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## A NEW APPROACH TO THE CONCEPT OF $A^I$ –STATISTICAL CONVERGENCE WITH THE NUMBER OF ALPHA

HAFİZE GÜMÜŞ

**ABSTRACT.** Following a very recent approach, we generalize recently introduced summability method, namely,  $I$ –statistical convergence. We use an infinite matrix of complex numbers and an  $\alpha$  number to do this generalization. We call this new convergence as  $A^I$ –statistical convergence of order  $\alpha$  with respect to a sequence of modulus functions. We also define two new spaces by using strong convergence and Cesáro summability and after giving these descriptions, we investigate their relationship and we obtain some results.

### 1. INTRODUCTION AND BACKGROUND

As is known, convergence is one of the most important notions in Mathematics. Statistical convergence extends the notion. We can easily show that any convergent sequence is statistically convergent, but not conversely. Let  $E$  be a subset of  $\mathbb{N}$ , the set of all natural numbers.  $d(E) := \lim_n \frac{1}{n} \sum_{j=1}^n \chi_E(j)$  is said to be natural density of  $E$  whenever the limit exists, where  $\chi_E$  is the characteristic function of  $E$ .

Statistical convergence was given by Zygmund in the first edition of his monograph published in Warsaw in 1935 ([25]). It was formally introduced by Fast ([10]) and Steinhaus ([22]) and later was reintroduced by Schoenberg ([21]). It has become an active area of research in recent years. This concept has applications in different fields of mathematics such as number theory by Erdős and Tenenbaum ([8]), measure theory by Miller ([16]), trigonometric series by Zygmund ([25]), summability theory by Freedman and Sember ([11]), etc. Statistical convergence is also applied to approximation theory by Gadjiev and Orhan ([12]), Anastassiou and Duman ([1]) and Sakaoğlu and Ünver ([19]).

**Definition 1.** ([10]) *A number sequence  $(x_k)$  is statistically convergent to  $x$  provided that for every  $\varepsilon > 0$ ,  $d\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\} = 0$  or equivalently there exists*

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a subset  $K \subseteq \mathbb{N}$  with  $d(E) = 1$  and  $n_0(\varepsilon)$  such that  $k > n_0(\varepsilon)$  and  $k \in K$  imply that  $|x_k - x| < \varepsilon$ . In this case we write  $st - \lim x_k = x$ .

Let  $\lambda = (\lambda_n)$  is a non-decreasing sequence of positive numbers tending to  $\infty$  such that

$$\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 1.$$

The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in J_n} x_k$$

where  $J_n = [n - \lambda_n + 1, n]$ .

A sequence  $x = (x_k)$  is said to be  $(V, \lambda)$ -summable to a number  $L$  if

$$t_n(x) \rightarrow \infty \quad \text{as} \quad n \rightarrow \infty.$$

If  $\lambda_n = n$ , then  $(V, \lambda)$ -summability reduces to  $(C, 1)$ -summability. We write

$$[C, 1] = \left\{ x = (x_n) : \exists L \in \mathbb{R}, \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |x_k - L| = 0 \right\}$$

and

$$[V, \lambda] = \left\{ x = (x_n) : \exists L \in \mathbb{R}, \lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \sum_{k \in J_n} |x_k - L| = 0 \right\}$$

$\Lambda$  denotes the set of all  $\lambda$  sequences described above.

Space of  $\lambda$ -statistically convergent sequences was defined by Mursaleen ([17]) and he denoted this new method by  $S_\lambda$  and found its relation to statistical convergence,  $[C, 1]$ -summability and  $[V, \lambda]$ -summability. Then many others discussed about this concept and they gave many applications in many areas. Gümüş and Savaş ([14]) gave the definition of  $S_\lambda^L(\mathcal{I})$ -asymptotically statistical equivalence and interested in some relations with  $V_\lambda^L(\mathcal{I})$  and  $C_1^L(\mathcal{I})$  spaces. Pandoh and Raj ([18]) defined  $\lambda$ -statistical convergence in intuitionistic fuzzy  $n$ -normed spaces.

**Definition 2.** ([17]) A number sequence  $x = (x_k)$  is said to be  $\lambda$ -statistically convergent to the number  $L$  if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} |\{k \in J_n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case we write  $S_\lambda - \lim x = L$ .

$\mathcal{I}$ -convergence has emerged as a kind of generalization form of many types of convergence. This means that, if we choose different ideals we will have different convergences. Koystro et. al. introduced this concept in a metric space ([15]). We will explain this situation with two examples later. Before defining  $\mathcal{I}$ -convergence, the definitions of ideal and filter will be needed.

An ideal is a family of sets  $\mathcal{I} \subseteq 2^{\mathbb{N}}$  such that (i)  $\emptyset \in \mathcal{I}$ , (ii)  $A, B \in \mathcal{I}$  implies  $A \cup B \in \mathcal{I}$ , (iii) For each  $A \in \mathcal{I}$  and each  $B \subseteq A$  implies  $B \in \mathcal{I}$ . An ideal is called

non-trivial if  $\mathbb{N} \notin \mathcal{I}$  and a non-trivial ideal is called admissible if  $\{n\} \in \mathcal{I}$  for each  $n \in \mathbb{N}$ .

A filter is a family of sets  $\mathcal{F} \subseteq 2^{\mathbb{N}}$  such that (i)  $\emptyset \notin \mathcal{F}$ , (ii)  $A, B \in \mathcal{F}$  implies  $A \cap B \in \mathcal{F}$ , (iii) For each  $A \in \mathcal{F}$  and each  $A \subseteq B$  implies  $B \in \mathcal{F}$ .

If  $\mathcal{I}$  is an ideal in  $\mathbb{N}$  then the collection,

$$F(\mathcal{I}) = \{A \subset \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$$

forms is a filter in  $\mathbb{N}$  which is called the filter associated with  $\mathcal{I}$ .

**Definition 3.** ([15]) A real sequence  $x = (x_k)$  is said to be  $\mathcal{I}$ -convergent to  $L \in \mathbb{R}$  if and only if for each  $\varepsilon > 0$  the set

$$A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

belongs to  $\mathcal{I}$ . The number  $L$  is called the  $\mathcal{I}$ -limit of the sequence  $x$ .

Now we can give the examples which were mentioned above.

**Example 1.** Define the set of all finite subsets of  $\mathbb{N}$  by  $\mathcal{I}_f$ . Then,  $\mathcal{I}_f$  is a non-trivial admissible ideal and  $\mathcal{I}_f$ -convergence coincides with the usual convergence.

**Example 2.** Define the set  $\mathcal{I}_d$  by  $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ . Then,  $\mathcal{I}_d$  is an admissible ideal and  $\mathcal{I}_d$ -convergence gives the statistical convergence.

Following the statistical convergence and  $\mathcal{I}$ -convergence located an important role in this area, in 2011, Das, Savaş and Ghosal ([6]) have introduced the concept of  $\mathcal{I}$ -statistical convergence as follows:

**Definition 4.** ([6]) A sequence  $x = (x_k)$  is said to be  $\mathcal{I}$ -statistically convergent to  $L$  for each  $\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

Now we recall that, a modulus  $f$  is a function from  $[0, \infty)$  to  $[0, \infty)$  such that (i)  $f(x) = 0$  if and only if  $x = 0$ , (ii)  $f(x + y) = f(x) + f(y)$  for  $x, y \geq 0$ , (iii)  $f$  is increasing and (iv)  $f$  is continuous from the right at 0. It follows that  $f$  must be continuous on  $[0, \infty)$ .

Now we are ready to explain  $A^{\mathcal{I}}$ -statistical convergence. Let  $A = (a_{ki})$  be an infinite matrix of complex numbers. We write  $Ax = (A_k(x))$  for a sequence  $x = (x_k)$  if  $A_k(x) = \sum_{i=1}^{\infty} a_{ki}x_i$  converges for each  $k$ . Some authors obtained more general results about statistical convergence by using  $A$  matrix and they called this new method by  $A^{\mathcal{I}}$ -statistical convergence. (Bilgin [3]; Yamanci, Gürdal and Saltan, [24]).

**Definition 5.** ([24]) Let  $A = (a_{ki})$  be an infinite matrix of complex numbers and  $(f_k)$  be a sequence of modulus functions. A sequence  $x = (x_k)$  is said to be  $A^{\mathcal{I}}$ -statistically convergent to  $L$  with respect to a sequence of modulus functions for each

$\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write  $x_k \rightarrow L (S^A(\mathcal{I}, F))$ .

Recently, many concepts that are considered essential in this area has been re-worked using the alpha number. In [2] and [4], a different direction was given to the statistical convergence, where the notion of statistical convergence of order  $\alpha$  ( $0 < \alpha < 1$ ) was introduced by using the notion of natural density of order  $\alpha$ . The behavior of this new convergence was not exactly parallel to that of statistical convergence. Some other applications of this concept are  $\lambda$ -statistical convergence of order  $\alpha$  by Çolak and Bektaş ([5]), lacunary statistical convergence of order  $\alpha$  by Şengül and Et ([23]), weighted statistical convergence of order  $\alpha$  and its applications by Ghosal ([13]) and almost statistical convergence of order  $\alpha$  by Et, Altın and Çolak ([9]).  $\mathcal{I}$ -statistical convergence and  $\mathcal{I}$ -lacunary statistical convergence of order  $\alpha$  which we use more introduced by Das and Savaş in 2014 ([7]). In all these studies, the authors gave a different direction to the study of statistical convergence where  $n$  is replaced by  $n^\alpha$  in the denominator in the definition of natural density.

**Definition 6.** ([7]) A sequence  $x = (x_k)$  is said to be  $\mathcal{I}$ -statistically convergent of order  $\alpha$  to  $L$  or  $S(\mathcal{I})^\alpha$ -convergent to  $L$  where  $0 < \alpha \leq 1$  if for each  $\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : |x_k - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

Our goal in this study is to introduce new and more general summability methods, namely,  $A_\lambda^\mathcal{I}$ -statistical convergence of order  $\alpha$ , strongly  $A_\lambda^\mathcal{I}$ -statistically convergence of order  $\alpha$  and strongly Cesàro  $A_\lambda^\mathcal{I}$ -summability of order  $\alpha$ . It should be mentioned that this concept has not been studied until now and our results effectively extend and improve all the existing results in [24], [4], [2], [7].

## 2. MAIN RESULTS

We now consider our main results. We begin with the following definitions. Throughout the paper  $A = (a_{ki})$  will be an infinite matrix of complex numbers,  $F$  will be the set of all modulus functions,  $(\lambda_n) \in \Lambda$  and  $\lambda_n^\alpha$  denote the  $\alpha^{th}$  power of  $(\lambda_n)^\alpha$  of  $\lambda_n$ , that is  $\lambda^\alpha = (\lambda_n)^\alpha = (\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_n^\alpha, \dots)$ .

**Definition 7.** A sequence  $x = (x_k)$  is said to be  $A^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$  with respect to a sequence of modulus functions or  $x \in [S_A(\mathcal{I}, F)]^\alpha$  for each  $\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

**Definition 8.** A sequence  $x = (x_k)$  is said to be  $A^{\mathcal{I}}$ -statistically convergent of order  $\alpha$  to the number  $L$  with respect to a sequence of modulus functions or  $x \in [S_A^{\lambda}(\mathcal{I}, F)]^{\alpha}$  for each  $\varepsilon > 0$  and  $\delta > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}.$$

The following theorem examines the relation between  $A^{\mathcal{I}}$ -statistical convergence of order  $\alpha$  and  $A_{\lambda}^{\mathcal{I}}$ -statistical convergence of order  $\alpha$ .

**Theorem 1.** If  $\liminf_{n \rightarrow \infty} \frac{\lambda_n^{\alpha}}{n^{\alpha}} > 0$  then  $[S_A(I, F)]^{\alpha} \subset [S_A^{\lambda}(I, F)]^{\alpha}$ .

*Proof.* Suppose that  $\frac{\lambda_n^{\alpha}}{n^{\alpha}} > 0$  then there exist a  $\eta > 0$  such that  $\frac{\lambda_n^{\alpha}}{n^{\alpha}} \geq \eta$  for sufficiently large  $n$ . For given  $\varepsilon > 0$  we have,

$$\frac{1}{n^{\alpha}} \{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\} \supseteq \frac{1}{n^{\alpha}} \{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}.$$

Therefore

$$\begin{aligned} \frac{1}{n^{\alpha}} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| &\geq \frac{1}{n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \\ &\geq \frac{\lambda_n^{\alpha}}{n^{\alpha}} \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \\ &\geq \eta \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \end{aligned}$$

then for any  $\delta > 0$  we get

$$\begin{aligned} &\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n^{\alpha}} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \eta \right\} \in \mathcal{I}. \end{aligned}$$

Then we have the proof.  $\square$

The next theorem shows when the inverse of theorem 1 is correct.

**Theorem 2.** If  $\liminf_{n \rightarrow \infty} \frac{\lambda_n^{\alpha}}{n^{\alpha}} = 1$  then  $[S_A^{\lambda}(I, F)]^{\alpha} \subset [S_A(I, F)]^{\alpha}$ .

*Proof.* Let  $\delta > 0$ . Since  $\liminf_{n \rightarrow \infty} \frac{\lambda_n^{\alpha}}{n^{\alpha}} = 1$  we can choose  $m \in \mathbb{N}$  such that  $\left| \frac{\lambda_n^{\alpha}}{n^{\alpha}} - 1 \right| < \frac{\delta}{2}$  for all  $n \geq m$ . For  $\varepsilon > 0$ ,

$$\begin{aligned} &\frac{1}{n^{\alpha}} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \\ &\leq \frac{n^{\alpha} - \lambda_n^{\alpha}}{n^{\alpha}} + \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \\ &\leq 1 - \left(1 - \frac{1}{\delta}\right) + \frac{1}{\lambda_n^{\alpha}} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \end{aligned}$$

for all  $n \geq m$ . Hence,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leq n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \\ & \subset \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \frac{\delta}{2} \right\} \cup \{1, 2, \dots, m\} \end{aligned}$$

This proves the theorem.  $\square$

**Definition 9.** A sequence  $x = (x_k)$  is said to be strongly  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$  with respect to a sequence of modulus functions or  $x \in [V_A^\lambda(\mathcal{I}, F)]^\alpha$  for each  $\varepsilon > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \geq \varepsilon \right\} \in \mathcal{I}.$$

After this definition, we investigate the relationship between strongly  $A_\lambda^\mathcal{I}$ -statistical convergence of order  $\alpha$  and  $A_\lambda^\mathcal{I}$ -statistical convergence of order  $\alpha$ .

**Theorem 3.** If  $x = (x_k)$  is strongly  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$  then  $x$  is  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$ .

*Proof.* Let  $x$  is strongly  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$ . Since  $f$  is a modulus function we can write for any  $\varepsilon > 0$ ,

$$\begin{aligned} \sum_{k \in J_n} f_k(|A_k(x) - L|) & \geq \sum_{\substack{k \in J_n \\ f_k(|A_k(x) - L|) \geq \varepsilon}} f_k(|A_k(x) - L|) \\ & \geq \varepsilon |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \end{aligned}$$

and so,

$$\frac{1}{\varepsilon} \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \geq \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}|.$$

Then for any  $\delta > 0$  we have,

$$\begin{aligned} & \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \delta \right\} \\ & \subset \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \geq \varepsilon \delta \right\} \end{aligned}$$

Since the right side is in ideal and this means  $x$  is  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$ .  $\square$

The following theorem shows that the inverse of the previous theorem is correct when  $x$  is bounded.

**Theorem 4.** If  $x$  is bounded and  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$  then  $x$  is strongly  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$ .

*Proof.* Let  $x$  is bounded then for each  $k$  there is an  $M$  such that,  $f_k(|A_k(x) - L|) \leq M$ . For each  $\varepsilon > 0$ ,

$$\begin{aligned}
\frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) &= \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in J_n \\ f_k(|A_k(x) - L|) \geq \varepsilon}} f_k(|A_k(x) - L|) \\
&+ \frac{1}{\lambda_n^\alpha} \sum_{\substack{k \in J_n \\ f_k(|A_k(x) - L|) < \varepsilon}} f_k(|A_k(x) - L|) \\
&\leq M \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \\
&+ \varepsilon \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) < \varepsilon\}| \\
&\leq \frac{M}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| + \varepsilon
\end{aligned}$$

then for any  $\delta > 0$ ,

$$\begin{aligned}
&\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \geq \varepsilon \right\} \\
&\subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} |\{k \in J_n : f_k(|A_k(x) - L|) \geq \varepsilon\}| \geq \frac{\varepsilon}{M} \right\} \in \mathcal{I}.
\end{aligned}$$

□

**Theorem 5.**  $[S_A^\lambda(I, F)]^\alpha \cap m(X) = [V_A^\lambda(I, F)]^\alpha \cap m(X)$  where  $m(X)$  is the space of all bounded sequences.

*Proof.* This readily follows from Theorem 3 and Theorem 4. □

**Definition 10.** A sequence  $x = (x_k)$  is said to be strongly Cesàro  $A_\lambda^{\mathcal{I}}$ -summable of order  $\alpha$  to the number  $L$  with respect to a sequence of modulus functions or  $x \in [\sigma_A^\lambda(\mathcal{I}, F)]^\alpha$  for each  $\varepsilon > 0$ ,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^n f_k(|A_k(x) - L|) \geq \varepsilon \right\} \in \mathcal{I}.$$

Finally we investigate the relationship between strongly  $A_\lambda^{\mathcal{I}}$ -statistical convergence of order  $\alpha$  and strongly Cesàro  $A_\lambda^{\mathcal{I}}$ -summability of order  $\alpha$  in Theorem 6.

**Theorem 6.** If  $x$  is strongly  $A_\lambda^{\mathcal{I}}$ -statistically convergent of order  $\alpha$  to the number  $L$  then  $x$  is strongly Cesàro  $A_\lambda^{\mathcal{I}}$ -summable of order  $\alpha$  to the number  $L$ .



*Proof.* Assume that  $x$  is strongly  $A_\lambda^\mathcal{I}$ -statistically convergent of order  $\alpha$  to the number  $L$  and  $\varepsilon > 0$ . Then

$$\begin{aligned} \frac{1}{n^\alpha} \sum_{k=1}^n f_k(|A_k(x) - L|) &= \frac{1}{n^\alpha} \sum_{k=1}^{n-\lambda_n} f_k(|A_k(x) - L|) + \frac{1}{n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \\ &\leq \frac{1}{\lambda_n^\alpha} \sum_{k=1}^{n-\lambda_n} f_k(|A_k(x) - L|) + \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \\ &\leq \frac{2}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \end{aligned}$$

and so,

$$\left\{ n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=1}^n f_k(|A_k(x) - L|) \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{\lambda_n^\alpha} \sum_{k \in J_n} f_k(|A_k(x) - L|) \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

□

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