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WEAK CONVERGENCE THEOREM BY A NEW EXTRAGRADIENT METHOD FOR FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

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Abstract. We introduce a new extragradient iterative process, motivated and inspired by [S. H. Khan, A Picard-Mann Hybrid Iterative Process, Fixed Point Theory and Applications, doi:10.1186/1687-1812-2013-69], for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality for an inverse strongly monotone mapping in a Hilbert space. Using this process, we prove a weak convergence theorem for the class of nonexpansive mappings in Hilbert spaces. Finally, as an application, we give some theorems by using resolvent operator and strictly pseudocontractive mapping.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, I be the idendity mapping on C, and P_C be the metric projection from H onto C.

Recall that a mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. For a mapping $A : C \to H$, it is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0,$$

L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \ \forall x, y \in C;$$

and α-inverse strongly monotone if

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$.

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Remark 1.1. It is obvious that any α-inverse strongly monotone mapping A is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Monotonicity conditions in the context of variational methods for nonlinear operator equations were used by Vainberg and Kacurovskii [1] and then many authors have studied on this subject.

In this paper, we consider the following variational inequality problem $VI(C,A)$: find a $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of $VI(C,A)$ is denoted by Ω, i.e.,

$$\Omega = \{x \in C : \langle Ax, y-x\rangle \geq 0, \forall y \in C\}.$$

In the context of the variational inequality problem it is easy to check that

$$x \in \Omega \Leftrightarrow x \in F(P_C(I-\lambda A)), \ \forall \lambda > 0.$$

Variational inequalities were initially studied by Stampacchia [2], [3]. Such a problem is connected with convex minimization problem, the complementarity problem, the problem of finding point $x \in C$ satisfying $0 \in A$ and etc. Fixed point problems are also closely related to the variational inequality problems.

For finding an element of $\mathcal{F} = F(T) \cap \Omega$, many authors have studied widely under suitable assumptions (see [4, 5, 6, 7, 8, 9]). For example, in 2006, Takahashi and Toyoda [10] introduced following iterative process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) TP_C(I-\lambda_n A)x_n, \ \forall n \geq 0, \end{cases} \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $A : C \to H$ is an α-inverse strongly monotone mapping, $P_C : H \to C$ is a metric projection, $T : C \to C$ is a nonexpansive mapping, $\{\alpha_n\} \subset [a,b]$ for some $a, b \in (0,1)$, and $\{\lambda_n\} \subset [c,d]$ for some $c, d \in (0, 2\alpha)$. They proved that if $\mathcal{F} = F(T) \cap \Omega$ is nonempty, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to some $z \in \mathcal{F}$ where $z = \lim_{n\to\infty} P_{\mathcal{F}} x_n$. In the same year, Nadezkhina and Takahashi [11] generalized the iterative process (1.1) and motivated by this process they introduced following iterative scheme for nonexpansive mapping S and monotone k-Lipschitzian mapping A

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n y_n) \\ y_n = P_C(I-\lambda_n A)x_n, \ \forall n \geq 0. \end{cases} \tag{1.2}$$

They proved the weak convergence of $\{x_n\}$ under the suitable conditions. Recently, independetly from the above processes, Khan [12] and Sahu [13], individually, introduced the following iterative process which Khan referred to as Picard-Mann hybrid iterative process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = Ty_n \\ y_n = \alpha_n x_n + (1-\alpha_n) Tx_n, \ \forall n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. Khan proved a strong and a weak convergence theorems in a Banach space for iterative process (1.3) under the suitable conditions where T is a nonexpansive mapping. Also, he proved that the iterative process given by (1.3) converges faster than the Picard, Mann and Ishikawa processes for the contraction mappings.

In this paper, motivated and inspired by the idea of extragradient method and the above processes, we introduce the following process:

$$\begin{cases} x_0 \in C \\ x_{n+1} = TP_C\left(I-\lambda_n A\right) y_n \\ y_n = \alpha_n x_n + (1-\alpha_n)\, TP_C\left(I-\lambda_n A\right) x_n, \ \forall n \geq 0, \end{cases} \tag{1.4}$$

where T is a nonexpansive mapping and P_C is a metric projection from H onto C. Our iterative process is independent from all of the above processes. Also, under the suitable conditions, we establish a weak convergence theorem.

2. Preliminaries

In this section, we collect some useful lemmas that will be used for our main result in the next section. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ for the strong convergence. It is well known that for any $x \in H$, there exists a unique point $y_0 \in C$ such that

$$\|x - y_0\| = \inf\{\|x-y\| : y \in C\}.$$

We denote y_0 by $P_C x$, where P_C is called the metric projection of H onto C. It is known that P_C has the following properties:

(i) $\|P_C x - P_C y\| \leq \|x-y\|$, for all $x, y \in H$,
(ii) $\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, for all $x \in H$, $y \in C$,
(iii) $\langle x - P_C x, y - P_C x\rangle \leq 0$, for all $x \in H$, $y \in C$,

On the other hand, it is known that a Hilbert space H satisfies the Opial condition that, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. [10] *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H. Suppose that, for all $z \in C$,*

$$\|x_{n+1} - z\| \leq \|x_n - z\|$$

for every $n = 0, 1, 2, \ldots$. Then, $\{P_C x_n\}$ converges strongly to some $u \in C$.

Lemma 2.2. [10] *Let C be a nonempty closed convex subset of a real Hilbert space H and let A be an α-inverse strongly monotone mapping of C into H. Then the set of solutions of $VI(C,A)$, Ω, is nonempty.*

For a set-valued mapping $S : H \to 2^H$, if the inequality

$$\langle f - g, u - v \rangle \geq 0$$

holds for all $u, v \in C, f \in Su, g \in Sv$, then S is called monotone mapping. A monotone mapping $S : H \to 2^H$ is maximal if the graph $G(S)$ of S is not properly contained in the graph of any other monotone mappings. It is known that a monotone mapping S is maximal if and only if, for $(u, f) \in H \times H$, $\langle u - v, f - w \rangle \geq 0$ for every $(v, w) \in G(S)$ implies $f \in Su$. Let A be an inverse strongly monotone mapping of C into H, let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\},$$

and define

$$Sv = \begin{cases} Av + N_C v & v \in C \\ \emptyset & v \notin C. \end{cases}$$

Then S is maximal monotone and $0 \in Sv$ if and only if $v \in \Omega$.

Lemma 2.3. [14] *Let C be a nonempty closed convex subset of a real Hilbert space H, and T be a nonexpansive self-mapping of C. If $F(T) \neq \emptyset$, then $I - T$ is demiclosed; that is whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

Lemma 2.4. [15] *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, \ldots,$ and let $\{x_n\}$ and $\{y_n\}$ be sequences of H such that*

$$\limsup_{n\to\infty} \|x_n\| \leq c, \quad \limsup_{n\to\infty} \|y_n\| \leq c \text{ and } \lim_{n\to\infty} \|\alpha_n x_n + (1 - \alpha_n) y_n\| = c,$$

for some $c > 0$. Then,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

3. Main result

In this section, we introduced a new extragradient method and proved that the sequence generated by this iteration method converges weakly to a fixed point of nonexpansive mapping and to a solution of variational inequality $VI(C, A)$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping and $T : C \to C$ be a nonexpansive mapping such that $\mathcal{F} = F(T) \cap \Omega \neq \emptyset$. For arbitrary initial value $x_0 \in H$, let $\{x_n\}$ be a sequence defined by (1.4) where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to a point $p \in \mathcal{F}$, where $p = \lim_{n\to\infty} P_{\mathcal{F}} x_n$.*

Proof. We devide our proof into four steps.

Step 1. Let $t_n = P_C (I - \lambda_n A) x_n$. First, we show that $\{x_n\}$ and $\{t_n\}$ are bounded sequences. Let $z \in F(T) \cap \Omega$, then, we have

$$\begin{aligned}
\|t_n - z\|^2 &= \|P_C (I - \lambda_n A) x_n - z\|^2 \\
&\leq \|(I - \lambda_n A) x_n - (I - \lambda_n A) z\|^2 \\
&= \|x_n - z - \lambda_n (Ax_n - Az)\|^2 \\
&\leq \|x_n - z\|^2 - 2\lambda_n \langle x_n - z, Ax_n - Az\rangle + \lambda_n^2 \|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 \qquad (3.1)
\end{aligned}$$

and from (3.1) we get

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|TP_C (I - \lambda_n A) y_n - z\|^2 \\
&= \|TP_C (I - \lambda_n A) y_n - TP_C (I - \lambda_n A) z\|^2 \\
&\leq \|y_n - z\|^2 \\
&= \|\alpha_n (x_n - z) + (1 - \alpha_n)(Tt_n - z)\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|Tt_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|t_n - z\|^2 \\
&\leq \alpha_n \|x_n - z\|^2 \\
&\quad + (1 - \alpha_n) \left[\|x_n - z\|^2 + \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2\right] \\
&= \|x_n - z\|^2 + (1 - \alpha_n) \lambda_n (\lambda_n - 2\alpha) \|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 + (1 - d) a (b - 2\alpha) \|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 .
\end{aligned}$$

Therefore, there exists $\lim_{n\to\infty} \|x_n - z\|$ and $Ax_n - Az \to 0$. Hence $\{x_n\}$ and $\{t_n\}$ are bounded.

Step 2. We will show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Before that, we shall show $\lim_{n\to\infty} \|Tt_n - x_n\| = 0$. From Step 1, we know that $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F(T) \cap \Omega$. Let $\lim_{n\to\infty} \|x_n - z\| = c$. Since

$$\|x_{n+1} - z\| \leq \|y_n - z\| \leq \|x_n - z\| ,$$

we get

$$\lim_{n\to\infty} \|y_n - z\| = c. \qquad (3.2)$$

On the other hand, since

$$\|Tt_n - z\| \leq \|t_n - z\| \leq \|x_n - z\| ,$$

we have

$$\limsup_{n\to\infty} \|Tt_n - z\| \leq c. \qquad (3.3)$$

Also, we know that

$$\limsup_{n\to\infty} \|x_n - z\| \leq c \tag{3.4}$$

and

$$\lim_{n\to\infty} \|y_n - z\| = \lim_{n\to\infty} \|\alpha_n (x_n - z) + (1-\alpha_n)(Tt_n - z)\| = c. \tag{3.5}$$

Hence, from (3.3), (3.4), (3.5), and Lemma 2.4 , we get that

$$\lim_{n\to\infty} \|x_n - Tt_n\| = 0. \tag{3.6}$$

We have also

$$\begin{aligned} \|x_n - y_n\| &= \|x_n - \alpha_n x_n - (1-\alpha_n) Tt_n\| \\ &= (1-\alpha_n) \|x_n - Tt_n\|. \end{aligned}$$

So, from (3.6) we obtain that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.7}$$

Since A is Lipschitz continuous, we have $Ax_n - Ay_n \to 0$.

Step 3. Next, we show that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Using the properties of metric projections, since

$$\begin{aligned} \|t_n - z\|^2 &= \|P_C (I-\lambda_n A) x_n - P_C (I-\lambda_n A) z\|^2 \\ &\leq \langle t_n - z, (I-\lambda_n A) x_n - (I-\lambda_n A) z \rangle \\ &= \frac{1}{2} \Big[\|t_n - z\|^2 + \|(I-\lambda_n A) x_n - (I-\lambda_n A) z\|^2 \\ &\quad - \|t_n - z - [(I-\lambda_n A) x_n - (I-\lambda_n A) z]\|^2 \Big] \\ &\leq \frac{1}{2} \Big[\|t_n - z\|^2 + \|x_n - z\|^2 - \|(t_n - x_n) + \lambda_n (Ax_n - Az)\|^2 \Big] \\ &= \frac{1}{2} \Big[\|t_n - z\|^2 + \|x_n - z\|^2 - \|t_n - x_n\|^2 \\ &\quad -2\lambda_n \langle t_n - x_n, Ax_n - Az \rangle - \lambda_n^2 \|Ax_n - Az\|^2 \Big], \end{aligned}$$

it follows that

$$\begin{aligned} \|t_n - z\|^2 \leq\; & \|x_n - z\|^2 - \|t_n - x_n\|^2 \\ & -2\lambda_n \langle t_n - x_n, Ax_n - Az \rangle - \lambda_n^2 \|Ax_n - Az\|^2. \end{aligned} \tag{3.8}$$

So, using the inequality (3.8) we get

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|TP_C(I-\lambda_n A)y_n - z\|^2 \\
&= \|TP_C(I-\lambda_n A)y_n - TP_C(I-\lambda_n A)z\|^2 \\
&\leq \|y_n - z\|^2 \\
&= \|\alpha_n(x_n-z)+(1-\alpha_n)(Tt_n-z)\|^2 \\
&\leq \alpha_n\|x_n-z\|^2+(1-\alpha_n)\|Tt_n-z\|^2 \\
&\leq \alpha_n\|x_n-z\|^2+(1-\alpha_n)\|t_n-z\|^2 \\
&\leq \|x_n-z\|^2-(1-\alpha_n)\|t_n-x_n\|^2 \\
&\quad -2\lambda_n(1-\alpha_n)\langle t_n-x_n, Ax_n-Az\rangle \\
&\quad -\lambda_n^2(1-\alpha_n)\|Ax_n-Az\|^2 \\
&\leq \|x_n-z\|^2-(1-d)\|t_n-x_n\|^2 \\
&\quad -2\lambda_n(1-\alpha_n)\langle t_n-x_n, Ax_n-Az\rangle \\
&\quad -\lambda_n^2(1-\alpha_n)\|Ax_n-Az\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_{n+1}-z\| = \lim_{n\to\infty}\|x_n-z\|$ and $Ax_n - Az \to 0$, we obtain

$$\lim_{n\to\infty}\|x_n-t_n\| = 0. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned}
\|Tx_n-x_n\| &\leq \|Tx_n-Tt_n\|+\|Tt_n-x_n\| \\
&\leq \|x_n-t_n\|+\|Tt_n-x_n\|.
\end{aligned}$$

So, it follows from (3.6) and (3.9) that

$$\lim_{n\to\infty}\|Tx_n-x_n\| = 0. \tag{3.10}$$

Step 4. Finally, we show that $\{x_n\}$ converges weakly to a $p \in F$. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to p. We need to show that p belongs to F. First, we show that $p \in \Omega$. From (3.9), we have $t_{n_i} \rightharpoonup p$. Let

$$Sv = \begin{cases} Av + N_C v & , \ v \in C, \\ \emptyset & , \ v \notin C. \end{cases}$$

Then S is maximal monotone mapping. Let $(v,w) \in G(S)$. Since $w - Av \in N_C v$ and $t_n \in C$, we get

$$\langle v - t_n, w - Av\rangle \geq 0. \tag{3.11}$$

On the other hand, from the definiton of t_n, we have that

$$\langle x_n - \lambda_n A x_n - t_n, t_n - v\rangle \geq 0$$

and hence,

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ax_n \right\rangle \geq 0.$$

Therefore, using (3.11), we get

$$\begin{aligned}
\langle v-t_{n_i},w\rangle &\geq \langle v-t_{n_i},Av\rangle \\
&\geq \langle v-t_{n_i},Av\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-x_{n_i}}{\lambda_{n_i}}+Ax_{n_i}\right\rangle \\
&= \left\langle v-t_{n_i},Av-Ax_{n_i}-\frac{t_{n_i}-x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v-t_{n_i},Av-At_{n_i}\rangle+\langle v-t_{n_i},At_{n_i}-Ax_{n_i}\rangle \\
&\quad -\left\langle v-t_{n_i},\frac{t_{n_i}-x_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v-t_{n_i},At_{n_i}-Ax_{n_i}\rangle-\left\langle v-t_{n_i},\frac{t_{n_i}-x_{n_i}}{\lambda_{n_i}}\right\rangle .
\end{aligned}$$

Hence, for $i\to\infty$ we have

$$\langle v-p,w\rangle\geq 0.$$

Since S is maximal monotone, we have $p\in S^{-1}0$ and hence $p\in\Omega$. Next, we show that $p\in F(T)$. From (3.10), Lemma 2.3 and by using $x_{n_i}\rightharpoonup p$, we have that $p\in F(T)$. So desired conclusion ($p\in F$) is obtained.

Now it remains to show that $\{x_n\}$ converges weakly to $p\in F$ and $p=\lim_{n\to\infty}P_F x_n$. Let assume that there exists an another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x_{n_j}\rightharpoonup p_0\in F$. We shall show that $p=p_0$. Conversely, let suppose that $p\neq p_0$. By using Opial condition, we obtain that

$$\begin{aligned}
\lim_{n\to\infty}\|x_n-p\| &= \liminf_{i\to\infty}\|x_{n_i}-p\| \\
&< \liminf_{i\to\infty}\|x_{n_i}-p_0\| \\
&= \lim_{n\to\infty}\|x_n-p_0\| \\
&= \liminf_{j\to\infty}\left\|x_{n_j}-p_0\right\| \\
&< \liminf_{j\to\infty}\left\|x_{n_j}-p\right\| \\
&= \lim_{n\to\infty}\|x_n-p\| .
\end{aligned}$$

This is a contradiction, so we get $p=p_0$. This implies that $x_n\rightharpoonup p\in F$.

Finally, we need to show $p=\lim_{n\to\infty}P_F x_n$. Since $p\in F$, we have

$$\langle p-P_F x_n,P_F x_n-x_n\rangle\geq 0.$$

By Lemma 2.1, $\{P_F x_n\}$ converges strongly to $u_0\in F$. Then, we get

$$\langle p-u_0,u_0-p\rangle\geq 0,$$

and hence $p=u_0$. So, proof is completed. □

Corollary 1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping such that $\Omega \neq \emptyset$. For arbitrary initial value $x_0 \in H$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = P_C\left(I - \lambda_n A\right) y_n \\ y_n = \alpha_n x_n + (1-\alpha_n) P_C\left(I - \lambda_n A\right) x_n, \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then, the sequence $\{x_n\}$ converges weakly to a point $p \in \Omega$ where $p = \lim_{n\to\infty} P_\Omega x_n$.

4. Applications

Let $B : H \to 2^H$ be a maximal monotone mapping. The resolvent of B of order $r > 0$ is the single valued mapping $J_r^B : H \to H$ defined by

$$J_r^B x = (I + rB)^{-1} x$$

for any $x \in H$. It is easy to check that $F\left(J_r^B\right) = B^{-1}0$. Moreover, the resolvent J_r^B is a nonexpansive mapping. So, we can give the following theorem.

Theorem 4.1. *Let H be a real Hilbert space. Let $\alpha > 0$, $A : H \to H$ be an α-inverse strongly monotone mapping and $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. For arbitrary initial value $x_0 \in H$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = J_r^B\left(y_n - \lambda_n A y_n\right) \\ y_n = \alpha_n x_n + (1-\alpha_n) J_r^B\left(x_n - \lambda_n A x_n\right), \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,2\alpha)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then the sequence $\{x_n\}$ converges weakly to a point $p \in A^{-1}0 \cap B^{-1}0$ where $p = \lim_{n\to\infty} P_{A^{-1}0\cap B^{-1}0} x_n$.

Proof. We have $A^{-1}0 = VI(H,A)$, $F\left(J_r^B\right) = B^{-1}0$ and $P_H = I$. Since the resolvent J_r^B is a nonexpansive mapping, we obtain the desired conclusion. $\square$

Now, we give a theorem for a pair of nonexpansive mapping and strictly pseudocontractive mapping. A mapping $S : C \to C$ is called k- strictly pseudocontractive mapping if there exists k with $0 \leq k < 1$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\left\|(I-S)x - (I-S)y\right\|^2$$

for all $x, y \in C$. Let $A = I - S$. Then, it is known that the mapping A is inverse strongly monotone mapping with $(1-k)/2$, i.e.,

$$\langle Ax - Ay, x - y\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2.$$

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping and $S : C \to C$ be a k- strictly pseudocontractive mapping such that $F(T) \cap F(S) \neq \emptyset$. For arbitrary initial value $x_0 \in H$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = T\left(\left(I - \lambda_n\right) y_n + \lambda_n S y_n\right) \\ y_n = \alpha_n x_n + \left(1 - \alpha_n\right) T\left(\left(I - \lambda_n\right) x_n + \lambda_n S x_n\right), \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1 - k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequence $\{x_n\}$ converges weakly to a point $p \in F(T) \cap F(S)$ where $p = \lim_{n\to\infty} P_{F(T)\cap F(S)} x_n$.

Proof. Let $A = I - S$. Then, we know that A is inverse strongly monotone mapping. Also, It is clear that $F(S) = VI(C, A)$. Since, A is a mapping from C into itself, we get

$$\left(I - \lambda_n\right) x_n + \lambda_n S x_n = x_n - \lambda_n \left(I - S\right) x_n = P_C\left(x_n - \lambda_n A x_n\right).$$

So, from Theorem 3.1, we obtain the desired conclusion. □

Theorem 4.3. *Let H be a real Hilbert space. Let $\alpha > 0$, $A : H \to H$ be an α-inverse strongly monotone mapping and $T : H \to H$ be a nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. For arbitrary initial value $x_0 \in H$, let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_{n+1} = T\left(y_n - \lambda_n A y_n\right) \\ y_n = \alpha_n x_n + \left(1 - \alpha_n\right) T\left(x_n - \lambda_n A x_n\right), \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequence $\{x_n\}$ converges weakly to a point $p \in VI(F(T), A)$ where $p = \lim_{n\to\infty} P_{F(T)\cap A^{-1}0} x_n$.

Proof. We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. Also, it is clear that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. So, by Theorem 3.1, we get the desired conclusion. □

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