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PSEUDO MATRIX MULTIPLICATION

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Abstract. In this paper, a new matrix multiplication is defined in $\mathbb{R}^{m,n} \times \mathbb{R}^{n,p}$ by using scalar product in $\mathbb{R}^n$, where $\mathbb{R}^{m,n}$ is set of matrices of m rows and n columns. With this multiplication it has been shown that $\mathbb{R}^{n,n}$ is an algebra with unit. By considering this new multiplication we define eigenvalues and eigenvectors of square $n \times n$ matrix A and also present some applications.

1. Introduction

In [1], Lorentzian matrix multiplication was introduced. For some applications related to this multiplication, we refer the papers [2-5].

In the present paper we aim to define a new matrix multiplication using scalar product on $\mathbb{R}^n$ of which index is ν. We generalize some properties given for ordinary matrix multiplication. As one of the most important properties of this new multiplication we don't need to use sign matrix to obtain orthogonal, symmetric matrix etc. In the third section, we examine the concepts of eigenvalues and its eigenvectors of square $n \times n$ matrix A. Finally, we study on diagonalizable matrix.

We start with some basic concepts and notations.

Let $\mathbb{R}^{m,n}$ be the set of all $m \times n$ matrices. $\mathbb{R}^{m,n}$ with the matrix addition and the scalar-matrix multiplication is a real vector space. More properties of the ordinary matrix multiplication can be found in [7].

Let $\mathbb{R}^n_\nu$ be pseudo-Euclidean space over the real field $\mathbb{R}$ equipped with a scalar product $\langle x, y \rangle_\nu$ which is symmetric, non degenerate bilinear form;

$$\langle x, y \rangle_\nu = -\sum_{i=1}^{\nu} x_i y_i + \sum_{i=\nu+1}^{n} x_i y_i$$

where $x, y \in \mathbb{R}^n$ and ν is an integer with $0 \leq \nu \leq n$ [8].

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2. Pseudo Matrix Multiplication and Properties

Let $A_1, \ldots, A_m$ denote the row vectors of $A = [a_{ij}] \in \mathbb{R}^{m,n}$ and $B_1, \ldots, B_p$ denote the column vectors of $B = [b_{jk}] \in \mathbb{R}^{n,p}$. Then we define a new matrix multiplication denoted by "$\bullet_\nu$", as

$$\begin{aligned} A \bullet_\nu B &= \begin{bmatrix} \langle A_1, B_1 \rangle_\nu & \langle A_1, B_2 \rangle_\nu & \cdots & \langle A_1, B_p \rangle_\nu \\ \langle A_2, B_1 \rangle_\nu & \langle A_2, B_2 \rangle_\nu & \cdots & \langle A_2, B_p \rangle_\nu \\ \vdots & \vdots & & \vdots \\ \langle A_m, B_1 \rangle_\nu & \langle A_m, B_2 \rangle_\nu & \cdots & \langle A_m, B_p \rangle_\nu \end{bmatrix} \\ &= \left[-\sum_{j=1}^{\nu} a_{ij} b_{jk} + \sum_{j=\nu+1}^{n} a_{ij} b_{jk} \right]. \end{aligned}$$

We call this multiplication as pseudo matrix multiplication and if we let A_i to be ith row of A and B^j to be jth column of B then (i,j) entry of $A \bullet_\nu B$ is $\left\langle A_i, B^j \right\rangle_\nu$. Note that $A \bullet_\nu B$ is an $m \times p$ matrix. We will denote $\mathbb{R}^{m,n}$ with pseudo matrix multiplication by $\mathbb{R}^{m,n}_\nu$. In the special case of ν we get followings:

(1) For $\nu = 0$, $A \bullet_\nu B$ coincide with usual matrix multiplication.
(2) For $\nu = 1$, $A \bullet_\nu B$ coincide with Lorentzian matrix multiplication defined in [1].

Also, in the results and definitions given in throughout the paper one can easily obtain the classical ones when $\nu = 0$.

In the sequel we present some properties of new type matrix multiplication and give the analogous of definitions, in the classical matrix multiplication, by $\bullet_\nu$.

Theorem 1. *The following statements are satisfied.*

i) For every $A \in \mathbb{R}^{m,n}_\nu, B \in \mathbb{R}^{n,p}_\nu, C \in \mathbb{R}^{p,r}_\nu$, $A \bullet_\nu (B \bullet_\nu C) = (A \bullet_\nu B) \cdot_L C$
ii) For every $A \in \mathbb{R}^{m,n}_\nu, B, C \in \mathbb{R}^{n,p}_\nu$, $A \bullet_\nu (B + C) = A \bullet_\nu B + A \bullet_\nu C$
iii) For every $A, B \in \mathbb{R}^{m,n}_\nu, C \in \mathbb{R}^{n,p}_\nu$, $(A + B) \bullet_\nu C = A \bullet_\nu C + B \bullet_\nu C$
iv) For every $k \in \mathbb{R}$, $A \in \mathbb{R}^{m,n}_\nu, B \in \mathbb{R}^{n,p}_\nu$, $k(A \bullet_\nu B) = (kA) \bullet_\nu B = A \bullet_\nu (kB)$

Definition 1. *$n \times n$ identity matrix according to pseudo matrix multiplication, denoted by $I_n = [\imath_{ij}]$, is defined by*

$$\imath_{ij} = \begin{cases} -1 & , \quad i = j \text{ and } 1 \le i, j \le \nu \\ 1 & , \quad i = j \text{ and } \nu + 1 \le i, j \le n \\ 0 & , \quad i \ne j \end{cases},$$

that is

$$I_n = \begin{bmatrix} -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that for every $A \in \mathbb{R}_\nu^{m,n}$, $I_m \bullet_\nu A = A \bullet_\nu I_n = A$.

Corollary 1. *$\mathbb{R}_\nu^{n,n}$ with pseudo matrix multiplication is an algebra with unit.*

Definition 2. *An $n \times n$ matrix A is called p-invertible, if there exists an $n \times n$ matrix B such that $A \bullet_\nu B = B \bullet_\nu A = I_n$. Then B is called p-inverse of A and is shown by A^{-1}.*

Definition 3. *Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the transpose of A is the $n \times m$ matrix A^T obtained by interchanging the rows and columns of A,so that the (i, j) th entry of A^T is a_{ji}.*

Theorem 2. *Let A and B be matrices of the appropriate sizes so that the following operations make sense, and c be a scalar. Then*

(1) $(A + B)^T = A^T + B^T$
(2) $\left(A \bullet_\nu B^T\right) = B^T \bullet_\nu A^T$
(3) $(cA)^T = cA^T$
(4) $\left(A^T\right)^T = A.$

Definition 4. *Let $A \in \mathbb{R}_\nu^{n,n}$. If $A^T = A$, $A^T = -A$ and $A^{-1} = A^T$ then A is said to be symmetric, skew-symmetric and p-orthogonal matrix, respectively.*

Based on this definition, we obtain the following result.

Theorem 3. *Let $A \in \mathbb{R}_\nu^{n,n}$. Then*

(1) *A is p-orthogonal if and only if the row vectors of A form an orthonormal basis of $\mathbb{R}_\nu^n$ under the scalar product; and*
(2) *A is p-orthogonal if and only if the column vectors of A form an orthonormal basis of $\mathbb{R}_\nu^n$ under the scalar product.*

Proof. We shall only prove (1), since the proof of (2) is almost identical. Let $A_1, \ldots, A_n$ denote the row vectors of A. Then

$$A \bullet_\nu A^T = \begin{bmatrix} \langle A_1, A_1 \rangle_\nu & \cdots & \langle A_1, A_n \rangle_\nu \\ \vdots & & \vdots \\ \langle A_n, A_1 \rangle_\nu & \cdots & \langle A_n, A_n \rangle_\nu \end{bmatrix}.$$

It follows that $A \bullet_\nu A^T = I_n$ if and only if for every $i, j = 1, \ldots, n$

$$\langle A_i, A_j \rangle = \begin{cases} -1 & , \quad i = j \text{ and } 1 \leq i, j \leq \nu \\ 1 & , \quad i = j \text{ and } \nu + 1 \leq i, j \leq n \\ 0 & , \quad i \neq j \end{cases} .$$

Then $\{A_1, \ldots, A_n\}$ is an orthonormal basis of $\mathbb{R}^n_\nu$. $\square$

Definition 5. *The determinant of a matrix* $A = [a_{ij}] \in \mathbb{R}^{n,n}_\nu$ *is denoted by* $\det A$ *and defined as*

$$\det A = \sum_{\sigma \in S_n} s(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n},$$

where S_n *is set of all permutations of the set* $\{1, 2, \cdots, n\}$ *and* $s(\sigma)$ *is sign of the permutation* σ*.*

Theorem 4. *For every* $A, B \in \mathbb{R}^{n,n}_\nu$, $\det(A \bullet_\nu B) = (-1)^\nu \det A \cdot \det B$.

Proof. Let $A = [a_{ij}]$, $B = [b_{jk}]$ and $A \bullet_\nu B = C$. Let us denote the ith column of matrix A by A^i and the kth column of matrix C by C^k

$$C^k = -b_{1k} A^1 - b_{2k} A^2 - \cdots - b_{\nu k} A^\nu + b_{(\nu+1)k} A^{\nu+1} + \cdots + b_{nk} A^n, \qquad 1 \leq k \leq n.$$

Then

$$\begin{aligned} \det(A \bullet_\nu B) &= \det[-b_{11} A^1 - b_{21} A^2 - \cdots - b_{\nu 1} A^\nu + b_{(\nu+1)1} A^{\nu+1} + \cdots + b_{n1} A^n, \\ &\quad \ldots, -b_{1n} A^1 - b_{2n} A^2 - \cdots - b_{\nu n} A^\nu + b_{(\nu+1)n} A^{\nu+1} + \cdots + b_{nn} A^n] \\ &= \sum_{\sigma \in S_n} (-1)^\nu b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \det[A^{\sigma(1)}, A^{\sigma(2)}, \cdots, A^{\sigma(n)}] \\ &= (-1)^\nu \sum_{\sigma \in S_n} s(\sigma) b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \det[A^1, A^2, \cdots, A^n] \\ &= (-1)^\nu \det A \sum_{\sigma \in S_n} s(\sigma) b_{\sigma(1)1} b_{\sigma(2)2} \cdots b_{\sigma(n)n} \\ &= (-1)^\nu \det A \det B. \end{aligned}$$

$\square$

3. Some Applications Of Pseudo Matrix Multiplication

Eigenvalues and eigenvectors play an important role in matrix theory because of its application in the areas of mathematics, physics and engineering. By this aim, we define the eigenvalues and eigenvectors of square $n \times n$ matrix A by pseudo matrix multiplication.

Definition 6. *Let* $A \in \mathbb{R}^{n,n}_\nu$*. An eigenvector of* A *is a nonzero vector* x *in* $\mathbb{R}^n_\nu$ *such that*

$$A \bullet_\nu x = \lambda x$$

for some scalar λ. The scalar λ is called an eigenvalue of the matrix A, and we say that the vector x is an eigenvector belonging to the eigenvalue λ.

Theorem 5. *The eigenvectors of a symmetric matrix $A \in \mathbb{R}_\nu^{n,n}$ corresponding to different eigenvalues are orthogonal to each other.*

Proof. For the eigenvectors x, y corresponding to two different eigenvalues λ, μ of the matrix A, we can say that $A \bullet_\nu x = \lambda x$ and $A \bullet_\nu y = \mu y$, so

$$y^T \bullet_\nu A \bullet_\nu x = \lambda y^T \bullet_\nu x = \lambda \langle x, y \rangle_\nu . \tag{3.1}$$

But numbers are always their own transpose, so

$$\begin{aligned} y^T \bullet_\nu A \bullet_\nu x &= x^T \bullet_\nu A \bullet_\nu y \\ &= x^T \bullet_\nu \mu y \\ &= \mu \left(x^T \bullet_\nu y \right) \\ y^T \bullet_\nu A \bullet_\nu x &= \mu \langle x, y \rangle_\nu . \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we get

$$(\lambda - \mu) \langle x, y \rangle_\nu = 0.$$

So $\lambda = \mu$ or $\langle x, y \rangle_\nu = 0$, and it isn't the former, so x and y are orthogonal. $\square$

Example 1. *Let*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}_2^{3,3}.$$

A is a symmetric matrix. Then the eigenvalues of A are $\lambda_1 = 0, \lambda_2 = \sqrt{2} - 1$ and $\lambda_3 = -\sqrt{2} - 1$. Some eigenvectors of A corresponding to λ_1, λ_2 and λ_3 are

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} -1 + \sqrt{2} \\ -2 + \sqrt{2} \\ 1 \end{bmatrix} \quad , \quad u_3 = \begin{bmatrix} -1 - \sqrt{2} \\ -2 - \sqrt{2} \\ 1 \end{bmatrix}$$

respectively. For $i \neq j$, we get

$$\langle u_i, u_j \rangle_2 = u_i^T \bullet_2 u_j = 0.$$

Then $\{u_1, u_2, u_3\}$ form an orthogonal basis of $\mathbb{R}_2^3$.

Definition 7. *A matrix A is diagonalizable if there exists a nonsingular matrix P and a diagonal matrix D such that*

$$D = P^{-1} \bullet_\nu A \bullet_\nu P.$$

Theorem 6. *Let all the eigenvalues of $A \in \mathbb{R}_\nu^{n,n}$ are real. Then A diagonalizable if and only if it has n linearly independent eigenvectors.*

Proof. Let $x_1, \dots, x_n$ be n linearly independent eigenvectors of A associated with the eigenvalues $\lambda_1, \dots, \lambda_n$. That is,

$$A \bullet_\nu x_i = \lambda_i x_i \quad , \quad i = 1, \dots, n \ .$$

Now, we denote $P = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$. Since the columns of P are linearly independent, P is invertible. Let D be $diag\,[-\lambda_1, \dots, -\lambda_\nu, \lambda_{\nu+1}, \dots, \lambda_n]$. Then

$$\begin{aligned} A \bullet_\nu P &= A \bullet_\nu \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \bullet_\nu diag\,[-\lambda_1, \dots, -\lambda_\nu, \lambda_{\nu+1}, \dots, \lambda_n] \\ &= P \bullet_\nu D. \end{aligned}$$

Since $A \bullet_\nu P = P \bullet_\nu D$, it follows that $D = P^{-1} \bullet_\nu A \bullet_\nu P$ which shows that A is diagonalizable.

To prove the other direction we assume that A is diagonalizable. Then there exists a nonsingular matrix P and a diagonal matrix $D = diag\,[-\lambda_1, \dots, -\lambda_\nu, \lambda_{\nu+1}, \dots, \lambda_n]$ such that

$$D = P^{-1} \bullet_\nu A \bullet_\nu P.$$

If we multiply above equation with P from the left, we get

$$A \bullet_\nu P = P \bullet_\nu D \tag{3.3}$$

which implies

$$A \bullet_\nu v_i = \lambda_i v_i \quad , \quad i = 1, \dots, n \tag{3.4}$$

where v_i are columns of P. The equations (3.4) show that $v_1, \dots, v_n$ are eigenvectors of A corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$. Furthermore, since P is invertible, $\{v_1, \dots, v_n\}$ are linearly independent. □

Example 2. *Consider the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ 1 & -2 & -2 \end{bmatrix} \in \mathbb{R}_2^{3.3}.$$

The eigenvalues of A are $\lambda_1 = -2, \lambda_2 = 0, \lambda_3 = -1$ and eigenvectors corresponding these eigenvalues are

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad , \quad u_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad , \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

respectively. Therefore

$$P = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix} \quad and \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & -2 & -2 \end{bmatrix} .$$

Finally

$$P^{-1} \bullet_\nu A \bullet_\nu P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

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