Available online: July 16, 2017

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 67, Number 1, Pages 235–241 (2018)
DOI: 10.1501/Commua1_0000000845
ISSN 1303–5991
http://communications.science.ankara.edu.tr/index.php?series=A1

SOME PROPERTIES OF SEQUENCE SPACE $\widehat{BV}_\theta(f,p,q,s)$

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Abstract. In this paper, we define the sequence space $\widehat{BV}_\theta(f,p,q,s)$ on a seminormed complex linear space, by using a Modulus function. We give various properties and some inclusion relations on this space.

1. INTRODUCTION

Let ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x=(x_n)$ normed by $\|x\| = \sup_n |x_n|$, respectively.

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^k(n) = \sigma\left(\sigma^{k-1}(n)\right)$, $k = 1, 2, \ldots$ .A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$mean if and only if

(i) $\varphi(x) \geq 0$ when $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and

(iii) $\varphi\left(\left\{x_{\sigma(n)}\right\}\right) = \varphi(\{x_n\})$ for all $x \in \ell_\infty$.

If σ is the translation mapping $n \to n+1$, a $\sigma-$mean is often called a Banach limit [3], and V_σ is the set of $\sigma-$convergent sequences, that is, the set of bounded sequences all of whose invariant means are equal, is the set $\hat{f}$ of almost convergent sequences [11].

It can be shown (see Schaefer [24]) that

$$V_\sigma = \left\{x = (x_n) : \lim_r t_{rn}(x) = Le \text{ uniformly in } n,\ L = \sigma - \lim x\right\}, \tag{1.1}$$

where

$$t_{rn}(x) = \frac{1}{r+1}\sum_{j=0}^{r} T^j x_n.$$

The special case of (1.1), in which $\sigma(n) = n+1$ was given by Lorentz [11].

Received by the editors: June 08, 2016; Accepted: February 01, 2017.
2010 *Mathematics Subject Classification.* 40A05, 40C05, 40D05.
Key words and phrases. Modulus function, sequence spaces, seminorm.

Subsequently invariant means have been studied by Ahmad and Mursaleen[1], Mursaleen ([16],[17]), Raimi [20], Altinok et al. [2], Mohiuddine [13],[14], Mohiuddine and Mursaleen [15] many others.

We may remark here that the concept $\widehat{BV}$ of almost bounded variation have been introduced and investigated by Nanda and Nayak [19] as follows:

$$\widehat{BV}=\left\{x:\sum_{r}\left|t_{rn}\left(x\right)\right| \text{ converges uniformly in } n\right\}$$

where

$$t_{rn}\left(x\right)=\frac{1}{r\left(r+1\right)}\sum_{v=1}^{r}v\left(x_{n+v}-x_{n+v-1}\right).$$

By a lacunary sequence $\theta=(k_r)_{r=0,1,2,...}^{\infty}$, where $k_0=0$, we shall mean an increasing sequence of non-negative integers with $k_r-k_{r-1}\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=\left(k_{r-1},k_r\right]$, and we let $h_r=k_r-k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will usually be denoted by q_r (see [7]) .

Karakaya and Savaş [10] were defined sequence spaces $\widehat{BV}_{\theta}\left(p\right)$ and $\widehat{\widehat{BV}}_{\theta}\left(p\right)$ as follows:

$$\widehat{BV}_{\theta}\left(p\right)=\left\{x:\sum_{r=1}^{\infty}\left|\varphi_{rn}\left(x\right)\right|^{p_r} \text{converges uniformly in } n\right\},$$

$$\widehat{\widehat{BV}}_{\theta}\left(p\right)=\left\{x:\sup_{n}\sum_{r=1}^{\infty}\left|\varphi_{rn}\left(x\right)\right|^{p_r}<\infty\right\},$$

where

$$\varphi_{r,n}\left(x\right)=\frac{1}{h_r+1}\sum_{j=k_{r-1}+1}x_{j+n}-\frac{1}{h_r}\sum_{j=k_{r-1}+1}^{k_r}x_{j+n}\ ,r>1.$$

Straightforward calculation shows that

$$\varphi_{r,n}\left(x\right)=\frac{1}{h_r\left(h_r+1\right)}\sum_{u=1}^{h_r}u\left(x_{k_{r-1}+u+1+n}-x_{k_{r-1}+u+n}\right),$$

and

$$\varphi_{r-1,n}\left(x\right)=\frac{1}{h_r\left(h_r-1\right)}\sum_{u=1}^{h_r-1}\left(x_{k_{r-1}+u+1+n}-x_{k_{r-1}+u+n}\right).$$

Note that for any sequences x,y and scalar λ, we have

$$\varphi_{r,n}\left(x+y\right)=\varphi_{r,n}\left(x\right)+\varphi_{r,n}\left(y\right) \text{ and } \varphi_{r,n}\left(\lambda x\right)=\lambda\varphi_{r,n}\left(x\right).$$

The notion of modulus function was introduced by Nakano [18] in 1953. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that

$i)$ $f\left(x\right)=0$ if and only if $x=0$,

(ii) $f(x+y) \le f(x) + f(y)$, for all $x \ge 0, y \ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

A modulus may be bounded or unbounded. For example, $f(x) = x^p, (0 < p \le 1)$ is unbounded but $f(x) = \frac{x}{1+x}$ is bounded. Maddox [12] and Ruckle[21], Bhardwaj [4], Et ([5], [6]), Işık ([8], [9]), Savas ([22], [23]) used a modulus function to construct some sequence spaces.

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \le 1$.

It is well known that a sequence space E is normal implies that E is monotone .

Definition 1.1 Let q_1, q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n) \to 0$, then also $q_2(x_n) \to 0$. If each is stronger than the others q_1 and q_2 are said to be equivalent (one may refer to Wilansky [25]).

Lemma 1.2 Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if and only if there exists a constant T such that $q_2(x) \le Tq_1(x)$ for all $x \in X$ (see for instance Wilansky [25]).

Let $p = (p_r)$ be a sequence of strictly positive real numbers, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q, f be a Modulus function and $s \ge 0$ be a fixed real number. Then we define the sequence space $\widehat{BV}_\theta(f,p,q,s)$ as follows:

$$\widehat{BV}_\theta(f,p,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{uniformly in } n, \right\}.$$

We get the following sequence spaces from $\widehat{BV}_\theta(f,p,q,s)$ by choosing some of the special p, f and s :
For $f(x) = x$, we get

$$\widehat{BV}_\theta(p,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{ uniformly in } n \right\},$$

for $p_r = 1$ for all $r \in \mathbb{N}$, we get

$$\widehat{BV}_\theta(f,q,s) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} r^{-s} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right] < \infty, \text{ uniformly in } n \right\},$$

for $s = 0$ we get

$$\widehat{BV}_\theta(f,p,q) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} \left[f\left(q\left(\varphi_{rn}(x) \right) \right) \right]^{p_r} < \infty, \text{ uniformly in } n \right\},$$

for $f(x) = x$ and $s = 0$ we get

$$\widehat{BV}_\theta(p,q) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} [(q(\varphi_{rn}(x)))]^{p_r} < \infty, \text{ uniformly in } n \right\},$$

for $p_r = 1$ for all $r \in \mathbb{N}$, and $s = 0$ we get

$$\widehat{BV}_\theta(f,q) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} [f(q(\varphi_{rn}(x)))] < \infty, \text{ uniformly in } n \right\},$$

for $f(x) = x$, $p_r = 1$ for all $r \in \mathbb{N}$, and $s = 0$ we have

$$\widehat{BV}_\theta(q) = \left\{ x = (x_k) \in X : \sum_{r=1}^{\infty} q(\varphi_{rn}(x)) < \infty, \text{ uniformly in } n \right\}.$$

The following inequalities will be used throughout the paper. Let $p = (p_r)$ be a bounded sequence of strictly positive real numbers with $0 < p_r \leq \sup p_r = H$, $D = \max\left(1, 2^{H-1}\right)$, then

$$|a_r + b_r|^{p_r} \leq D\left\{|a_r|^{p_r} + |b_r|^{p_r}\right\}, \tag{1.2}$$

where a_r, $b_r \in \mathbb{C}$.

2. MAIN RESULTS

In this section we will prove the general results of this paper on the sequence space $\widehat{BV}_\theta(f,p,q,s)$, those characterize the structure of this space.

Theorem 2.1 The sequence space $\widehat{BV}_\theta(f,p,q,s)$ is a linear space over the field $\mathbb{C}$ of complex numbers.

Proof. Let $x, y \in \widehat{BV}_\theta(f,p,q,s)$. For $\lambda, \mu \in \mathbb{C}$ there exists M_λ and N_μ integers such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Since f is subadditive, q is a seminorm

$$\begin{aligned}
&\sum_{r=1}^{\infty} r^{-s}\left[f(q(\lambda\varphi_{rn}(x) + \mu\varphi_{rn}(y)))\right]^{p_r} \\
&\leq \sum_{r=1}^{\infty} r^{-s}\left[f(|\lambda|\, q(\varphi_{rn}(x))) + f(q(|\mu|\,\varphi_{rn}(y)))\right]^{p_r} \\
&\leq D(M_\lambda)^H \sum_{r=1}^{\infty} r^{-s}\left[f(q(\varphi_{rn}(x)))\right]^{p_r} + D(N_\mu)^H \sum_{r=1}^{\infty} r^{-s}\left[f(q(\varphi_{rn}(y)))\right]^{p_r} < \infty.
\end{aligned}$$

This proves that $\widehat{BV}_\theta(f,p,q,s)$ is a linear space.

Theorem 2.2 $\widehat{BV}_\theta\,(f,p,q,s)$ is a paranormed space (not necessarily totally paranormed), paranormed by

$$g\,(x)=\left(\sum_{r=1}^{\infty}r^{-s}\left[f\left(q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}\right)^{\frac{1}{M}},$$

where $M=\max\left(1,\sup p_r\right),\ H=\sup_r p_r<\infty.$

Proof. It is clear that $g\left(\bar{\theta}\right)=0$ and $g\,(x)=g\,(-x)$ for all $x\in\widehat{BV}_\theta\,(f,p,q,s)$, where $\bar{\theta}=(\theta,\theta,\theta,...)$. It also follows from (1.2), Minkowski's inequality and definition f that g is subadditive and

$$g\,(\lambda x)\leq K_\lambda^{H\backslash M}g\,(x)\,,$$

where K_λ is an integer such that $|\lambda|<K_\lambda$. Therefore the function $(\lambda,x)\to\lambda x$ is continuous at $x=\bar{\theta}$ and that when λ is fixed, the function $x\to\lambda x$ is continuous at $x=\bar{\theta}$. If x is fixed and $\varepsilon>0$, we can choose r_0 such that

$$\sum_{r=r_0}^{\infty}r^{-s}\left[f\left(q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}<\frac{\varepsilon}{2}.$$

and $\delta>0$ so that $|\lambda|<\delta$ and definition of f gives

$$\sum_{r=1}^{r_0}r^{-s}\left[f\left(q\left(\lambda\varphi_{rn}\,(x)\right)\right)\right]^{p_r}=\sum_{r=1}^{r_0}r^{-s}\left[f\left(|\lambda|\,q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}<\frac{\varepsilon}{2}.$$

Therefore $|\lambda|<\min\left(1,\delta\right)$ implies that $g\,(\lambda x)<\varepsilon$. Thus the function $(\lambda,x)\to\lambda x$ is continuous at $\lambda=0$ and $\widehat{BV}_\theta\,(f,p,q,s)$ is paranormed space

Theorem 2.3 Let f,f_1,f_2 be modulus functions q,q_1,q_2 seminorms and $s,s_1,s_2\geq 0$. Then

(i) $\widehat{BV}_\theta\,(f_1,p,q,s)\cap\widehat{BV}_\theta\,(f_2,p,q,s)\subseteq\widehat{BV}_\theta\,(f_1+f_2,p,q,s)\,,$

(ii) If $s_1\leq s_2$ then $\widehat{BV}_\theta\,(f,p,q,s_1)\subseteq\widehat{BV}_\theta\,(f,p,q,s_2)\,,$

(iii) $\widehat{BV}_\theta\,(f,p,q_1,s)\cap\widehat{BV}_\theta\,(f,p,q_2,s)\subseteq\widehat{BV}_\theta\,(f,p,q_1+q_2,s)\,,$

(iv) If q_1 is stronger than q_2 then $\widehat{BV}_\theta\,(f,p,q_1,s)\subseteq\widehat{BV}_\theta\,(f,p,q_2,s)\,.$

Proof. i) The proof follows from the following inequality

$$r^{-s}\left[(f_1+f_2)\left(q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}\leq Dr^{-s}\left[f_1\left(q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}+Dr^{-s}\left[f_2\left(q\left(\varphi_{rn}\,(x)\right)\right)\right]^{p_r}.$$

ii), iii) and iv) follow easily.

Corollary 2.4 Let f be a modulus function, then we have

(i) If $q_1\cong$(equivalent to) q_2, then $\widehat{BV}_\theta\,(f,p,q_1,s)=\widehat{BV}_\theta\,(f,p,q_2,s)\,,$

(ii) $\widehat{BV}_\theta\,(f,p,q)\subseteq\widehat{BV}_\theta\,(f,p,q,s)\,,$

(iii) $\widehat{BV}_\theta\,(f,q)\subseteq\widehat{BV}_\theta\,(f,q,s)\,.$

Theorem 2.5. Suppose that $0 < m_r \leq t_r < \infty$ for each $r \in \mathbb{N}$. Then $\widehat{BV_\theta}(f, m, q) \subseteq \widehat{BV_\theta}(f, t, q)$.

Proof. Let $x \in \widehat{BV_\theta}(f, m, q)$. This implies that

$$[f(q(\varphi_{rn}(x)))]^{m_r} \leq 1$$

for sufficiently large values of k, say $k \geq k_0$ for some fixed $k_0 \in \mathbb{N}$. Since f is non decreasing, we have

$$\sum_{r=k_0}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{t_r} \leq \sum_{r=k_0}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{m_r}.$$

It gives $x \in \widehat{BV_\theta}(f, t, q)$.

The following result is a consequence of the above result.

Corollary 2.6

(i) If $0 < p_r \leq 1$ for each r, then $\widehat{BV_\theta}(f, p, q) \subseteq \widehat{BV_\theta}(f, q)$,

(ii) If $p_r \geq 1$ for all r, then $\widehat{BV_\theta}(f, q) \subseteq \widehat{BV_\theta}(f, p, q)$.

Theorem 2.7 The sequence space $\widehat{BV_\theta}(f, p, q, s)$ is solid.

Proof. Let $x \in \widehat{BV_\theta}(f, p, q, s)$, i.e.

$$\sum_{r=1}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{p_r} < \infty.$$

Let (α_r) be sequence of scalars such that $|\alpha_r| \leq 1$ for all $r \in \mathbb{N}$. Then the result follows from the following inequality.

$$\sum_{r=1}^{\infty} r^{-s}[f(q(\alpha_r\varphi_{rn}(x)))]^{p_r} \leq \sum_{r=1}^{\infty} r^{-s}[f(q(\varphi_{rn}(x)))]^{p_r}.$$

Corollary 2.8 The sequence space $\widehat{BV_\theta}(f, p, q, s)$ is monotone.

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