(i)})$ satisfies the following conditions:

(i) $\mathcal{A}$ is regular,
(ii) $\|\mathcal{A}\| < \infty$,
(iii) $\lim_k \sup_{i,n} \sum_{j=k}^{\infty} |a_{n,j+1}^{(i)} - a_{nj}^{(i)}| = 0$.

2. Main results

In this section, using a sequence of infinite matrices we give a theorem analogous to one of Cohen [3].

We now present a lemma which will be used in the proof of the main theorem.

Lemma 2. *Let T and $A_n^{(i)}$ be bounded linear operators on a Banach space X into itself such that $TA_n^{(i)} = A_n^{(i)}T$ for all n and i. If*

$$\lim_{n\to\infty} A_n^{(i)}(x - Tx) = 0, \quad \textit{(uniformly in } i\textit{)}, \tag{3}$$

and

$$A_n^{(i)}x \to x_0(w), \quad (n \to \infty, \textit{ uniformly in } i),$$

then $Tx_0 = x_0$, where (w) indicates the weak convergence.

Proof. By X' we denote the dual space of X. Let $f \in X'$. Then, by weak convergence (uniformly in i) of $(A_n^{(i)}x)$ we have

$$\lim_n \sup_i f(A_n^{(i)}x - x_0) = 0. \tag{4}$$

Since T is a linear and continuous operator on X, we also have

$$\lim_n \sup_i f(TA_n^{(i)}x - Tx_0) = 0. \tag{5}$$

It follows from (3) and the fact that $f \in X'$,

$$\lim_{n\to\infty} \sup_i f(A_n^{(i)}x - A_n^{(i)}Tx) = 0. \tag{6}$$

Using the commutativity $TA_n^{(i)} = A_n^{(i)}T$ for each n and i, one may write

$$f(x_0 - Tx_0) = f(x_0 - A_n^{(i)}x) + f(A_n^{(i)}x - A_n^{(i)}Tx) + f(TA_n^{(i)}x - Tx_0). \tag{7}$$

Applying the operator $\lim\limits_n \sup\limits_i$ to both sides of (7) we get that

$$\begin{aligned}\left|\lim_n \sup_i f(x_0 - Tx_0)\right| &\le \left|\lim_n \sup_i f(x_0 - A_n^{(i)}x)\right| + \left|\lim_n \sup_i f(A_n^{(i)}x - A_n^{(i)}Tx)\right| \\ &\quad + \left|\lim_n \sup_i f(TA_n^{(i)}x - Tx_0)\right|. \end{aligned} \tag{8}$$

Then by (4), (5), (6) and (8), we conclude that $f(x_0 - Tx_0) = 0$ for all $f \in X'$. This implies that $Tx_0 = x_0$. □

We now present the main result of the paper.

Theorem 3. *Let X be a Banach space and $T : X \to X$ be a bounded linear operator. Suppose that there exists an $H > 0$ such that $\|T^j\| \le H$ for all $j \in \mathbb{N}$. Suppose that the sequence of infinite matrices $(A^{(i)}) = (a_{nj}^{(i)})$ satisfies the conditions (i)-(iii) and define $A_n^{(i)}x = \sum\limits_{j=1}^{\infty} a_{nj}^{(i)}T^jx$. Assume that there exists a subsequence $\{A_{n_p}^{(i)}x\} \subset \{A_n^{(i)}x\}$ such that*

$$\lim_p \sup_i A_{n_p}^{(i)}x = x_0(w), \tag{9}$$

where $x_0 \in X$. Then, $Tx_0 = x_0$ and $\lim_{n\to\infty} A_n^{(i)}x = x_0$ (uniformly in i). Denote by P the strong limit in $B(X)$ of $\{A_n^{(i)}x\}$. Then it is the projection onto the space $N(I-T)$ of T-fixed points corresponding to the ergodic decomposition $X = \overline{R(I-T)} \oplus N(I-T)$ and $P = P^2 = TP = PT$.

Proof. From the hypothesis there exists an $H > 0$ such that $\|T^j\| \le H$ for all $j \in \mathbb{N}$. Since $\|\mathcal{A}\| < \infty$, for $x \in X$ we have

$$\left\|A_n^{(i)}x\right\| = \left\|\sum_{j=1}^{\infty} a_{nj}^{(i)} T^j x\right\| \le H\,\|x\| \sum_{j=1}^{\infty} |a_{nj}^{(i)}| \le H\,\|x\| \sup_{n,i} \sum_{j=1}^{\infty} |a_{nj}^{(i)}| < H\,\|x\|\,\|\mathcal{A}\|. \tag{10}$$

Since X is complete, each $\{A_n^{(i)}x\}$ is defined on X. By taking supremum over $\|x\| = 1$ in both sides of (10), we get, for all n and i, that

$$\|A_n^{(i)}\| \le H\|\mathcal{A}\|. \tag{11}$$

Also we have

$$TA_n^{(i)}x = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^{j+1}x = A_n^{(i)}Tx. \tag{12}$$

By the hypothesis, we have for any $\varepsilon > 0$ that there exists a $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for all $k \ge k_0$

$$\sup_{i,n} \sum_{j=k}^{\infty} |a_{n,j+1}^{(i)} - a_{nj}^{(i)}| < \varepsilon.$$

Hence, we get, for each $x \in X$, that

$$\begin{aligned}\left\|A_n^{(i)}(x - Tx)\right\| &= \left\|a_{n1}^{(i)}Tx + \sum_{j=1}^{\infty} (a_{n,j+1}^{(i)} - a_{nj}^{(i)}) T^{j+1}x\right\| \\ &\le H\,\|x\|\,(\sup_i |a_{n1}^{(i)}| + \sup_i \sum_{j=1}^{k_0-1} |a_{n,j+1}^{(i)} - a_{nj}^{(i)}| + \sup_{i,n} \sum_{j=k_0}^{\infty} |a_{n,j+1}^{(i)} - a_{nj}^{(i)}|) \\ &\le H\,\|x\| \left(2 \sup_i \sum_{j=1}^{k_0} |a_{nj}^{(i)}| + \varepsilon\right).\end{aligned}$$

Then, for $n > n_\varepsilon$ we also have $\sup_i \sum_{j=1}^{k_0} |a_{nj}^{(i)}| < \varepsilon$ which yields

$$\left\|A_n^{(i)}(x - Tx)\right\| \le H\,\|x\|\,3\varepsilon.$$

This implies

$$\lim_{n\to\infty} A_n^{(i)}(x - Tx) = 0, \quad \text{(uniformly in } i). \tag{13}$$

Furthermore, from (9), (12) and (13), the conditions of Lemma 2 are satisfied. Thus, one can get $Tx_0 = x_0$.

Now, we consider the linear subspace X_0 spanned by $x - Tx$ for $x \in X$. We will show that $x_0 - x \in X_0$. To achieve this, we follow the idea given by Cohen [3]. Assume that $x_0 - x \notin X_0$. Then, one can easily see that there exists an $f \in X'$ such that

$$f(u) = 0, \quad u \in X_0; \quad f(x - x_0) = 1.$$

Since $T^k x - T^{k+1}x \in X_0$ for $k = 0, 1, 2, \ldots$, we have $f(T^k x - T^{k+1}x) = 0$. Then, it is easy to show that $f(x - T^j x) = 0$. So we obtain

$$f(x) = f(T^j x), \quad j = 1, 2, \ldots \tag{14}$$

Moreover, from (11) and (13), it follows that

$$\lim_n \sup_i A_n^{(i)} u = 0, \quad u \in X_0. \tag{15}$$

Since $f \in X'$, one can get by (14) that

$$f(A_n^{(i)}x) = \sum_{j=1}^{\infty} a_{nj}^{(i)} f(T^j x) = \left(\sum_{j=1}^{\infty} a_{nj}^{(i)}\right) f(x)$$

which yields

$$\lim_n \sup_i f(A_n^{(i)}x) = f(x). \tag{16}$$

By (9) and (16) we obtain

$$\begin{aligned} 0 = \lim_p \sup_i f(A_{n_p}^{(i)}x - x_0) &= \lim_p \sup_i (f(A_{n_p}^{(i)}x) - f(x_0)) \\ &= f(x) - f(x_0) = f(x - x_0). \end{aligned}$$

This is a contradiction. Then we necessarily have $x_0 - x \in X_0$. Since $Tx_0 = x_0$ we have $T^j x_0 = x_0$ for $j = 1, 2, \ldots$. Hence we have

$$A_n^{(i)}x_0 = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^j x_0 = \left(\sum_{j=1}^{\infty} a_{nj}^{(i)}\right) x_0 \tag{17}$$

from which we immediately get

$$\lim_n \sup_i A_n^{(i)} x_0 = x_0. \tag{18}$$

Since $x = x_0 + (x - x_0)$, we get from (15) and (18) that

$$\lim_n \sup_i A_n^{(i)} x = x_0,$$

which proves the first claim.

We can write $x = x_0 + (x - x_0)$ such that $x_0 \in N(I - T)$ and $(x - x_0) \in R(I - T) \subset$

$\overline{R(I-T)}$. Now let $\varepsilon > 0$ and let $z \in \overline{R(I-T)} \cap N(I-T)$. Following [4] we then have $\|z-(u-Tu)\| < \varepsilon/(3H\,\|\mathcal{A}\|)$ for $u \in X$. Hence

$$\left\|A_n^{(i)}(z-(u-Tu))\right\| < \left\|\sum_{j=1}^{\infty} a_{nj}^{(i)} T^j\right\| \|z-(u-Tu)\| < \frac{\varepsilon}{3}. \tag{19}$$

Since $z \in \overline{R(I-T)} \cap N(I-T)$, we observe that

$$A_n^{(i)} z = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^j z = \sum_{j=1}^{\infty} a_{nj}^{(i)} z \tag{20}$$

from which we get

$$\lim_n \sup_i A_n^{(i)} z = z. \tag{21}$$

By (15), (19) and (21), we conclude that

$$\|z\| = \left\|z - A_n^{(i)} z + A_n^{(i)} z\right\| \leq \left\|z - A_n^{(i)} z\right\| + \left\|A_n^{(i)}(z-(u-Tu))\right\| + \left\|A_n^{(i)}(u-Tu)\right\| < \varepsilon.$$

Hence, we find that $\overline{R(I-T)} \cap N(I-T) = \{0\}$, which implies that

$$X = \overline{R(I-T)} \oplus N(I-T).$$

On the other hand, we know that $\lim_n \sup_i A_n^{(i)} x = x_0$. Let $Px := \lim_n \sup_i A_n^{(i)} x = x_0$. Then, since $Tx_0 = x_0$ and $Px = x_0$ one can obtain, for all $x \in X$, that

$$Tx_0 = TPx = x_0 = Px,$$

which yields $TP = P$. Also, we have $T^j P = P$ for all $j \in \mathbb{N}$. Hence, we observe that

$$A_n^{(i)} Px = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^j Px = \sum_{j=1}^{\infty} a_{nj}^{(i)} Px$$

Applying the operator $\lim_n \sup_i$ to both sides we find $P^2 = P$.

In addition, from (15) we obtain $Px = PTx$ for all $x \in X$, that is $P = PT$. This concludes the proof. $\square$

Remark 4. *If we define the sequence of matrices* $(A^{(i)}) = (a_{nj}^{(i)})$ *by*

$$a_{nj}^{(i)} = \begin{cases} \dfrac{1}{n+1} & , \quad i \leq j \leq i+n, \\ 0 & , \quad \textit{otherwise} \end{cases}$$

then $\mathcal{A}$ *reduces to almost convergence method of Lorentz* [6]*. Observe that* $(a_{nj}^{(i)})$ *defined as above satisfies the conditions (i)-(iii) imposed in Section 1. Some results concerning the almost convergence of the sequence of operators may be found in* [1] *and* [7].

Given a sequence $\mathcal{A}$ of matrices $(A^{(i)}) = (a_{nj}^{(i)})$, if the limit of $\left\{A_n^{(i)}x\right\}$ exists then we call the operator T an $\mathcal{A}$-ergodic operator. Motivated by that of Proposition 2.2 in [5] we have the following

Theorem 5. *Let X be a Banach space, T be a bounded linear operator on X into itself. Assume that there exists an $H > 0$ such that $\|T^j\| \leq H$ for all $j \in \mathbb{N}$. Let $(A^{(i)}) = (a_{nj}^{(i)})$ be a sequence of infinite matrices satisfying the conditions (i)-(iii). Then, the operator T is $\mathcal{A}$-ergodic if and only if $(I-T)\overline{(I-T)X} = (I-T)X$.*

Proof. Let the operator T be $\mathcal{A}$-ergodic. Then, by Theorem 3 we have

$$X = \overline{R(I-T)} \oplus N(I-T).$$

The necessity is proved by applying the operator $(I-T)$.

Assume that $(I-T)\overline{(I-T)X} = (I-T)X$. We have, for $x \in N(I-T)$, that

$$A_n^{(i)}x = \sum_{j=1}^{\infty} a_{nj}^{(i)} T^j x = \sum_{j=1}^{\infty} a_{nj}^{(i)} x.$$

Hence, we get

$$\|A_n^{(i)}x - x\| \to 0, \quad (n \to \infty, \text{ uniformly in } i). \tag{22}$$

Now, let $x \in \overline{R(I-T)}$. Hence, there exists $x_k \in R(I-T)$ so that $x_k \to x$. One can get

$$\left\|A_n^{(i)}x\right\| \leq \left\|A_n^{(i)}x_k\right\| + \left\|A_n^{(i)}(x_k - x)\right\|.$$

If we choose k in order to make $\|x_k - x\|$ sufficiently small, we find that $\|A_n(x_k - x)\|$ is also sufficiently small (no matter what n may be) because of the fact that $\mathcal{A}$ satisfies (ii) and T is power bounded.Combining this with (15), we observe, for $x \in \overline{R(I-T)}$, that

$$\|A_n^{(i)}x\| \to 0, \quad (n \to \infty, \text{ uniformly in } i). \tag{23}$$

Thus, by (22) and (23) the sequence $\{A_n^{(i)}\}$ is strongly convergent on $\overline{R(I-T)} \oplus N(I-T)$. Since $(I-T)\overline{(I-T)X} = (I-T)X$, for $y \in X$ there exists $z \in \overline{R(I-T)}$ such that $(I-T)z = (I-T)y$. We then get $h = y - z \in N(I-T)$. Since we have $y = h + z$ such that $h \in N(I-T)$ and $z \in \overline{R(I-T)}$, the proof is completed. $\square$

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