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# MODELING DEPENDENT FINANCIAL ASSETS BY DYNAMIC COPULA AND PORTFOLIO OPTIMIZATION BASED ON CVAR

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Abstract. This paper is concerned with the statistical modeling of the dependence structure of multivariate financial data using copula. Since financial data is greatly affected by the economic factors, it often varies according to the time. Therefore, dynamic copula model is used that takes into account the time-varying. In addition, portfolio optimization based on Mean-CVaR model is applied with Monte Carlo simulation. As an application, a portfolio with four different Indexes is constructed from the Turkish financial markets. The marginal distributions of assets in the portfolio are estimated and parameter estimates are given for the different copula models. The portfolio optimization based on CVaR is made for the portfolio created from the specified copula model.

## 1. INTRODUCTION

Modern Portfolio Theory is an approach which had emerged after the 1950s and has enabled the creation of portfolios, taking into account the relationships between assets. Markowitz [1] presented the mean variance model and laid the foundations of modern portfolio theory. Portfolio risk is measured by variance in the mean variance model. When the return data is not distributed normally, it is more appropriate to use other risk measurements instead of measuring risk by variance. Value at Risk (VaR) and Conditional Value at Risk (CVaR) are the most known risk measurements in the field of finance. In the first studies carried out in finance, VaR was used as risk measurement. However, VaR could not ensure the attributes of being subadditive and convex, which are required in a risk measurement. It is of importance to ensure these attributes, particularly in a portfolio optimization. CVaR has been recommended as opposed to VaR, since it can ensure these attributes [2]. In this study, risk measurement was adopted as CVaR, and the mean-CVaR model was used for portfolio optimization.

In order to estimate the portfolio CVaR in a more accurate way, the nonlinear dependency between the tails of return on assets should be revealed. In this study,

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the copula model that was first mentioned in Sklar [3] was used in order to model the dependency structure between the return series. However, since the financial data exploited varied with respect to time, the dynamic copula model was applied to the data instead of a known copula model [4].

In order to conduct a portfolio optimization based on CVaR , the copula parameter should primarily be estimated. In the literature, there are different methods for the copula parameter estimation. The Inference Function for Margins (IFM) method, based on the marginal distribution, Kendall's $\tau$ based on sequence independent of marginal functions and Maximum Pseudo Likelihood (MPL) method, are the most known estimation methods. In this study, Two-Stage IFM method was used. In this method, the parameter of marginal distribution is estimated in the first phase, and copula parameter is estimated in the second phase [5]. For the test of marginal and copula goodness of fit, AIC and BIC criteria were applied. Finally, portfolio optimization based on the CVaR risk measurement was made for the data generated from this model with the Monte Carlo simulation method, and the weights in the portfolio were determined.

Comprehensive information regarding copula is available in the study of Joe [6] and Nelsen [7].

In the field of finance, copula was first mentioned by Embrechts et al. [8]. The first applications of this subject were given in Cherubini [5]. Jondeau and Rockinger [9] used the Copula-GARCH model in order to determine the dependence structure among stocks. Other studies that included this model have been presented in Wei and Zhang [10], Ozun and Cifter [11], and Huang et al. [12]. Optimization studies based on CVaR were recommended in Uryasev and Rockafellar [2] for the first time. The application of the copula-GARCH model for portfolio risk analysis was given in Wu and Chen [13], and Wang et al. [14]. He and Li [4] have conducted studies regarding the dynamic copula model based on CVaR and copula.

In the second part of the study, the dynamic copula model is introduced, and methods to find the estimations for parameters are given. In the third part, the definition of CVaR risk measurement is presented nd portfolio optimization based on CVaR is formed. In the fourth part, for a sample of Turkish financial data is tested and the results are presented. Finally, in the fifth part, the results are discussed and further studies that could be conducted in the future are mentioned.

## 2. DYNAMIC COPULA MODEL AND PARAMETER ESTIMATION

Copula functions are frequently used in modeling the dependency structure among the risk variables in the field of finance and actuary. Sklar [3] states that there is only one expression of an n-dimensional $C(\cdot, \ldots \cdot)$ copula for any continuous $(X_1, ..., X_n)$ random vector:

$$F\left(x_1, ...x_n\right) = C\left(F_1\left(x_1\right), ..., F_N\left(x_n\right)\right) \tag{2.1}$$

where $F_1(.), ..., F_N(.)$ and $F(., ..., .)$ respectively show the marginal and joint distribution functions of $x_1, x_2, ..., x_n$ random variables.

In this study, static copula models (Gaussian, Student-t, Clayton, and Symmetrized Joe Clayton-SJC) and corresponding dynamic copula models (GDCC, tDCC, tvC, tvSJC) are used. The static copulas exploited are defined as follows [5, 7, 15]

Table 1. Bivariate copula functions and parameter space of the considered copulas

| Copula | $C(u_1, u_2)$ | Parameter Space |
|---|---|---|
| Gaussian | $\Phi_\rho\left(\phi^{-1}(u_1), \phi^{-1}(u_2)\right) = \int_{-\infty}^{\phi^{-1}(u_1)}\int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right) dsdt$ | $\rho \in (-1,1)$ |
| Student's t | $t_{d,\rho}\left(t_d^{-1}(u_1), t_d^{-1}(u_2)\right) = \int_{-\infty}^{t_d^{-1}(u_1)}\int_{-\infty}^{t_d^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1+\frac{s^2-2\rho st+t^2}{d(1-\rho^2)}\right) dsdt$ | $\rho \in (-1,1)$ $d \in (0,\infty)$ |
| Clayton | $\left[u_1^{-w} + u_2^{-w} - 1\right]^{-\frac{1}{w}}$ | $w \in (0,\infty)$ |
| SJC | $\frac{1}{2}\left(C_{JC}\left(u_1, u_2 \vert \tau^U, \tau^L\right) + C_{JC}\left(1-u_1, 1-u_2 \vert \tau^L, \tau^U\right) + u_1 + u_2 - 1\right)$ | Upper tail: $\tau^U \in (0,1)$ Lower tail: $\tau^L \in (0,1)$ |

Also note that $\phi^{-1}$and $t_d^{-1}$are the inverse of the Normal and t-student c.d.f., where the parameters $\rho$ and $d$ are the coefficient of linear correlation and the degrees of freedom, respectively. Joe-Clayton copula is defined as $C_{JC}\left(u_1, u_2|\tau^U, \tau^L\right) = 1 - \left(1 - \left\{[1-(1-u_1)^\kappa]^{-\gamma} + [1-(1-u_2)^\kappa]^{-\gamma} - 1\right\}^{-1/\gamma}\right)^{1/\kappa}$, where $\kappa = \frac{1}{\log_2(2-\tau^U)}$ and $\gamma = \frac{1}{\log_2(\tau^L)}$

The used dynamic copula are created by applying the static copulas which are defined in Table 1. According to the Sklar [3] theorem, the dynamic copula model is expressed as below [11].

$$F\left(X_{1t}, ..., X_{nt} \mid \xi_t\right) = C_t\left(F_{1t}\left(X_{1t} \mid \xi_t\right), F_{2t}\left(X_{2t} \mid \xi_t\right) ..., F_{nt}\left(X_{nt} \mid \xi_t\right)\right) \qquad (2.2)$$

where $\xi_t = \sigma\left\{X_{1t-1}, X_{2t-2}, ..., X_{nt-t}, ...\right\}$ for $t = 1, 2, ..., T$ represents the historical data until t time.

The first step to create the joint distribution of portfolio assets with the dynamic copula method is generating the marginal distribution of each asset. In the literature, since the financial data varies according to the time, it is stated that marginal distributions are in compliance with the GARCH model, which was first run by Bollerslev [16]. The GJR model was acquired by adding the dummy variable to the GARCH model.

$GARCH(1,1)-n$ and $GARCH(1,1)-t$ models can be expressed as below:

$$\begin{aligned} x_t &= \mu + a_t \\ a_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2 \\ \varepsilon_t &\sim N(0,1) \text{ or } \varepsilon_t \sim t_d \end{aligned} \tag{2.3}$$

Here, provided that $\mu$ indicates conditional mean of return series, and $\sigma_{t-1}^2$ indicates conditional variance, it is defined as below. Moreover, $\Omega_{t-1}$ indicates the information set and $d$ indicates the degree of freedom.

$$\begin{aligned} \mu &= E(x_t) = E(E(x_t|\Omega_{t-1})) = E(\mu_t) = \mu \\ \sigma_t^2 &= Var(x_t|\Omega_{t-1}) = Var(a_t|\Omega_{t-1}) \\ \alpha_0 &> 0, \alpha_1 \geq 0, \beta \geq 0 \text{ ve } \alpha_1 + \beta < 1 \end{aligned}$$

GJR-n and GJR- t models are defined as:

$$\begin{aligned} x_t &= \mu + a_t \\ a_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma s_{t-1} a_{t-1}^2 \\ \varepsilon_t &\sim N(0,1) \text{ or } \varepsilon_t \sim t_d \end{aligned} \tag{2.4}$$

where,$s_{t-1} = \begin{cases} 1, & a_{t-1} < 0 \\ 0, & a_{t-1} \geq 0 \end{cases}$ is a dummy variable which is equal to 0 as opposed to 1 while $\varepsilon_t$ is negative. Moreover, parameter intervals are given as $\alpha_0 > 0, \alpha_1 \geq 0, \beta \geq 0$ , $\beta + \gamma \geq 0$ and $\alpha_1 + \beta + \frac{1}{2}\gamma < 1$.

The parameters of the GARCH and GJR models are estimated with MLE method. The joint density function can be expressed as

$f(a_1, ..., a_t) = f(a_t|\Omega_{t-1}) f(a_{t-1}|\Omega_{t-2}) ... f(a_1|\Omega_0) f(a_0)$

where $\Omega_{t-1} = \{a_0, a_1, ..., a_{t-1}\}$. Given data $a_1, ..., a_t$, the log-likelihood function is given as:

$$LL = \sum_{k=0}^{n-1} f(a_{n-k}|\Omega_{n-k-1})$$

Here, the conditional marginal distribution of $X_{t+1}$ for the $GARCH(1,1)$ model is defined as follows:

$$\begin{aligned} P(X_{t+1} \leq x|\Omega_t) &= P(a_{t+1} \leq (x-\mu)|\Omega_t) \\ &= P\left(\varepsilon_{t+1} \leq \frac{(x-\mu)}{\sqrt{\alpha_0 + \alpha_1 a_t^2 + \beta\sigma_t^2}}|\Omega_t\right) \\ &\begin{cases} N\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2}}|\Omega_t\right), & \varepsilon \sim N(0,1) \\ t_d\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2}}|\Omega_t\right), & \varepsilon \sim t_d \end{cases} \end{aligned}$$

For the $GJR(1,1)$ model:

$$\begin{aligned} P\left(X_{t+1} \leq x|\Omega_t\right) &= P\left(a_{t+1} \leq (x-\mu)\,|\Omega_t\right) \\ &= P\left(\varepsilon_{t+1} \leq \frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t a_t^2}}|\Omega_t\right) \\ & \begin{cases} N\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t a_t^2}}|\Omega_t\right), & \varepsilon \sim N(0,1) \\ t_d\left(\frac{(x-\mu)}{\sqrt{\alpha_0+\alpha_1 a_t^2+\beta\sigma_t^2+\gamma s_t a_t^2}}|\Omega_t\right), & \varepsilon \sim t_d \end{cases} \end{aligned}$$

The copula parameter should also be estimated in order to conduct portfolio optimization based on CVaR. Therefore, the IFM method that is composed of two phases based on the method of Maximum Likelihood Estimation (MLE) was used. The IFM estimates the marginal distribution parameters separately from the copula parameters. The estimation procedure of the IFM method consists of two steps [5].

In the first step, the parameters of the marginals are estimated:

$$\widehat{\theta}_1 = \arg\max \theta_1 \sum_{t=1}^{T}\sum_{j=1}^{n} \ln f_j\left(x_{jt};\theta_1\right). \tag{2.5}$$

In the second step, the parameter of the copula model is estimated, given $\theta_1$:

$$\widehat{\theta}_2 = \arg\max \theta_2 \sum_{t=1}^{T} \ln c\left(F_1\left(x_{1t}\right), F_2\left(x_{2t}\right), ..., F_n\left(x_{nt}\right); \theta_2, \widehat{\theta}_1\right). \tag{2.6}$$

The IFM estimator is defined as(Cherubini et.al.,[5])

$$\widehat{\theta}_{IFM} = \left(\widehat{\theta}_1, \widehat{\theta}_2\right)' \tag{2.7}$$

To make the goodness of fit tests for marginal and copula, AIC (Akaike Information Criterion) [17] and BIC (Bayesian Information Criterion) [18, 19] criteria were used. AIC and BIC values are,

$$AIC = -2 * LL + 2k \tag{2.8}$$

$$BIC = -2 * LL + \ln(n) * k \tag{2.9}$$

where LL is the log-likelihood at its maximum point of the model estimated and k is the number of copula parameters in the model. According to these criteria, the best choice is the model with minimum AIC or BIC value.

## 3. PORTFOLIO OPTIMIZATION BASED ON CVAR

CVaR measurement was introduced to the literature by the development of VaR measurement by Rockafellar and Uryasev [2]. Let $x = (x_1, x_2, ..., x_n)^T$ vector be a portfolio with n assets, and let $y = (y_1, y_2, ..., y_m)^T$ be m type loss factor with $p(y)$ density distribution. The loss factor of portfolio $f(x, y)$ is given as follows:

$$\Psi(x,\alpha) = \int_{f(x,y)\leq\alpha} p(y)dy, \qquad \alpha \in R$$

For the given $\beta \in (0, 1)$ confidence level

$$\begin{aligned}\alpha_\beta(x) &= \min\{\alpha \in R : \Psi(x,\alpha) \geq \beta\}\\ \Phi_\beta(x) &= E\left[f(x,y)|f(x,y) \geq \alpha_\beta(x)\right]\\ &= \frac{1}{1-\beta}\int_{f(x,y)\geq\alpha_\beta(x)} f(x,y)p(y)dy\end{aligned}$$

$\alpha_\beta(x)$ and $\Phi_\beta(x)$ show $VaR$ and $CVaR$ at $\beta$ confidence level, respectively. Rockafellar and Uryasev [2] converted the minimization problem of $CVaR$ to that of $F_\beta(x, \alpha)$ . Here, $F_\beta(x, \alpha)$ is a constantly differentiable function, which is composed of convex composition of $VaR$ and $CVaR$.

$$F_\beta(x,\alpha) = \alpha + \frac{1}{1-\beta}\int_{y\in R^m} [f(x,y)-\alpha]^+ \, p(y)dy \tag{3.1}$$

Since the analytical expression of $p(y)$ is difficult, or it is not known generally in practice, y's are derived through the Monte Carlo simulation method. Accordingly, the (3.1) equation can be written as follows:

$$\widetilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{m(1-\beta)}\sum_{j=1}^{m}\left[-x^T y_j - \alpha\right]^+ \tag{3.2}$$

Here, provided that $m$ indicates the number of simulations, $x = (x_1, x_2, ..., x_n)$ indicates the weights of assets in the portfolio, and $y_j = (y_{j1}, y_{j2}, ..., y_{jn})$ indicates the derived returns, the portfolio selection model based on $CVaR$ optimization is given as below:

$$\begin{aligned}\min \widetilde{F}_\beta(x,\alpha) &= \alpha + \frac{1}{m(1-\beta)}\sum_{j=1}^{m} z_j\\ z_j &= \left[-x^T y_j - \alpha\right]^+\\ &\left\{\begin{array}{c} x^T y_j + \alpha + z_j \geq 0\\ z_j \geq 0\\ \frac{1}{q}x^T\sum_{j=1}^{m} y_j \geq \rho\\ \sum_{i=1}^{m} x_i = 1, \quad x \geq 0\end{array}\right\}\end{aligned} \tag{3.3}$$

Here, $\rho$ is the return expected by investor. $CVaR$ optimization problem can be solved as a linear programming problem. Returns generated from the dynamic copula model are obtained from the Monte Carlo simulation.

## 4. NUMERICAL EXAMPLE

In this study, in order to demonstrate the performance of the copula-GARCH model, 1649 daily stock final quotations between January 4, 2007 and August 1, 2013, as well as USD and EURO currency data, were used. Stock final quotations were obtained from the website of BIST [20], and USD and EURO currency data were retrieved from the website of the Turkish Central Bank [21]. Daily market returns of BIST30, BIST100, USD and EURO are shown in Figure 1.

When $r_t$ indicates the returns, it is given as:

$$r_t = 100 \times \ln\left(\frac{P_t}{P_{t-1}}\right)$$

Here $P_t$ indicates the index value at t time. The existence of volatility clustering is seen in Figure 1. Therefore, the results are given in Table 2, after quadratic returns that are named as ARCH effects are tested with the Engle test (LM(4), LM(6), LM(8), and LM(10)) to reveal whether there is a relation to the series.

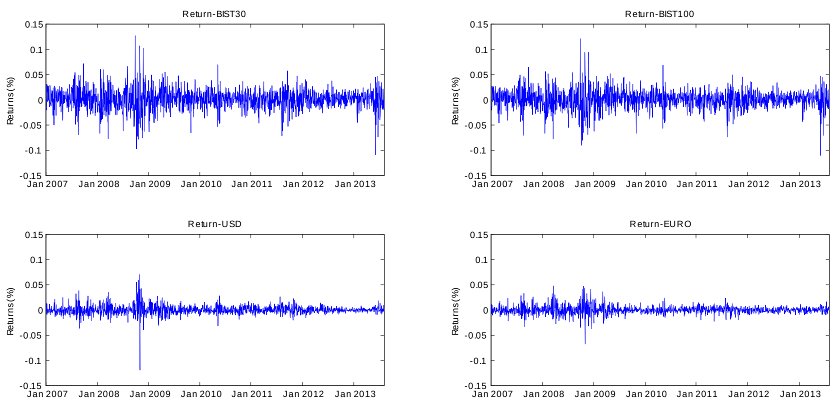


Figure 1. Daily returns of BIST30, BIST100, USD and EURO

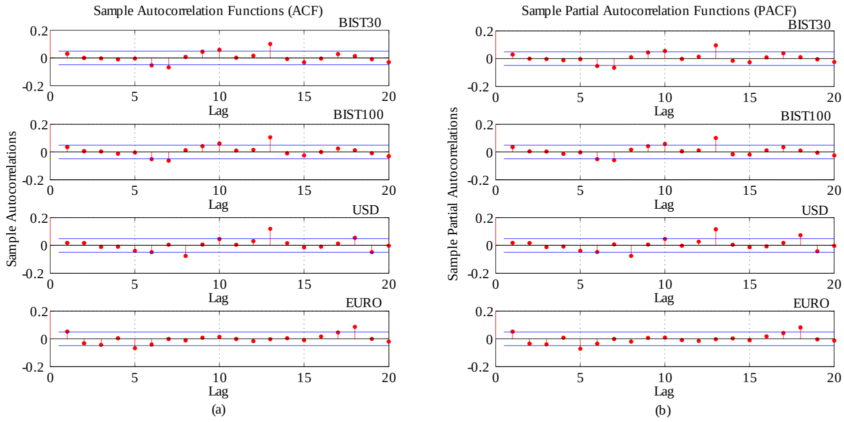


Figure 2. Autocorrelations of BIST30, BIST100, USD and EURO indexes (a) and partial autocorrelation measurements (b)

Table 2. Descriptive statistics and tests

| Statistics | BIST30 | | BIST100 | | USD | | EURO | |
|---|---|---|---|---|---|---|---|---|
| Sample Number | 1646 | | 1646 | | 1646 | | 1646 | |
| Mean | 0.000462 | | 0.000479 | | 0.000191 | | 0.000195 | |
| Standart deviation | 0.0200 | | 0.0185 | | 0.0090 | | 0.0081 | |
| Skewness | -0.1174 | | -0.2545 | | -0.5112 | | 0.2224 | |
| Kurtosis | 3.7106 | | 4.1608 | | 24.03122 | | 7.4139 | |
| Tests | Q-stat | p-value | Q-stat | p-value | Q-stat | p-value | Q-stat | p-value |
| Jarque-Bera | 940.51 | 0.0010 | 1195.9 | 0.0010 | 39427 | 0.0010 | 3756.8 | 0.0010 |
| LM(4) | 118.5178 | 0.0000 | 124.0513 | 0.0000 | 227.9964 | 0.0000 | 191.0501 | 0.0000 |
| LM(6) | 129.5706 | 0.0000 | 132.7788 | 0.0000 | 228.3458 | 0.0000 | 199.3508 | 0.0000 |
| LM(8) | 160.8098 | 0.0000 | 152.2154 | 0.0000 | 286.3049 | 0.0000 | 258.9005 | 0.0000 |
| LM(10) | 166.1320 | 0.0000 | 155.6451 | 0.0000 | 287.7057 | 0.0000 | 268.2120 | 0.0000 |

Table 2 consists of summary statistics of financial returns and statistical test information in relation to ARCH effects. Thus, it is revealed that BIST30, BIST100, and USD have negative skewness (-0.1174, -0.2545 and -0.5112, respectively), the EURO has a positive skewness (0.2224), and marginal are not distributed normally according to JB statistics ($p\ value < 0.05$). It should be looked to the Engle test results to determine whether there is correlation between the data related to each financial return. According to the Engle test results, LM statistics indicate that there are ARCH effects ($p\ value < 0.05$). Furthermore, the correlations for raw returns in Figure 2 were evaluated with autocorrelation functions (ACF) and partial autocorrelation functions (PACF). Here, it was observed that all autocorrelations are not zero after zero lag. Since financial returns are considered to be time series data, their marginal distributions were regarded as GARCH and GJR models to conduct copula estimation. Another result that supports the selection of these models is that kurtosis coefficients that were computed for each of the financial returns were found to be greater than 3. As a consequence, empiric observations in regard to returns have greater leptokurtosis than the normal distribution. Goodness of fit for the distribution with regard to the residual series of each of the financial returns (BIST30, BIST100, USD, and EURO) were made for the GARCH and GJR models by selecting Normal and Student-t, respectively.

Table 3 and Table 4 respectively show maximum likelihood results, estimated parameter values, AIC (Akaike Information Criterion) and BIC (Bayesian Information Criterion) values for GARCH and GJR models selection.

In order to determine the model that best explains financial returns, minimum AIC and BIC values should be sought in the tables. Table 3 and Table 4 demonstrate that the GJR-t (AIC:-8612.8174,-8895.0195) model can be used for BIST 30 and BIST 100, and the GARCH-t (AIC: -11681.6623, -11889.9500) model can be used for USD and EURO.

An appropriate copula model was selected by using the estimated parameter values for the identified marginal distributions. Four copula functions were created for the application. The copula selection model was examined in two situations: static (Gaussian, Student-t, Clayton, SJC) and dynamic (GDCC, tDCC, tvC, tvSJC). Table 5 demonstrates the parameter estimation results for the IFM method in regard to these situations, and AIC and BIC values. From the table, it is understood that in both situations, the Student-t copula is the best copula model in order to explain the dependency structure of financial returns with four variables. When examined separately for static and dynamic situations, the dependency structure of GJR marginal distributions with the minimum AIC (-8831.6529) in a dynamic situation was modeled ideally with tDCC copula.

It was identified that financial data marginals that were used fit GJR-t, and joint dependency structure fit the tDCC dynamic copula model. By using the parameter values for selected model, the Monte Carlo simulation was performed 10,000 times, and portfolio optimization was achieved based on CVaR risk measurement. VaR and CVaR risk measurement values for different confidence levels, and results including the weights for each financial asset are given in Table 6. Accordingly, mean loss above a certain value, that is, CVAR value would reach the minimum when an investor allocates 35% of his/her assets in BIST30, 15% in BIST100, 30% in USD, and 20% in EURO within a 99% confidence level.

Table 3. Parameter estimates of GARCH-n and GARCH-t models and statistic tests

GARCH-n

| | BIST30 | | BIST100 | | USD | | EURO | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Value | Std | Value | Std | Value | Std | Value | Std |
| $\mu$ | 0.0015 | 0.0000 | 0.0016 | 0.0000 | 0.0001 | 0.0000 | 0.0000 | 0.0000 |
| $\alpha_0$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\alpha_1$ | 0.1198 | 0.0320 | 0.1325 | 0.0380 | 0.0994 | 0.0150 | 0.1028 | 0.0400 |
| $\beta$ | 0.8548 | 0.0390 | 0.8452 | 0.0410 | 0.9005 | 0.0150 | 0.8970 | 0.0380 |
| LL | 4274.4950 | | 4407.2970 | | 5816.9220 | | 5917.6170 | |
| AIC | -8540.9905 | | -8806.5938 | | -11625.8442 | | -11827.2338 | |
| BIC | -8519.3661 | | -8784.9694 | | -11604.2198 | | -11805.6094 | |

GARCH-t

| | BIST30 | | BIST100 | | USD | | EURO | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Value | Std | Value | Std | Value | Std | Value | Std |
| $\mu$ | 0.0013 | 0.0000 | 0.0014 | 0.0000 | -0.0001 | 0.0000 | 0.0000 | 0.0000 |
| $\alpha_0$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| $\alpha_1$ | 0.1080 | 0.0340 | 0.1237 | 0.0420 | 0.1000 | 0.0170 | 0.1471 | 0.0390 |
| $\beta$ | 0.8613 | 0.0500 | 0.8485 | 0.0510 | 0.8985 | 0.0170 | 0.8529 | 0.0400 |
| $d$ | 8.0756 | 1.825 | 7.3133 | 1.7420 | 8.4006 | 1.2500 | 4.6384 | 0.2690 |
| LL | 4305.663 | | 4446.708 | | 5845.831 | | 5949.975 | |
| AIC | -8601.3251 | | -8883.4169 | | **-11681.6623** | | **-11889.9500** | |
| BIC | -8574.2946 | | -8856.3854 | | -11654.6317 | | -11862.9194 | |

Table 4. Parameter estimates of GJR-n and GJR-t models and statistic tests

GJR-n

| | BIST30 | | BIST100 | | USD | | EURO | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Value | Std | Value | Std | Value | Std | Value | Std |
| $\mu$ | 0.0010 | 0.0000 | 0.0014 | 0.0000 | 0.0000 | 4825134.081 | -0.0001 | 0.001 |
| $\alpha_0$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 15973.427 | 0.0000 | 0.0000 |
| $\alpha_1$ | 0.0480 | 0.0015 | 0.0502 | 0.0140 | 0.2017 | 683963044.071 | 0.2889 | 0.0000 |
| $\beta$ | 0.8501 | 0.0510 | 0.8487 | 0.0400 | 0.6650 | 675977556.066 | 0.5453 | 1.2460 |
| $\gamma$ | 0.1255 | 0.0610 | 0.1490 | 0.0610 | 0.0000 | 5416656809.538 | 0.1305 | 1.9300 |
| LL | 4285.5600 | | 4416.5190 | | 5756.1670 | | 5862.2690 | |
| AIC | -8561.1200 | | -8823.0377 | | -11502.3339 | | -11714.5376 | |
| BIC | -8534.0895 | | -8796.0071 | | -11475.3034 | | -11687.5071 | |

GJR-t

| | BIST30 | | BIST100 | | USD | | EURO | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Value | Std | Value | Std | Value | Std | Value | Std |
| $\mu$ | 0.0010 | 0.0000 | 0.0011 | 0.0000 | -0.0002 | 1406475.109 | 0.0000 | 28078131.946 |
| $\alpha_0$ | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 8879.443 | 0.0000 | 221202.851 |
| $\alpha_1$ | 0.0532 | 0.0170 | 0.0618 | 0.0180 | 0.3678 | 276992226.475 | 0.2539 | 29560250106.651 |
| $\beta$ | 0.8371 | 0.0700 | 0.8198 | 0.0610 | 0.6226 | 426936136.846 | 0.7147 | 2212994007.407 |
| $\gamma$ | 0.1294 | 0.0800 | 0.1291 | 0.0660 | 0.0000 | 2473886510.168 | 0.0003 | 47746730949.042 |
| $d$ | 7.9437 | 1.5840 | 7.2247 | 1.2100 | 5.2103 | 5946045507.02 | 3.9473 | 13182520678.426 |
| LL | 4312.1270 | | 4453.5100 | | 5810.7600 | | 5937.0560 | |
| AIC | **-8612.2540** | | **-8895.0195** | | -11609.5192 | | -11862.1122 | |
| BIC | -8579.8174 | | -8862.5829 | | -11577.0826 | | -11829.6756 | |

Table 5. Parameter estimates for static and dynamic copula families and model selection statistic

| Static Copula | Parameter | GARCH-n | GARCH-t | GJR-n | GJR-t |
|---|---|---|---|---|---|
| Gaussian | $\rho$ | 0.0221 | 0.0297 | 0.0179 | 0.0360 |
| | $d$ | 0.9731 | 0.9686 | 0.9664 | 0.9611 |
| | LL | 4351.2600 | 4329.1890 | 4055.8200 | 4347.8940 |
| | AIC | -8704.5208 | -8654.3780 | -8107.6392 | -8691.7872 |
| | BIC | -8693.7086 | -8643.5658 | -8096.8270 | -8680.9750 |
| Student-t | $d$ | 6.5491 | 8.0109 | 23.9706 | 5.5313 |
| | LL | 4290.433 | 4243.688 | 3981.715 | 4330.352 |
| | AIC | -8578.8664 | -8485.9692 | -7961.4291 | -8658.7042 |
| | BIC | -8573.4603 | -8479.9692 | -7956.0230 | -8653.2981 |
| Clayton | $\omega$ | 0.85 | 0.85 | 0.85 | 0.85 |
| | LL | 3103.124 | 3103.277 | 2833.643 | 3101.575 |
| | AIC | -6204.2483 | -6204.5538 | -5665.2859 | -6201.1500 |
| | BIC | -6198.8422 | -6199.1477 | -5659.8798 | -6195.7439 |
| SJC | $\lambda$ | 0.5353 0.5322 | 0.8109 0.8144 | 0.5348 0.5388 | 0.8299 0.8319 |
| | LL | 1413.471 | 2299.055 | 1443.225 | 2424.352 |
| | AIC | -2822.9424 | -4594.1105 | -2882.4507 | -4844.7030 |
| | BIC | -2812.1302 | -4583.2983 | -2871.6385 | -4833.8908 |

| Dynamic Copula | Parameter | GARCH-n | GARCH-t | GJR-n | GJR-t |
|---|---|---|---|---|---|
| GDCC | $\alpha$ | 0.0316 | 0.0297 | 0.0179 | 0.036 |
| | $\beta$ | 0.9663 | 0.9686 | 0.9663 | 0.9611 |
| | LL | 4358.363 | 4329.189 | 4055.820 | 4347.894 |
| | AIC | -8712.7252 | -8654.3780 | -8107.6391 | -8691.7870 |
| | BIC | -8701.9130 | -8643.5658 | -8096.8269 | -8680.9748 |
| tDCC | $d$ | 7.5376 | 9.5010 | 24.9955 | 5.9672 |
| | $\alpha$ | 0.028 | 0.0266 | 0.0194 | 0.0286 |
| | $\beta$ | 0.9703 | 0.9718 | 0.9684 | 0.9688 |
| | LL | 4409.837 | 4365.207 | 4083.280 | 4418.826 |
| | AIC | -8813.6744 | -8724.4134 | -8160.56 | **-8831.6529** |
| | BIC | -8797.4561 | -8708.1951 | -8144.3417 | -8815.4346 |
| tvC | $\omega$ | 12.8923 | 12.8913 | 12.8748 | 12.8905 |
| | $\alpha$ | 0.2197 | 0.2223 | 0.2343 | 0.2222 |
| | $\beta$ | 12.7842 | 12.7839 | 12.704 | 12.7857 |
| | LL | 2430.023 | 2433.630 | 2403.856 | 2423.371 |
| | AIC | -4854.0459 | -4861.2597 | -4801.7120 | -4840.7423 |
| | BIC | -4837.8276 | -4845.0414 | -4785.4937 | -4824.5240 |
| tvSJC | $\lambda$ | 9.1147 1.7757 | -2.3995 1.5160 | -0.4491 1.1956 | 1.4481 9.9382 |
| | $\alpha$ | 1.0295 8.4754 | -8.3109 7.6800 | -0.3626 1.2357 | 9.9486 9.6797 |
| | $\beta$ | 3.228 3.9021 | 2.9388 5.0730 | -0.4471 -0,3567 | 2.6305 3.0602 |
| | LL | 1966.988 | 1706.809 | 1431.022 | 1976.633 |
| | AIC | -3921.9770 | -3401.6179 | -2850.0436 | -3941.2664 |
| | BIC | -3889.5404 | -3369.1812 | -2817.6070 | -3908.8298 |

Table 6. The results of portfolio optimization based on CVaR

| $\beta$ | $VaR$ | $CVaR$ | $x_1-$BIST30 | $x_2-$BIST100 | $x_3-$USD | $x_4-$EURO |
|---|---|---|---|---|---|---|
| 90% | 0.0057 | 0.0102 | 0.3888 | 0.1552 | 0.2676 | 0.1884 |
| 95% | 0.0084 | 0.0136 | 0.3760 | 0.1611 | 0.2715 | 0.1913 |
| 99% | 0.0159 | 0.0233 | 0.3535 | 0.1486 | 0.2983 | 0.1996 |

## 5. CONCLUSION AND DISCUSSION

In this study, the dynamic Copula model was introduced and applied to four different sets of financial data (BIST30, BIST100, USD, EURO) in Turkey. In order to shape the model, first, the marginal distributions of the data series were determined as GJR-t, and by using identified marginal distributions, the copula model that established the dependency structure between the data series was selected as dynamic t (tDCC). By conducting the simulation, portfolio optimization was achieved based on CVaR risk measurement. Therefore, it can be inferred that investment in BIST30 is the best at each confidential level in accordance with the portfolio optimization results based on the minimization of CVaR. Other best investments are USD, EURO, and BIST100 respectively.

It is possible to add extreme value theory to the model due to the fat tail of financial data. Another possibility is to take into account the change points; we can offer different models for every period. Those will be topics of the future work.

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[0]Başlık:Bağımlı finansal varlıkların dinamik bağ ve CVaR'a dayalı portföy optimizasyonu ile modellenmesi

Anahtar Kelimeler:Dinamik Bağ, CVaR, portföy optimizasyonu