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## HARARY ENERGY OF COMPLEMENT OF LINE GRAPHS OF REGULAR GRAPHS

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ABSTRACT. The Harary matrix of a graph G is defined as  $H(G) = [h_{ij}]$ , where  $h_{ij} = \frac{1}{d(v_i, v_j)}$ , if  $i \neq j$  and  $h_{ij} = 0$ , otherwise, where  $d(v_i, v_j)$  is the distance between the vertices  $v_i$  and  $v_j$  in G. The H-energy of G is defined as the sum of the absolute values of the eigenvalues of Harary matrix. Two graphs are said to be H-equienergetic if they have same H-energy. In this paper we obtain the H-energy of the complement of line graphs of certain regular graphs in terms of the order and regularity of a graph and thus constructs pairs of H-equienergetic graphs of same order and having different H-eigenvalues.

## 1. INTRODUCTION

Let G be a simple, undirected, connected graph with n vertices and m edges. Let the vertices of G be labeled as  $v_1, v_2, \ldots, v_n$ . The *adjacency matrix* of a graph G is the square matrix  $A(G) = [a_{ij}]$ , in which  $a_{ij} = 1$  if  $v_i$  is adjacent to  $v_j$  and  $a_{ij} = 0$ , otherwise. The eigenvalues of A(G) are the *adjacency eigenvalues* of G, and they are labeled as  $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ . These form the *adjacency spectrum* of G [4]. Two graphs are said to be cospectral if they have same spectra.

The distance between the vertices  $v_i$  and  $v_j$ , denoted by  $d(v_i, v_j)$ , is the length of the shortest path joining  $v_i$  and  $v_j$ . The diameter of a graph G, denoted by diam(G), is the maximum distance between any pair of vertices of G. A graph Gis said to be *r*-regular graph if all of its vertices have same degree equal to r.

The Harary matrix [9] of a graph G is a square matrix  $H(G) = [h_{ij}]$  of order n, where

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$$h_{ij} = \begin{cases} \frac{1}{d(v_i, v_j)}, & \text{if} \quad i \neq j \\ 0, & \text{if} \quad i = j. \end{cases}$$

The Harary matrix was used in the study of molecules in the quantitative structure property relationship (QSPR) models [9].

The *Harary index* defined as the sum of the reciprocal of the distances between all pairs of vertices and it can be derived from the Harary matrix. It has interesting properties in structure-property correlations [11, 16].

The eigenvalues of H(G) labeled as  $\xi_1 \ge \xi_2 \ge \cdots \ge \xi_n$  are said to be the Harary eigenvalues or H-eigenvalues of G and their collection is called Harary spectrum or H-spectrum of G. Two non-isomorphic graphs are said to be H-cospectral if they have same H-spectra.

The Harary energy or H-energy of a graph G, denoted by HE(G), is defined as [5]

$$HE(G) = \sum_{i=1}^{n} |\xi_i|$$
 (1)

The Harary energy is defined in full analogy with the ordinary graph energy E(G), defined as [6]

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$
(2)

The ordinary graph energy has a relation with the total  $\pi$ -electron energy of a molecule in quantum chemistry [10]. Bounds for the Harary energy of a graph are reported in [3,5].

Two connected graphs  $G_1$  and  $G_2$  are said to be *Harary equienergetic* or *H*-equienergetic if  $HE(G_1) = HE(G_2)$ . The *H*-equienergetic graphs are reported in [12, 13]. The distance energy of complements of iterated line graphs of regular graphs has been obtained in [8]. In this paper we use similar technique of [8] to obtain the *H*-energy of the complement of line graphs of certain regular graphs and thus construct *H*-equienergetic graphs having different *H*-spectra.

The complement of a graph G is a graph  $\overline{G}$ , with vertex set same as of G and two vertices in  $\overline{G}$  are adjacent if and only if they are not adjacent in G. The line graph of G, denoted by L(G) is the graph whose vertices corresponds to the edges of G and two vertices of L(G) are adjacent if and only if the corresponding edges are adjacent in G. For k = 1, 2, ... the k-th iterated line graph of G is defined as  $L^k(G) = L(L^{k-1}(G))$ , where  $L^0(G) = G$  and  $L^1(G) = L(G)$  [7].

If G is a regular graph of order  $n_0$  and of degree  $r_0$  then the line graph L(G) is a regular graph of order  $n_1 = (n_0 r_0)/2$  and of degree  $r_1 = 2r_0 - 2$ . Consequently the order and degree of  $L^k(G)$  are [1,2]

$$n_k = \frac{r_{k-1}n_{k-1}}{2} \tag{3}$$

and

$$r_k = 2r_{k-1} - 2, (4)$$

where  $n_i$  and  $r_i$  stands for order and degree of  $L^i(G)$ , i = 0, 1, ...Therefore

r

$$r_k = 2^k r_0 - 2^{k+1} + 2 \tag{5}$$

and

$$n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2)$$
(6)

We need following results.

**Theorem 1.** [4] If G is an r-regular graph, then its maximum adjacency eigenvalue is equal to r.

**Theorem 2.** [15] If  $\lambda_1, \lambda_2, \ldots, \lambda_n$  are the adjacency eigenvalues of a regular graph G of order n and of degree r, then the adjacency eigenvalues of L(G) are

$$\lambda_i + r - 2,$$
  $i = 1, 2, ..., n,$  and  
-2,  $n(r-2)/2$  times.

**Theorem 3.** [14] Let G be an r-regular graph of order n. If  $r, \lambda_2, \ldots, \lambda_n$  are the adjacency eigenvalues of G, then the adjacency eigenvalues of  $\overline{G}$ , the complement of G, are n - r - 1 and  $-\lambda_i - 1$ ,  $i = 2, 3, \ldots, n$ .

**Theorem 4.** [3] Let G be an r-regular graph of order n and let  $diam(G) \leq 2$ . If  $r, \lambda_2, \ldots, \lambda_n$  are the adjacency eigenvalues of G, then its H-eigenvalues are  $\frac{1}{2}(n + r - 1)$  and  $\frac{1}{2}(\lambda_i - 1)$ ,  $i = 2, 3, \ldots, n$ .

**Lemma 5.** [8] Let G be an r-regular graph of order n. If  $r \leq \frac{n-1}{2}$ , then diam  $\left(\overline{L^k(G)}\right) = 2, k \geq 1$ .

## 2. Results

**Theorem 6.** Let G be an r-regular graph of order n. If  $r \leq \frac{n-1}{2}$ , then

$$HE\left(\overline{L(G)}\right) = r(n-2)$$

*Proof.* Let the adjacency eigenvalues of G be  $r, \lambda_2, \ldots, \lambda_n$ . From Theorem 2, the adjacency eigenvalues of L(G) are

$$\begin{array}{cccc}
2r - 2, & \text{and} \\
\lambda_i + r - 2, & i = 2, 3, \dots, n, \\
-2, & n(r-2)/2 \text{ times.} \end{array}$$
(7)

From Theorem 3 and Eq. (7), the adjacency eigenvalues of  $\overline{L(G)}$  are

$$\left. \begin{array}{ccc} (nr/2) - 2r + 1, & \text{and} \\ & & \\ -\lambda_i - r + 1, & i = 2, 3, \dots, n, \\ & & \\ 1, & n(r-2)/2 \text{ times.} \end{array} \right\}$$
(8)

The graph  $\overline{L(G)}$  is a regular graph of order nr/2 and of degree (nr/2) - 2r + 1. Since  $r \leq \frac{n-1}{2}$ , by Lemma 5,  $diam\left(\overline{L(G)}\right) = 2$ . Therefore by Theorem 4 and Eq. (8), the *H*-eigenvalues of  $\overline{L(G)}$  are

$$(nr - 2r)/2,$$
 and  
 $-(\lambda_i + r)/2,$   $i = 2, 3, ..., n,$  and  
 $0, n(r-2)/2$  times. (9)

All adjacency eigenvalues of a regular graph of degree r satisfy the condition  $-r \leq \lambda_i \leq r$  [4]. Therefore  $\lambda_i + r \geq 0$ , i = 1, 2, ..., n. Therefore by (9),

$$HE\left(\overline{L(G)}\right) = \frac{nr-2r}{2} + \sum_{i=2}^{n} \frac{(\lambda_i+r)}{2} + |0| \times \frac{n(r-2)}{2}$$
$$= r(n-2) \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.$$



FIGURE 1. Cycle  $C_6$  and  $\overline{L(C_6)}$ .

**Example 7.** Consider the cycle  $C_6$ . It satisfies the conditions of Theorem 6. Complement of  $L(C_6)$  is shown in the Figure 1. The H-eigenvalues of  $\overline{L(C_6)}$  are 4, 0, -0.5, -0.5, -1.5, -1.5. Hence  $HE(\overline{L(C_6)}) = 8$  and by Theorem 6 also,  $HE(\overline{L(C_6)}) = 8$ .

**Corollary 8.** Let G be a regular graph of order  $n_0$  and of degree  $r_0$ . Let  $n_k$  and  $r_k$  be the order and degree respectively of the k-th iterated line graph  $L^k(G)$ ,  $k \ge 1$ . If  $r_0 \le \frac{n_0-1}{2}$ , then

$$HE(\overline{L^k(G)}) = r_{k-1}(n_{k-1}-2).$$

*Proof.* If  $r_0 \leq \frac{n_0-1}{2}$ , then by Eqs. (3) and (4), we have

$$r_1 = 2r_0 - 2 \le n_0 - 3 \le \frac{1}{2} \left(\frac{n_0 r_0}{2} - 1\right) = \frac{n_1 - 1}{2}.$$

Hence

$$r_{k-1} \le \frac{n_{k-1} - 1}{2}.$$

Therefore by Theorem 6,

$$HE\left(\overline{L^{k}(G)}\right) = HE\left(\overline{L(L^{k-1}(G))}\right) = r_{k-1}(n_{k-1}-2).$$

**Corollary 9.** Let G be a regular graph of order  $n_0$  and of degree  $r_0$ . Let  $n_k$  and  $r_k$  be the order and degree respectively of the k-th iterated line graph  $L^k(G)$ ,  $k \ge 1$ . If  $r_0 \le \frac{n_0-1}{2}$ , then

$$HE\left(\overline{L^{k}(G)}\right) = \left[\frac{n_{0}}{2^{k-1}}\prod_{i=0}^{k-1}(2^{i}r_{0}-2^{i+1}+2)\right] - 2(2^{k-1}r_{0}-2^{k}+2).$$

### 3. *H*-Equienergetic graphs

If  $G_1$  and  $G_2$  are the regular graphs of same order and of same degree. Then  $L(G_1)$  and  $L(G_2)$  are of the same order and of same degree. Further their complements are also of same order and of same degree.

**Lemma 10.** Let  $G_1$  and  $G_2$  be regular graphs of the same order n and of the same degree r. If  $r \leq \frac{n-1}{2}$ , then  $\overline{L(G_1)}$  and  $\overline{L(G_2)}$  are H-cospectral if and only if  $G_1$  and  $G_2$  are cospectral.

*Proof.* Follows from Eqs. (7), (8) and (9).

**Lemma 11.** Let  $G_1$  and  $G_2$  be regular graphs of the same order n and of the same degree r. If  $r \leq \frac{n-1}{2}$ , then for  $k \geq 1$ ,  $\overline{L^k(G_1)}$  and  $\overline{L^k(G_2)}$  are H-cospectral if and only if  $G_1$  and  $G_2$  are cospectral.

**Theorem 12.** Let  $G_1$  and  $G_2$  be regular, non *H*-cospectral graphs of the same order *n* and of the same degree *r*. If  $r \leq \frac{n-1}{2}$ , then for  $k \geq 1$ ,  $\overline{L^k(G_1)}$  and  $\overline{L^k(G_2)}$  form a pair of non *H*-cospectral, *H*-equienergetic graphs of equal order and of equal number of edges.

Proof. Follows from Lemma 11 and Corollary 9.

#### H. S. RAMANE, K. ASHOKA

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