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INEXTENSIBLE FLOW OF A SEMI-REAL QUATERNIONIC CURVE IN SEMI-EUCLIDEAN SPACE $\mathbb{R}_2^4$

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Abstract. In this paper, we investigate a general formulation for inextensible flows of semi-real quaternionic curve in $\mathbb{R}_2^4$. We obtain necessary and sufficient conditions for inextensible flow of semi-real quaternionic curves. Moreover, we give the evolution equation of curvatures as a partial differential equation.

1. Introduction

The Irish mathematician Hamilton wanted to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties and then he discovered the quaternion in 1843. Quaternions can be represented as the sum of a scalar and a vector. They are applied to mechanic and physics. The quaternions set Q is isomorphic to $\mathbb{R}^4$, which is a four-dimensional vector space over $\mathbb{R}$. The Serret-Frenet formulae for a quaternionic curves was given in $\mathbb{R}^3$ by K. Bharathi and M. Nagara [7]. Also, they defined Serret-Frenet formulae for quaternionic curves in $\mathbb{R}^4$, by using the formulae in $\mathbb{R}^3$. After these studies, a lot of articles about quaternionic curves are published in $\mathbb{R}^3$ and $\mathbb{R}^4$. And then Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna [2]. Gök et al and Kahraman et al defined a new kind of slant helix in Euclidean space $\mathbb{R}^4$ [6] and semi-Euclidean space $\mathbb{R}_2^4$ [5]. It called quaternionic B_2-slant and semi-real quaternionic B_2-slant helix, respectively. Güngör and Tosun studied quaternionic rectifying curves in $\mathbb{R}^4$ [8]. Moreover, Yıldız and Karakuş examined quaternionic normal curves in $\mathbb{R}^4$ [11].

In differential geometry studies, contrary what is known the time parameter plays an important role. One of the most important of these studies is envolving curve, which is the family of curves parametrized by time. Also, the time evolution of curve can be treated as flow of curve. Inextensible flows of curves and developable

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surfaces were studied in $\mathbb{R}^3$ by Kwon and Park [3, 4]. Inextensible flows of curves were investigated by according to Darboux frame in $\mathbb{R}^3$ [9] and were examined in Lie Group [10]. Uçum et al have studied flows for partially null and pseudo null curves. Moreover, Körpınar and Baş have investigated inextensible flows of quaternionic curves in Euclidean space $\mathbb{R}^4$ [12].

Our aim is to study inextensible flow of semi-real quaternionic curves in $\mathbb{R}^4_2$. We give necessary and sufficient conditions for inextensible flows of semi-real quaternionic curves. Also, we give the evolution equation of curvatures as a partial differential equation.

2. Preliminaries

In this section, we will give a brief summary of the semi-real quaternion in the semi Euclidean space $\mathbb{R}^4_2$. A semi real quaternion q is expressed as $q = ae_1 + be_2 + ce_3 + d$ such that

$$\begin{aligned} e_i \times e_i &= -\varepsilon_{e_i}, \qquad (1 \le i \le 3), \\ e_i \times e_j &= \varepsilon_{e_i}\varepsilon_{e_j} e_k, \qquad \text{in } \mathbb{R}^3_1, \\ e_i \times e_j &= -\varepsilon_{e_i}\varepsilon_{e_j} e_k, \quad \text{in } \mathbb{R}^4_2, \end{aligned}$$

where (ijk) is an even permutation of (123) and $a, b, c, d \in \mathbb{R}$ [2]. Further, any quaternion can be written as $q = S_q + V_q$ where $S_q = d$ and $V_q = ae_1 + be_2 + ce_3$ denote scalar and vector part of q, respectively. The multiplication of two semi-real quaternions p and q is defined as $p \times q = S_p S_q + < V_p, V_q > + S_p V_q + S_q V_p + V_p \wedge V_q$, for every $p, q \in \mathbb{R}^4_2$ where $<,>$ and $\wedge$ are scalar and cross product in $\mathbb{R}^3_1$, respectively. The conjugate of q is $\gamma q = S_q - V_q$. By using this, for every $p, q \in \mathbb{R}^4_2$ the symmetric, non-degenerate, bilinear form h is defined as follows

$$\begin{aligned} h &: \mathbb{R}^4_2 \times \mathbb{R}^4_2 \to \mathbb{R}, \\ h(p,q) &= \frac{1}{2}\left[\varepsilon_p \varepsilon_{\gamma q}(p \times \gamma q) + \varepsilon_q \varepsilon_{\gamma p}(q \times \gamma p)\right] \end{aligned}$$

The norm of q is denoted by

$$\|q\|^2 = -a^2 - b^2 + c^2 + d^2.$$

If $q \times \gamma q = 0$ then q is called a semi-real spatial quaternion. If $h(p,q) = 0$ then p and q are called h-orthogonal where $p, q \in \mathbb{R}^4_2$ [2]. If $\|q\|^2 = 1$, then q is called a semi-real unit quaternion.

Now, we give the definition of semi-real quaternionic curve and its Serret-Frenet apparatus. $\mathbb{R}^4_2$ is identified with the space of unit semi-quaternions and is denoted by Q_v. Let

$$\beta : I \subset R \longrightarrow Q_v, \qquad \beta(s) = \sum_{i=1}^{4} \beta_i(s) e_i, \qquad e_4 = 1,$$

be a smooth curve β with nonzero curvatures $\{K, k, (r - \varepsilon_t\varepsilon_T\varepsilon_N K)\}$ defined over the interval $I = [0, 1]$. Let the parameter s be chosen such that the tangent $T = \beta'(s) = \sum_{i=1}^{4} \beta_i'(s)e_i$ has unit magnitude and the Serret-Frenet apparatus of β are $\{T, N, B_1, B_2\}$. The Frenet equations are

$$\begin{aligned} T'(s) &= \varepsilon_N K N(s) \\ N'(s) &= -\varepsilon_t\varepsilon_N K T(s) + \varepsilon_n k B_1(s) \\ B_1'(s) &= -\varepsilon_t k N(s) + \varepsilon_n (r - \varepsilon_t\varepsilon_T\varepsilon_N K) B_2(s) \\ B_2'(s) &= -\varepsilon_b (r - \varepsilon_t\varepsilon_T\varepsilon_N K) B_1, \end{aligned} \tag{2.1}$$

where $h(T, T) = \varepsilon_T, h(N, N) = \varepsilon_T, h(B_1, B_1) = \varepsilon_n\varepsilon_T$ and $h(B_1, B_1) = \varepsilon_b\varepsilon_T$ [1]

3. Flow of Semi-Real Quaternionic Curve

Throughout this paper, we assume that $\alpha : [0, l] \times [0, w] \to Q_v$ is a one parameter family of smooth semi-real quaternionic curve in Q_v where l is arclength of initial curve and u is the curve parametrization variable, $0 \le u \le l$. Let $\alpha(u, t)$ be the position vector of the semi-real quaternionic curve at time t. The arclength variation of $\alpha(u, t)$ is given by

$$s(u, t) = \int_0^u \left\| \frac{\partial \alpha}{\partial u} \right\| du = \int_0^u v du. \tag{3.1}$$

The operator $\frac{\partial}{\partial s}$ is given in term of u by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$.

Definition 1. *Let α be smooth semi-real quaternionic curve with the Frenet frame $\{T, N, B_1, B_2\}$. Any flow of α can be given by*

$$\frac{\partial \alpha}{\partial t} = f_1 T + f_2 N + f_3 B_1 + f_4 B_2 \tag{3.2}$$

where f_1, f_2, f_3 and f_4 are scalar speed functions of α.

In Q_v, the inextensible condition of the length of the curve can be expressed by [4]

$$\frac{\partial}{\partial t} s(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0. \tag{3.3}$$

Definition 2. *A semi-real quaternionic curve evolution $\alpha(u, t)$ and its flow $\frac{\partial \alpha}{\partial t}$ in Q_v are said to be inextensible if*

$$\frac{\partial}{\partial t} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0. \tag{3.4}$$

Lemma 1. *The evolution equation for the speed v is given by*

$$\frac{\partial v}{\partial t}=\varepsilon_T\frac{\partial f_1}{\partial u}-\varepsilon_t\varepsilon_T\varepsilon_N v\kappa f_2 \tag{3.5}$$

Proof. As $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $v^2=h\left(\frac{\partial\alpha}{\partial u},\frac{\partial\alpha}{\partial u}\right)$, we have

$$2v\frac{\partial v}{\partial t}=\frac{\partial}{\partial t}h\left(\frac{\partial\alpha}{\partial u},\frac{\partial\alpha}{\partial u}\right)=2h\left(\frac{\partial\alpha}{\partial u},\frac{\partial}{\partial u}\left(\frac{\partial\alpha}{\partial t}\right)\right).$$

Thus, we obtain

$$\frac{\partial v}{\partial t}=\varepsilon_T\frac{\partial f_1}{\partial u}-\varepsilon_t\varepsilon_T\varepsilon_N v\kappa f_2$$

□

Theorem 1. *The flow of semi-real quaternionic curve is inextensible if and only if*

$$\frac{\partial f_1}{\partial s}=\varepsilon_t\varepsilon_N\kappa f_2 \tag{3.6}$$

Proof. Let the flow of semi-real quaternionic curve be inextensible. From equation (3.3) and (3.5), we have

$$\frac{\partial}{\partial t}s\left(u,t\right)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u\left(\varepsilon_T\frac{\partial f_1}{\partial u}-\varepsilon_t\varepsilon_T\varepsilon_N v\kappa f_2\right)du=0$$

This clearly forces

$$\frac{\partial f_1}{\partial s}=\varepsilon_t\varepsilon_N\kappa f_2.$$

□

Lemma 2. *Let the flow of $\alpha\left(u,t\right)$ be inextensible. Derivatives of the elements of Frenet frame with respect to evolution parameter can be given as follows;*

$$\begin{aligned}
\frac{\partial T}{\partial t}&=\left(\varepsilon_N f_1\kappa+\frac{\partial f_2}{\partial s}-\varepsilon_t f_3 k\right)N+\left(\varepsilon_n f_2 k+\frac{\partial f_3}{\partial s}-\varepsilon_b f_4\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)B_1\\
&\quad+\left(\varepsilon_n f_3\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)+\frac{\partial f_4}{\partial s}\right)B_2,\\
\frac{\partial N}{\partial t}&=-\left(\varepsilon_T f_1\kappa+\varepsilon_T\varepsilon_N\frac{\partial f_2}{\partial s}-\varepsilon_t\varepsilon_T\varepsilon_N f_3 k\right)T+\varepsilon_n\varepsilon_T\psi_1 B_1+\varepsilon_b\varepsilon_T\psi_2 B_2,\\
\frac{\partial B_1}{\partial t}&=-\left(f_2 k+\varepsilon_n\frac{\partial f_3}{\partial s}-\varepsilon_n\varepsilon_b f_4\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)T-\varepsilon_N\psi_1 N+\varepsilon_b\varepsilon_T\psi_3 B_2,\\
\frac{\partial B_2}{\partial t}&=-\left(\varepsilon_n\varepsilon_b f_3\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)+\varepsilon_b\frac{\partial f_4}{\partial s}\right)T-\varepsilon_N\psi_2 N-\varepsilon_n\varepsilon_T\psi_3 B_1.
\end{aligned}$$

where $\psi_1=h\left(\frac{\partial N}{\partial t},B_1\right)$, $\psi_2=h\left(\frac{\partial N}{\partial t},B_2\right)$, $\psi_3=h\left(\frac{\partial B_1}{\partial t},B_2\right)$.

Proof. Let $\frac{\partial\alpha}{\partial t}$ be inextensible. Then, considering that $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, we get

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial\alpha}{\partial t}\right) = \frac{\partial}{\partial s}\left(f_1T + f_2N + f_3B_1 + f_4B_2\right)$$

substituting (3.6) in the last equation, we have

$$\begin{aligned} \frac{\partial T}{\partial t} &= \left(\varepsilon_N f_1\kappa + \frac{\partial f_2}{\partial s} - \varepsilon_t f_3 k\right)N + \left(\varepsilon_n f_2 k + \frac{\partial f_3}{\partial s} - \varepsilon_b f_4\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)B_1 \\ &+ \left(\varepsilon_n f_3\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) + \frac{\partial f_4}{\partial s}\right)B_2. \end{aligned} \tag{3.7}$$

Now, if we consider orthogonality of $\{T, N, B_1, B_2\}$, then we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}h\left(T,N\right) = h\left(\frac{\partial T}{\partial t}, N\right) + h\left(T, \frac{\partial N}{\partial t}\right) \\ &= \varepsilon_N\left(\varepsilon_N f_1\kappa + \frac{\partial f_2}{\partial s} - \varepsilon_t f_3 k\right) + h\left(T, \frac{\partial N}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t}h\left(T,B_1\right) = h\left(\frac{\partial T}{\partial t}, B_1\right) + h\left(T, \frac{\partial B_1}{\partial t}\right) \\ &= \varepsilon_n\varepsilon_T\left(\varepsilon_n f_2 k + \frac{\partial f_3}{\partial s} - \varepsilon_b f_4\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right) + h\left(T, \frac{\partial B_1}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t}h\left(T,B_2\right) = h\left(\frac{\partial T}{\partial t}, B_2\right) + h\left(T, \frac{\partial B_2}{\partial t}\right) \\ &= \varepsilon_b\varepsilon_T\left(\varepsilon_n f_3\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) + \frac{\partial f_4}{\partial s}\right) + h\left(T, \frac{\partial B_2}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t}h\left(N,B_1\right) = h\left(\frac{\partial N}{\partial t}, B_1\right) + h\left(N, \frac{\partial B_1}{\partial t}\right) \\ &= \varepsilon_n\varepsilon_T\psi_1 + h\left(N, \frac{\partial B_1}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t}h\left(N,B_2\right) = h\left(\frac{\partial N}{\partial t}, B_2\right) + h\left(N, \frac{\partial B_2}{\partial t}\right) \\ &= \varepsilon_b\varepsilon_T\psi_2 + h\left(N, \frac{\partial B_2}{\partial t}\right), \\ 0 &= \frac{\partial}{\partial t}h\left(B_1,B_2\right) = h\left(\frac{\partial B_1}{\partial t}, B_2\right) + h\left(B_1, \frac{\partial B_2}{\partial t}\right) \\ &= \varepsilon_b\varepsilon_T\psi_3 + h\left(B_1, \frac{\partial B_2}{\partial t}\right), \end{aligned}$$

which brings about that

$$\begin{aligned}
\frac{\partial N}{\partial t} &= -\left(\varepsilon_T f_1 \kappa + \varepsilon_T \varepsilon_N \frac{\partial f_2}{\partial s} - \varepsilon_t \varepsilon_T \varepsilon_N f_3 k\right) T + \varepsilon_n \varepsilon_T \psi_1 B_1 + \varepsilon_b \varepsilon_T \psi_2 B_2, \\
\frac{\partial B_1}{\partial t} &= -\left(f_2 k + \varepsilon_n \frac{\partial f_3}{\partial s} - \varepsilon_n \varepsilon_b f_4 \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)\right) T - \varepsilon_N \psi_1 N + \varepsilon_b \varepsilon_T \psi_3 B_2, \\
\frac{\partial B_2}{\partial t} &= -\left(\varepsilon_n \varepsilon_b f_3 \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right) + \varepsilon_b \frac{\partial f_4}{\partial s}\right) T - \varepsilon_N \psi_2 N - \varepsilon_n \varepsilon_T \psi_3 B_1
\end{aligned}$$

where $\psi_1 = h\left(\frac{\partial N}{\partial t}, B_1\right)$, $\psi_2 = h\left(\frac{\partial N}{\partial t}, B_2\right)$, $\psi_3 = h\left(\frac{\partial B_1}{\partial t}, B_2\right)$. □

Theorem 2. *Let the flow of $\alpha(u,t)$ be inextensible. Then the evolution equation of κ is*

$$\begin{aligned}
\frac{\partial \kappa}{\partial t} = {} & \varepsilon_t \varepsilon_N f_2 \kappa^2 + f_1 \frac{\partial \kappa}{\partial s} + \varepsilon_N \frac{\partial^2 f_2}{\partial s^2} - 2\varepsilon_t \varepsilon_N \frac{\partial f_3}{\partial s} k - \varepsilon_t \varepsilon_N f_3 \frac{\partial k}{\partial s} \\
& - \varepsilon_t \varepsilon_n \varepsilon_N f_2 k^2 + \varepsilon_t \varepsilon_b \varepsilon_N f_4 k \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)
\end{aligned}$$

Proof. Since $\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right)$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right) = {} & \left(-\varepsilon_t f_1 \kappa^2 - \varepsilon_t \varepsilon_N \frac{\partial f_2}{\partial s} \kappa + \varepsilon_N f_3 \kappa k\right) T \\
+ {} & \left(\varepsilon_t f_2 \kappa^2 + \varepsilon_N f_1 \frac{\partial \kappa}{\partial s} + \frac{\partial^2 f_2}{\partial s^2} - 2\varepsilon_t \frac{\partial f_3}{\partial s} k - \varepsilon_t f_3 \frac{\partial k}{\partial s} - \varepsilon_t \varepsilon_n f_2 k^2\right. \\
+ {} & \left.\varepsilon_t \varepsilon_b f_4 k \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)\right) N \\
+ {} & \left(\varepsilon_n \varepsilon_N f_1 \kappa k + 2\varepsilon_n \frac{\partial f_2}{\partial s} k - \varepsilon_t \varepsilon_n f_3 k^2 + \varepsilon_n f_2 \frac{\partial k}{\partial s} + \frac{\partial^2 f_3}{\partial s^2}\right. \\
- {} & \left. 2\varepsilon_b \frac{\partial f_4}{\partial s} \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right) - \varepsilon_b f_4 \frac{\partial \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)}{\partial s} - \varepsilon_n \varepsilon_b f_3 \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)^2\right) B_1 \\
+ {} & \left(f_2 k \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right) + 2\varepsilon_n \frac{\partial f_3}{\partial s} \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right) - \varepsilon_n \varepsilon_b f_4 \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)^2\right. \\
+ {} & \left. \varepsilon_n f_3 \frac{\partial \left(r - \varepsilon_t \varepsilon_T \varepsilon_N \kappa\right)}{\partial s} + \frac{\partial^2 f_4}{\partial s^2}\right) B_2
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right) &= \frac{\partial}{\partial t}\left(\varepsilon_N \kappa N\right) = \varepsilon_N \frac{\partial \kappa}{\partial t} N + \varepsilon_N \kappa \frac{\partial N}{\partial t} \\
&= \left(-\varepsilon_T \varepsilon_N f_1 \kappa^2 - \varepsilon_T \frac{\partial f_2}{\partial s} \kappa + \varepsilon_t \varepsilon_T f_3 \kappa k\right) T + \varepsilon_N \frac{\partial \kappa}{\partial t} N + \varepsilon_n \varepsilon_T \varepsilon_N \psi_1 \kappa B_1 \\
&\quad + \varepsilon_b \varepsilon_T \varepsilon_N \psi_2 \kappa B_2.
\end{aligned}$$

From equality of the component of N in above two equations, we obtain

$$\begin{aligned}\frac{\partial \kappa}{\partial t} &= \varepsilon_t\varepsilon_N f_2\kappa^2 + f_1\frac{\partial \kappa}{\partial s} + \varepsilon_N\frac{\partial^2 f_2}{\partial s^2} - 2\varepsilon_t\varepsilon_N\frac{\partial f_3}{\partial s}k - \varepsilon_t\varepsilon_N f_3\frac{\partial k}{\partial s} - \varepsilon_t\varepsilon_n\varepsilon_N f_2 k^2 \\ &+ \varepsilon_t\varepsilon_b\varepsilon_N f_4 k\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right).\end{aligned}$$

□

Corollary 1. *In theorem (2), from rest of the equality, we get*

$$\begin{aligned}\kappa\psi_1 &= \varepsilon_T f_1 k\kappa + 2\varepsilon_T\varepsilon_N\frac{\partial f_2}{\partial s}k - \varepsilon_t\varepsilon_T\varepsilon_N f_3 k^2 + \varepsilon_T\varepsilon_N f_2\frac{\partial k}{\partial s} + \varepsilon_n\varepsilon_T\varepsilon_N\frac{\partial^2 f_3}{\partial s^2} \\ &- 2\varepsilon_n\varepsilon_b\varepsilon_T\varepsilon_N\frac{\partial f_4}{\partial s}\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) - \varepsilon_n\varepsilon_b\varepsilon_T\varepsilon_N f_4\frac{\partial\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial s} \\ &- \varepsilon_b\varepsilon_T\varepsilon_N f_3\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)^2 \\ \kappa\psi_2 &= \varepsilon_t\varepsilon_b f_2 k\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) + 2\varepsilon_t\varepsilon_n\varepsilon_b\frac{\partial f_3}{\partial s}\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) - \varepsilon_t\varepsilon_n f_4\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)^2 \\ &+ \varepsilon_t\varepsilon_n\varepsilon_b f_3\frac{\partial\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial s} + \varepsilon_t\varepsilon_b\frac{\partial^2 f_4}{\partial s^2}\end{aligned}$$

Theorem 3. *Let the flow of $\alpha(u,t)$ be inextensible. Then the evolution equation of k is*

$$\begin{aligned}\frac{\partial k}{\partial t} &= \varepsilon_t\varepsilon_N f_2\kappa k + \varepsilon_t\varepsilon_n\varepsilon_N\frac{\partial f_3}{\partial s}\kappa - \varepsilon_t\varepsilon_n\varepsilon_b\varepsilon_N f_4\kappa\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right) + \varepsilon_T\frac{\partial \psi_1}{\partial s} \\ &- \varepsilon_n\varepsilon_T\psi_2\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right).\end{aligned}$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial N}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial N}{\partial s}\right)$, it is seen that

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial N}{\partial t}\right) &= \left(-\varepsilon_t\varepsilon_T\varepsilon_N f_2\kappa^2 - \varepsilon_T f_1\frac{\partial \kappa}{\partial s} - \varepsilon_T\varepsilon_N\frac{\partial^2 f_2}{\partial s^2} + \varepsilon_t\varepsilon_T\varepsilon_N\frac{\partial f_3}{\partial s}k + \varepsilon_t\varepsilon_T\varepsilon_N f_3\frac{\partial k}{\partial s}\right)T \\ &+ \left(-\varepsilon_T\varepsilon_N f_1\kappa^2 - \varepsilon_T\frac{\partial f_2}{\partial s}\kappa + \varepsilon_t\varepsilon_T f_3\kappa k - \varepsilon_t\varepsilon_n\varepsilon_T\psi_1 k\right)N \\ &+ \left(\varepsilon_n\varepsilon_T\frac{\partial \psi_1}{\partial s} - \varepsilon_T\psi_2\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)B_1 \\ &+ \left(\varepsilon_T\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\psi_1 + \varepsilon_b\varepsilon_T\frac{\partial \psi_2}{\partial s}\right)B_2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial N}{\partial s}\right) &= \frac{\partial}{\partial t}\left(-\varepsilon_t\varepsilon_N\kappa T+\varepsilon_n k B_1\right)\\
&= \left(-\varepsilon_t\varepsilon_N\frac{\partial\kappa}{\partial t}-\varepsilon_n f_2 k^2-\frac{\partial f_3}{\partial s}k+\varepsilon_b f_4 k\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)T\\
&+\left(-\varepsilon_t f_1\kappa^2-\varepsilon_t\varepsilon_N\frac{\partial f_2}{\partial s}\kappa+\varepsilon_N f_3\kappa k-\varepsilon_n\varepsilon_N\psi_1 k\right)N\\
&+\left(-\varepsilon_t\varepsilon_n\varepsilon_N f_2 k\kappa-\varepsilon_t\varepsilon_N\frac{\partial f_3}{\partial s}\kappa+\varepsilon_t\varepsilon_b\varepsilon_N f_4\kappa\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)+\varepsilon_n\frac{\partial k}{\partial t}\right)B_1\\
&+\left(-\varepsilon_t\varepsilon_n\varepsilon_N f_3\kappa\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)-\varepsilon_t\varepsilon_N\frac{\partial f_4}{\partial s}\kappa+\varepsilon_n\varepsilon_b\varepsilon_T\psi_3 k\right)B_2\end{aligned}$$

From above equations, we get

$$\begin{aligned}\frac{\partial k}{\partial t} &= \varepsilon_t\varepsilon_N f_2\kappa k+\varepsilon_t\varepsilon_n\varepsilon_N\frac{\partial f_3}{\partial s}\kappa-\varepsilon_t\varepsilon_n\varepsilon_b\varepsilon_N f_4\kappa\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)+\varepsilon_T\frac{\partial\psi_1}{\partial s}\\
&-\varepsilon_n\varepsilon_T\psi_2\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right).\end{aligned}$$

□

Corollary 2. *In theorem (3), from rest of the equality, we obtain*

$$\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\psi_1=-\varepsilon_b\frac{\partial\psi_2}{\partial s}-\varepsilon_t\varepsilon_n\varepsilon_T\varepsilon_N f_3\kappa\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)-\varepsilon_t\varepsilon_T\varepsilon_N\frac{\partial f_4}{\partial s}\kappa+\varepsilon_n\varepsilon_b\psi_3 k$$

Theorem 4. *Let the flow of $\alpha(u,t)$ be inextensible. Then the evolution equation of $\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)$ is*

$$\frac{\partial\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial t}=\varepsilon_n\varepsilon_b\varepsilon_N\psi_2 k+\varepsilon_n\varepsilon_b\varepsilon_T\frac{\partial\psi_3}{\partial s}$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial B_1}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial B_1}{\partial s}\right)$, it is seen that

$$\begin{aligned}\frac{\partial}{\partial s}\left(\frac{\partial B_1}{\partial t}\right) &= \left(-\frac{\partial f_2}{\partial s}k-f_2\frac{\partial k}{\partial s}-\varepsilon_n\frac{\partial^2 f_3}{\partial s^2}+\varepsilon_n\varepsilon_b\frac{\partial f_4}{\partial s}\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right.\\
&\left.+\varepsilon_n\varepsilon_b f_4\frac{\partial\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial s}+\varepsilon_t\psi_1\kappa\right)T\\
&+\left(-\varepsilon_N f_2\kappa k-\varepsilon_n\varepsilon_N\frac{\partial f_3}{\partial s}\kappa+\varepsilon_n\varepsilon_b\varepsilon_N f_4\kappa\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)-\varepsilon_N\frac{\partial\psi_1}{\partial s}\right)N\\
&+\left(-\varepsilon_n\varepsilon_N\psi_1 k-\varepsilon_T\psi_3\left(r-\varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)B_1+\left(\varepsilon_b\varepsilon_T\frac{\partial\psi_3}{\partial s}\right)B_2\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial B_1}{\partial s}\right) &= \frac{\partial}{\partial t}\left(-\varepsilon_t kN + \varepsilon_n\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)B_2\right)\\
&= \Bigg(\varepsilon_t\varepsilon_T f_1 k\kappa + \varepsilon_t\varepsilon_T\varepsilon_N\frac{\partial f_2}{\partial s}k - \varepsilon_T\varepsilon_N f_3 k^2\\
&\quad -\varepsilon_b f_3\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)^2 - \varepsilon_n\varepsilon_b\frac{\partial f_4}{\partial s}\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\Bigg)T\\
&\quad + \left(-\varepsilon_t\frac{\partial k}{\partial t} - \varepsilon_n\varepsilon_N\psi_2\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)N\\
&\quad + \left(-\varepsilon_t\varepsilon_n\varepsilon_T\psi_1 k - \varepsilon_T\psi_3\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)\right)B_1\\
&\quad + \left(-\varepsilon_t\varepsilon_b\varepsilon_T\psi_2 k + \varepsilon_n\frac{\partial\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial t}\right)B_2
\end{aligned}
$$

From above equations, we obtain

$$
\frac{\partial\left(r - \varepsilon_t\varepsilon_T\varepsilon_N\kappa\right)}{\partial t} = \varepsilon_n\varepsilon_b\varepsilon_N\psi_2 k + \varepsilon_n\varepsilon_b\varepsilon_T\frac{\partial\psi_3}{\partial s}.
$$

□

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