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PAGES: 9-14

ORIGINAL PDF URL: <https://dergipark.org.tr/tr/download/article-file/772555>

THE EIGENVECTORS OF A COMBINATORIAL MATRIX

ILKER AKKUS

ABSTRACT. In this paper, we derive the eigenvectors of a combinatorial matrix whose eigenvalues studied by Kilic and Stanica. We follow the method of Cooper and Melham since they considered the special case of this matrix.

1. INTRODUCTION

In [7], Peele and Stănică studied $n \times n$ matrices with the (i, j) entry the binomial coefficient $\binom{i-1}{j-1}$, respectively, $\binom{i-1}{n-j}$ and derived many interesting results on powers of these matrices. In [8], one of them found that the same is true for a much larger class of what he called *netted matrices*, namely matrices with entries satisfying a certain type of recurrence among the entries of all 2×2 cells.

Let R_n be the matrix whose (i, j) entries are $a_{i,j} = \binom{i-1}{n-j}$, which satisfy

$$a_{i,j-1} = a_{i-1,j-1} + a_{i-1,j}. \quad (1.1)$$

The previous recurrence can be extended for $i \geq 0$, $j \geq 0$, using the boundary conditions $a_{1,n} = 1$, $a_{1,j} = 0$, $j \neq n$. Remark the following consequences of the boundary conditions and recurrence (1.1): $a_{i,j} = 0$ for $i + j \leq n$, and $a_{i,n+1} = 0$, $1 \leq i \leq n$.

The matrix R_n was firstly studied by Carlitz [2] who gave explicit forms for the eigenvalues of R_n . Let $f_{n+1}(x) = \det(xI - R_n)$ be the characteristic polynomial of R_n . Thus

$$f_n(x) = \sum_{r=0}^{n+1} (-1)^{r(r+1)/2} \binom{n}{r}_F x^{n-r}$$

where $\binom{n}{r}_F$ denote the Fibonomial coefficient, defined (for $n \geq r > 0$) by

$$\binom{n}{r}_F = \frac{F_1 F_2 \dots F_n}{(F_1 F_2 \dots F_r)(F_1 F_2 \dots F_{n-r})}$$

Received by the editors Feb. 15, 2011, Accepted: March 24, 2011.

2000 *Mathematics Subject Classification.* 11B39, 15A15, 15A18.

Key words and phrases. Fibonomial coefficients, binomial coefficients, eigenvector, Pascal matrix.

with $\binom{n}{n}_F = \binom{n}{0}_F = 1$. Carlitz showed that

$$f_n(x) = \prod_{j=0}^{n-1} (x - \phi^j \bar{\phi}^{n-j})$$

where $\phi, \bar{\phi} = (1 \pm \sqrt{5})/2$. Thus the eigenvalues of R_n are $\phi^n, \phi^{n-1}\bar{\phi}, \dots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.

In [7] it was proved that the entries of the power R_n^e satisfy the recurrence

$$F_{e-1}a_{i,j}^{(e)} = F_e a_{i-1,j}^{(e)} + F_{e+1}a_{i-1,j-1}^{(e)} - F_e a_{i,j-1}^{(e)}, \quad (1.2)$$

where F_e is the Fibonacci sequence. Closed forms for *all* entries of R_n^e were not found, but several results concerning the generating functions of rows and columns were obtained (see [7, 8]). Further, the generating function for the (i, j) -th entry of the e -th power of a generalization of R_n , namely

$$Q_n(a, b) = \left(a^{i+j-n-1} b^{n-j} \binom{i-1}{n-j} \right)_{1 \leq i, j \leq n}$$

is

$$B_n^{(e)}(x, y) = \frac{(U_{e-1} + U_e y)(b U_{e-1} + y U_e)^{n-1}}{U_{e-1} + U_e y - x(U_e + U_{e+1}y)}.$$

Regarding this generalization, in [3], the authors gave the characteristic polynomial of $Q_n(a, b)$ and the trace of k th power of $Q_n(a, b)$, that is, $\text{tr}(Q_n^k(a, b))$, by using the method of Carlitz [2]:

$$\text{tr}(Q_n^k(a, b)) = \frac{U_{kn}}{U_k}, \quad (1.3)$$

where $\binom{n}{i}_U$ stands for the generalized Fibonomial coefficient, defined by

$$\binom{n}{r}_U = \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_r)(U_1 U_2 \dots U_{n-r})},$$

for $n \geq i > 0$, where $\binom{n}{n}_U = \binom{n}{0}_U = 1$.

In [7], the authors proposed a conjecture on the eigenvalues of matrix R_n , which was proven independently in [1] and the unpublished manuscript [9]. Also, in [6], they found the eigenvectors of R_n .

In [7, 8], it was shown that the inverse of R_n is the matrix

$$R_n^{-1} = \left((-1)^{n+i+j+1} \binom{n-i}{j-1} \right)_{1 \leq i, j \leq n},$$

and, in general, the inverse of $Q_n(a, b)$ is

$$Q_n^{-1}(a, b) = \left((-1)^{n+i+j+1} a^{n+1-i-j} b^{i-n} \binom{n-i}{j-1} \right)_{1 \leq i, j \leq n}. \quad (1.4)$$

Let $\phi = \frac{1+\sqrt{5}}{2}$, $\bar{\phi} = \frac{1-\sqrt{5}}{2}$ be the golden section and its conjugate. The eigenvalues R_n are $\phi^n, \phi^{n-1}\bar{\phi}, \dots, \phi\bar{\phi}^{n-1}, \bar{\phi}^n$.

Let the sequences $\{u_n\}$, $\{v_n\}$ be defined by

$$\begin{aligned} u_n &= au_{n-1} + bu_{n-2} \\ v_n &= av_{n-1} + bv_{n-2}, \end{aligned}$$

for $n > 1$, where $u_0 = 0, u_1 = 1$, and $v_0 = 2, v_1 = a$, respectively. Let α, β be the roots of the associated equation $x^2 - ax - b = 0$. The next lemma can be found in [4].

Lemma 1.1. *For $k \geq 1$ and $n > 1$,*

$$\begin{aligned} u_{kn} &= v_k u_{k(n-1)} + (-1)^{k+1} b^k u_{k(n-2)} \\ v_{kn} &= v_k v_{k(n-1)} + (-1)^{k+1} b^k v_{k(n-2)}. \end{aligned} \quad (1.5)$$

In [5], using the sequence v_k , they defined the $n \times n$ matrix $H_n(v_k, b^k)$ as follows:

$$H_n(v_k, b^k) = \left(v_k^{i+j-n-1} \left(-(-b)^k \right)^{n-j} \binom{i-1}{n-j} \right)_{1 \leq i, j \leq n}.$$

As in equation (1.4), they also found the inverse of the matrix H_n , namely

$$H_n^{-1}(v_k, b^k) = \left((-1)^{j+1} (-b)^{-k(n-i)} v_k^{n+1-i-j} \binom{n-i}{j-1} \right)_{i,j}.$$

It is well known that for $n \geq -1$,

$$u_{n+1} = \sum_r \binom{r}{n-r} a^{2r-n} b^{n-r}. \quad (1.6)$$

Thus the authors [5] generalized this identity as well as they gave the following results:

Lemma 1.2. *For $k > 0$ and $n \geq -1$,*

$$\frac{u_{k(n+1)}}{u_k} = \sum_r \binom{r}{n-r} v_k^{2r-n} \left(-(-b)^k \right)^{n-r}.$$

Lemma 1.3. *For all $m > 0$,*

$$\text{tr}(H_n^m(v_k, b^k)) = \frac{u_{knm}}{u_k}.$$

Theorem 1.4. *The eigenvalues of $H_n(v_k, b^k)$ are*

$$\alpha^{kn}, \alpha^{k(n-1)} \beta^k, \dots, \alpha^k \beta^{k(n-1)}, \beta^{kn}.$$

2. THE EIGENVECTORS OF $H_n(v_k, b^k)$

In [6], the authors considered the matrix $Q_n(a, b)$ and gave its eigenvectors. In this section, we consider the generalization of matrix $Q_n(a, b)$ namely $H_n(v_k, b^k)$ and then determine its eigenvectors by using the method given in [6].

Let $0 \leq p \leq n-1$ be a fixed integer,

$$f(x) = (x - \alpha^k)^p (x - \beta^k)^{n-1-p} = \sum_{r=0}^{n-1} s_r x^r,$$

and

$$\mathbf{s} = (s_0, s_1, \dots, s_{n-1})^T.$$

Theorem 2.1. For $m \geq 0$

$$f^{(m)}(x) = m! \frac{f(x)}{(x - \alpha^k)^m (x - \beta^k)^m} \sum_{j=0}^m \binom{p}{m-j} \binom{n-1-p}{j} (x - \alpha^k)^j (x - \beta^k)^{m-j}.$$

Proof. It can be proved with the use of Leibniz's formula for the m -th derivative of a product of two functions. We recall the Leibniz's formula: For $m \geq 0$

$$\frac{d^m}{dx^m} g(x) h(x) = \sum_{j=0}^m \binom{m}{j} g^{(m-j)}(x) h^{(j)}(x). \quad (2.1)$$

We use the notation $x^{\underline{n}}$ to denote the falling factorial, and hence

$$\begin{aligned} f^{(m)}(x) &= \sum_{j=0}^m \binom{m}{j} p^{\underline{m-j}} (x - \alpha^k)^{p-m+j} (n-1-p)^{\underline{j}} (x - \beta^k)^{n-1-p-j} \\ &= m! \frac{f(x)}{(x - \alpha^k)^m (x - \beta^k)^m} \sum_{j=0}^m \binom{p}{m-j} \binom{n-1-p}{j} (x - \alpha^k)^j (x - \beta^k)^{m-j}, \end{aligned}$$

as claimed. \square

Lemma 2.2. Suppose that $0 \leq m \leq n-1$ be a fixed integer. Then,

$$s_{n-1-m} = \sum_{j=0}^m (-1)^m \binom{p}{m-j} \binom{n-1-p}{j} \alpha^{k(m-j)} \beta^{kj}$$

and

$$(H_n(v_k, b^k) \mathbf{s})_{n-1-m} = \sum_{r=m}^{n-1} (-1)^m (-b)^{km} \binom{r}{m} v_k^{r-m} s_r.$$

Proof. The proof of the first equation can be followed from computing the coefficient of x^{n-1-m} of $f(x)$ by multiplying $(x - \alpha^k)^p$ times $(x - \beta^k)^{n-1-p}$. The second proof can be seen by computing the product of $H_n(v_k, b^k)$ and \mathbf{s} . \square

Theorem 2.3.

$$H_n(v_k, b^k) \mathbf{s} = (\alpha^k)^{n-1-p} \beta^{kp} \mathbf{s}. \quad (2.2)$$

Proof. Consider

$$\begin{aligned} (H_n(v_k, b^k) \mathbf{s})_{n-1-m} &= \sum_{r=m}^{n-1} (-1)^m (-b)^{km} \binom{r}{m} v_k^{r-m} s_r \\ &= \frac{(-1)^m (-b)^{km}}{m!} \sum_{r=m}^{n-1} s_r r^m v_k^{r-m} \\ &= \frac{(-1)^m (-b)^{km}}{m!} f^{(m)}(v_k) \\ &= \frac{(-1)^m (-b)^{km}}{m!} \frac{(v_k - \alpha^k)^p (v_k - \beta^k)^{n-1-p}}{(v_k - \alpha^k)^m (v_k - \beta^k)^m} \times \\ &\quad m! \sum_{j=0}^m \binom{p}{m-j} \binom{n-1-p}{j} (v_k - \alpha^k)^j (v_k - \beta^k)^{m-j} \\ &= (-1)^m (\alpha\beta)^{km} \beta^{k(p-m)} \alpha^{k(n-1-p-m)} \times \\ &\quad \sum_{j=0}^m \binom{p}{m-j} \binom{n-1-p}{j} \beta^{kj} \alpha^{k(m-j)} \\ &= \alpha^{k(n-1-p)} \beta^{kp} \sum_{j=0}^m (-1)^m \binom{p}{m-j} \binom{n-1-p}{j} \alpha^{k(m-j)} \beta^{kj} \\ &= \alpha^{k(n-1-p)} \beta^{kp} s_{n-1-m}. \end{aligned}$$

Thus the proof is complete. \square

ÖZET: Bu çalışmada, Kilic ve Stanica tarafından özdeğerleri verilen bir kombinatorial matrisin özvektörleri, Cooper ve Melham'ın metodu takip edilerek elde edilmiştir.

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