https://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 69, Number 2, Pages 1522–1536 (2020)
DOI: 10.31801/cfsuasmas.750568
ISSN 1303–5991 E-ISSN 2618–6470

Received by the editors: June 10, 2020; Accepted:November 11, 2020

APPROXIMATION BY BÉZIER VARIANT OF JAKIMOVSKI-LEVIATAN-PĂLTĂNEA OPERATORS INVOLVING SHEFFER POLYNOMIALS

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Abstract. In the present paper, the Bézier variant of Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials is introduced and the degree of approximation by these operators is investigated with the aid of Ditzian-Totik modulus of smoothness, Lipschitz type space and for functions with derivatives of bounded variations.

Introduction

Approximation theory is a crucial branch of Mathematical analysis. The fundamental property of approximation theory is to approximate a function f by another functions which have better properties than f. In 1950, Szasz [14] introduced a generalization of Bernstein polynomials on the infinite interval $[0, \infty)$ and established the convergence properties of these operators. Subsequently, Jakimovski-Leviatan [8] generalised the Szász operators as

$$P_n(f;x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{0.1}$$

by means of Appell polynomials which are generated by:

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{0.2}$$

where $g(u) = \sum_{k=0}^{\infty} a_k u^k, a_0 \neq 0$ is an analytic function, on the disk $|u| < r$ $(r > 1)$, under the assumption $p_k(x) \geq 0$, for $x \in [0, \infty)$.

2020 *Mathematics Subject Classification.* 41A25, 41A36, 41A30, 26A15.
Keywords and phrases. Positive linear operators, rate of convergence, modulus of continuity, total variation, Sheffer polynomials.
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In 2008, Păltănea [11] defined a generalisation of the Phillips operators [12] based on a parameter $\rho > 0$, as

$$G_n^\rho(f;x) = \sum_{k=1}^{\infty} s_{n,k}(x) \int_0^\infty \Phi_{n,k}^\rho(t) f(t)dt + e^{-nx} f(0), \quad x \in [0,\infty), \tag{0.3}$$

where $s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$ and $\Phi_{n,k}^\rho(t) = \frac{n\rho^{k\rho}}{\Gamma(k\rho)} e^{-n\rho t}(nt)^{k\rho-1}$, which includes Szász operators for $\rho \to \infty$ and Phillips operators for $\rho = 1$. For $f \in C[0,\infty)$, Verma and Gupta [15] defined the Jakimovski-Leviatan-Păltănea operator as follows:

$$P_{n,\rho}^*(f;x) = \sum_{k=1}^{\infty} L_{n,k}(x) \int_0^\infty Q_{n,k}^\rho(t) f(t)dt + L_{n,0}(x) f(0), \quad \rho > 0, \tag{0.4}$$

where $L_{n,k}(x) = \frac{e^{-nx}}{g(1)} p_k(nx)$ and $Q_{n,k}^\rho(t) = \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}$ and established an asymptotic formula and rate of convergence for these operators. Goyal and Agrawal [4] defined the Bézier variant of these operators (0.4) and established the degree of approximation using Ditzian-Totik modulus of smoothness, Lipschitz type space and for functions having a derivative of bounded variation.

Let $C(z) = \sum_0^\infty c_k z^k, (c_0 \neq 0)$ and $D(z) = \sum_{k=1}^\infty d_k z^k, (d_1 \neq 0)$ be analytic functions on the disc $|z| < r,\ r > 1$ where c_k and d_k are real. The Sheffer type polynomials $\{p_k(x)\}$ are given by the generating functions of the form

$$C(z)e^{tD(z)} = \sum_{k=0}^{\infty} p_k(t) z^k, \quad |z| < r. \tag{0.5}$$

Under the following assumptions:

(i) for $t \in [0,\infty)$, $p_k(t) \geq 0$, $k = 0, 1, 2, \cdots$
(ii) $C(1) \neq 0$ and $D'(1) = 1$,

Ismail [6] defined another generalisation of the Szász operators and the Jakimovski-Leviatan operators [8] using the Sheffer polynomials as

$$T_n(f;x) = \frac{e^{-nxD(1)}}{C(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right), \tag{0.6}$$

and estabilished some approximation properties of these operators. For the special case $D(t) = t$ and $C(t) = 1$, we find $p_k(x) = \frac{x^k}{k!}$, therefore (0.6) reduces to Szasz operators and for the case $D(t) = t$, the operators $T_n(f;x)$ yield the operators $P_n(f;x)$ defined in (0.1). Inspired by the work of Verma and Gupta [15], Mursaleen et al. [9] defined the Jakimovski-Leviatan-Păltănea operators by means of Sheffer polynomials, and integral modification of the operators given by (0.6), as

$$M_{n,\rho}(f;x) = \sum_{k=1}^{\infty} L_{n,k}(x) \int_0^\infty Q_{n,k}^\rho(t) f(t)dt + L_{n,0}(x) f(0), \quad \rho > 0, \tag{0.7}$$

where $L_{n,k}(x) = \frac{e^{-nxD(1)}}{C(1)} p_k(nx)$ and $Q^{\rho}_{n,k}(t) = \frac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho-1}$ and established some convergence properties of these operators with the help of the Korovkin-type theorem, rate of convergence by using Ditzian-Totik modulus of smoothness and approximation properties for the functions having derivatives of bounded variation.

Since the Bézier curves have important applications in computer aided graphics and applied mathematics, Zeng and Piriou [16] initiated the study of a Bézier variant of Bernstein operators. Zeng [17] introduced the Szasz-Bézier operators and discussed the rate of convergence of these operators for the functions of bounded variations. Subsequently several researchers defined the Bézier variants of some other sequences of positive linear operators and studied their approximation properties (see, e.g., [1, 2, 4, 5, 7, 13]).
Motivated by the above work, we introduce the Bézier variant of the operators defined in (0.7). Let $\lambda > 0$ and $C_{\lambda}[0,\infty) := \{f \in C[0,\infty) : f(t) = O(e^{\lambda t}) \text{ as } t \to \infty\}$. For $\beta \geq 1$ and $f \in C_{\lambda}[0,\infty)$, the Bézier variant of (0.7) is defined as

$$M^{\beta}_{n,\rho}(f;x) = \sum_{k=1}^{\infty} N^{(\beta)}_{n,k}(x) \int_0^{\infty} Q^{\rho}_{n,k}(t) f(t)dt + N^{(\beta)}_{n,0}(x) f(0), \quad \rho > 0, \tag{0.8}$$

where $N^{(\beta)}_{n,k}(x) = \left[J_{n,k}(x)\right]^{\beta} - \left[J_{n,k+1}(x)\right]^{\beta}$, $\beta \geq 1$; $L_{n,k}(x) = \frac{e^{-nxD(1)}}{C(1)} p_k(nx)$ and $J_{n,k}(x) = \sum_{j=k}^{\infty} L_{n,j}(x)$ with the following properties:

(1) $J_{n,k}(x) - J_{n,k+1}(x) = L_{n,k}(x)$, $\quad k = 0, 1, 2, \cdots$,
(2) $J_{n,0}(x) > J_{n,1}(x) > J_{n,2}(x) > \cdots J_{n,n}(x), \quad x \in [0,\infty)$.

In particular,

(i) if $\beta = 1$, the operators $M^{\beta}_{n,\rho}(f;x)$ include the operators given by (0.7),
(ii) if $\beta = 1$ and $D(t) = t$, the operators $M^{\beta}_{n,\rho}(f;x)$ reproduce the operators defined in [15],
(iii) if $C(t) = 1$, $D(t) = t$, $\rho = 1$ and $\beta = 1$, the operators $M^{\beta}_{n,\rho}(f;x)$ reduce to the well known Phillips operators [12].

The organization of the paper as follows: In Section 1, the Bézier variant of Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials has been introduced. In Section 2, some auxiliary results such as moments, central moments and lemmas have been presented. In Section 3, the rate of convergence by using Ditzian-Totik modulus of smoothness and Lipschitz type space have been discussed. In Section 4, the approximation result for the functions having derivatives of bounded variation has been discussed.

1. Auxiliary Results

Lemma 1.1. *The r^{th} order moments $M_{n,\rho}(t^r;x)$, for $r = 0, 1, 2$, are given by the following identities:*

(i) $M_{n,\rho}(1;x) = 1$;
(ii) $M_{n,\rho}(t;x) = x + \frac{C'(1)}{nC(1)}$;

(iii) $M_{n,\rho}(t^2;x) = x^2 + \frac{x}{n}\bigg(1 + \frac{1}{\rho} + \frac{2C'(1)}{C(1)} + D''(1)\bigg) + \frac{1}{n^2\rho}\bigg(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\bigg)$.

As a consequence of the above lemma, we obtain

Lemma 1.2. *The central moments $M_{n,\rho}((t-x)^r;x)$, $r = 1,2$, are given by the following equalities:*

(i) $M_{n,\rho}(t-x;x) = \frac{C'(1)}{nC(1)}$;

(ii) $M_{n,\rho}((t-x)^2;x) = \frac{x}{n}\bigg(1 + \frac{1}{\rho} + D''(1)\bigg) + \frac{1}{n^2\rho}\bigg(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\bigg)$.

In what follows, we denote $M_{n,\rho}((t-x)^2;x) = \xi_{n,\rho}(x)$.

Remark 1.3. *For sufficiently large n and $\mu > 2$, one has*

$$M_{n,\rho}((t-x)^2;x) \le \frac{\mu x}{n}\bigg(1 + \frac{1}{\rho} + D''(1)\bigg). \tag{1.1}$$

Let $C_B[0,\infty)$ be the family of all continuous and bounded functions defined on $[0,\infty)$.

Lemma 1.4. *For every $f \in C_B[0,\infty)$, we have*

$$\|M_{n,\rho}(f;x)\| \le \|f\|. \tag{1.2}$$

Proof. The proof of this lemma is readily follow with the help of Lemma 1.1(i). Hence, the details are omitted. □

Lemma 1.5. *For $\lambda > 0$, let $f \in C_\lambda[0,\infty)$. Then*

$$|M^{\beta}_{n,\rho}(f;x)| \le \beta M_{n,\rho}(|f|;x). \tag{1.3}$$

Proof. For $0 \le u, v \le 1$ and $\beta \ge 1$, the following inequality holds

$$|u^\beta - v^\beta| \le \beta|u-v|. \tag{1.4}$$

Since, $N^{(\beta)}_{n,k}(x) = \big[J_{n,k}(x)\big]^\beta - \big[J_{n,k+1}(x)\big]^\beta$, for all $\beta \ge 1$ and

$$J_{n,k}(x) = \sum_{j=k}^{\infty} L_{n,j}(x) \le \sum_{j=0}^{\infty} L_{n,j}(x) = 1,$$

in view of the inequality (1.4), we have

$$\bigg|N^{(\beta)}_{n,k}(x)\bigg| = \bigg|\big[J_{n,k}(x)\big]^\beta - \big[J_{n,k+1}(x)\big]^\beta\bigg| \le \beta|J_{n,k}(x) - J_{n,k+1}(x)| = \beta L_{n,k}(x). \tag{1.5}$$

Further,

$$\bigg|M^{\beta}_{n,\rho}(f;x)\bigg| \le \sum_{k=1}^{\infty}\bigg|N^{(\beta)}_{n,k}(x)\bigg|\int_0^\infty Q^{\rho}_{n,k}(t)\big|f(t)\big|dt + \bigg|N^{(\beta)}_{n,0}(x)\bigg||f(0)|, \quad \rho > 0. \tag{1.6}$$

From (1.5) and (1.6), we get the desired result. □

2. Main Results

For $t \geq 0$, $x > 0$, and $0 < \alpha \leq 1$, the Lipschitz type space [10] is defined as:

$$Lip_K^*(\alpha) := \left\{ f \in C[0,\infty) : |f(x) - f(t)| \leq K \frac{|x-t|^\alpha}{(x+t)^{\frac{\alpha}{2}}} \right\},$$

where K is some positive constant.

In the next theorem, we investigate the rate of convergence of the operators $M_{n,\rho}^\beta(\cdot;x)$ for the function $f \in Lip_K^*(\alpha)$.

Theorem 2.1. *Let $f \in Lip_K^*(\alpha)$. Then for each $x > 0$, we have*

$$|M_{n,\rho}^\beta(f;x) - f(x)| \leq \frac{\beta K}{x^{\frac{\alpha}{2}}} \left(\xi_{n,\rho}(x)\right)^{\frac{\alpha}{2}}.$$

Proof. In view of Lemma 1.5 and the fact that, $M_{n,\rho}^\beta(1;x) = 1$, we have

$$\begin{aligned} |M_{n,\rho}^\beta(f;x) - f(x)| &\leq |M_{n,\rho}^\beta(f(t) - f(x);x)| \\ &\leq \beta M_{n,\rho}(|f(t) - f(x)|;x) \\ &\leq \beta K M_{n,\rho}\left(\frac{|x-t|^\alpha}{(x+t)^{\frac{\alpha}{2}}};x\right) \\ &\leq \frac{\beta K}{x^{\frac{\alpha}{2}}} M_{n,\rho}\Big(|x-t|^\alpha;x\Big). \end{aligned} \tag{2.1}$$

Now, applying Hölder's inequality by setting $p = 2/\alpha$ and $q = 2/(2-\alpha)$ and using Lemma 1.1

$$\begin{aligned} M_{n,\rho}(|x-t|^\alpha;x) &\leq \Big(M_{n,\rho}((x-t)^2;x)\Big)^{\frac{\alpha}{2}} \Big(M_{n,\rho}(1^{\frac{2}{2-\alpha}};x)\Big)^{\frac{2-\alpha}{2}} \\ &\leq \Big(M_{n,\rho}((x-t)^2;x)\Big)^{\frac{\alpha}{2}} = \left(\xi_{n,\rho}(x)\right)^{\frac{\alpha}{2}}. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we get the required result. □

Let us recall the definitions of the Peetre's K-functional and the Ditzian-Totik first order modulus of smoothness. Let $\phi(x) = \sqrt{x}$ and $f \in C_B[0,\infty)$.

Definition 2.1. *[3] The Ditzian-Totik first order modulus of smoothness $\omega_\phi(f;\delta)$, $\delta > 0$, is defined by*

$$\omega_\phi(f;\delta) := \sup_{0<h\leq\delta} \left| f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right) \right|, \quad \forall\, x \pm \frac{h\phi(x)}{2} \in [0,\infty).$$

Definition 2.2. *[3] The Peetre's K-functional is defined by*

$$K_\phi(f;\delta) := \inf\{\|f-g\| + \delta\|\phi g'\| + \delta^2\|g'\|,\ \delta > 0\}, \quad \forall\, g \in W_\phi,$$

where $W_\phi := \{g : g \in AC_{loc}, \|\phi g'\| < \infty, \|g'\| < \infty\}$ and $g \in AC_{loc}$ means that g is a locally absolutely continuous function in $[0,\infty)$.

From [3], it is known that $\omega_\phi(f;\delta) \sim K_\phi(f;\delta)$*, i.e. there exists a constant* $\gamma > 0$*, such that*

$$\gamma^{-1}\omega_\phi(f;\delta) \leq K_\phi(f;\delta) \leq \gamma\omega_\phi(f;\delta). \tag{2.3}$$

In the next theorem, Ditzian-Totik first order modulus of smoothness is used to establish a direct approximation theorem.

Theorem 2.2. *Let* $f \in C_B[0,\infty)$ *and* $\phi(x) = \sqrt{x}$*, then for every* $x \in [0,\infty)$ *we have*

$$|M_{n,\rho}^{\beta}(f;x) - f(x)| \leq C\omega_\phi\left(f;\frac{1}{\sqrt{n}}\right),$$

where C *is a constant and independent on* f *and* n*.*

Proof. Let $x \in [0,\infty)$ be arbitrary but fixed. For $g \in W_\phi$, we have the following representation

$$g(t) = g(x) + \int_x^t g'(u)du.$$

Applying the operator $M_{n,\rho}^{\beta}(f;x)$ on both sides of the above equation, we obtain

$$\begin{aligned} M_{n,\rho}^{\beta}(g;x) - g(x) &= M_{n,\rho}^{\beta}\left(\int_x^t g'(u)du;x\right), \\ |M_{n,\rho}^{\beta}(g;x) - g(x)| &= \left|M_{n,\rho}^{\beta}\left(\int_x^t g'(u)du;x\right)\right| \leq M_{n,\rho}^{\beta}\left(\left|\int_x^t g'(u)du\right|;x\right) \end{aligned} \tag{2.4}$$

In view of Lemma 1.5, we have

$$M_{n,\rho}^{\beta}((t-x)^2;x) = |M_{n,\rho}^{\beta}((t-x)^2;x)| \leq \beta M_{n,\rho}((t-x)^2;x).$$

Hence, using Lemma 1.2, we get

$$M_{n,\rho}^{\beta}((t-x)^2;x) \leq \beta\left\{\frac{x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)+\frac{1}{n^2\rho}\left(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\right)\right\}. \tag{2.5}$$

To estimate the right hand side of (2.4), we split our domain $[0,\infty)$ into two parts $A = [0,1/n]$ and $B = (1/n,\infty]$.

Case-I:

If $x \in [0,1/n]$, then from (2.5), for sufficiently large n, we have $M_{n,\rho}^{\beta}((t-x)^2;x) \sim \frac{\beta}{n^2\rho}\left(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\right)$, i.e. there exists some $k_1 > 0$, such that

$$M_{n,\rho}^{\beta}((t-x)^2;x) \leq \frac{k_1\beta}{n^2\rho}\left(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\right).$$

Hence, applying Cauchy-Schwarz inequality in equation (2.4), we have

$$|M_{n,\rho}^{\beta}(g;x) - g(x)| \quad \leq \quad \|g'\|M_{n,\rho}^{\beta}\left(|t-x|;x\right)$$

$$\begin{aligned} &\leq \|g'\|\bigg(M^{\beta}_{n,\rho}((t-x)^2;x)\bigg)^{1/2} \\ &\leq \|g'\|\bigg\{\frac{k_1\beta}{n^2\rho}\bigg(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\bigg)\bigg\}^{1/2} \\ &= \frac{\Delta_1}{n}\|g'\|, \end{aligned} \tag{2.6}$$

where $\Delta_1=\bigg\{\frac{k_1\beta}{\rho}\bigg(\frac{(1+\rho)C'(1)+\rho C''(1)}{C(1)}\bigg)\bigg\}^{1/2}$.

Case-II: If $x\in(1/n,\infty]$, then from (2.5), we obtain $M^{\beta}_{n,\rho}((t-x)^2;x)\sim\frac{\beta x}{n}\bigg(1+\frac{1}{\rho}+D''(1)\bigg)$. Hence, there exists some constant $k_2>0$, such that

$$M^{\beta}_{n,\rho}((t-x)^2;x)\leq\frac{k_2\beta x}{n}\bigg(1+\frac{1}{\rho}+D''(1)\bigg).$$

Since

$$\bigg|\int_x^t g'(u)du\bigg|\leq\|\phi g'\|\bigg|\int_x^t\frac{1}{\phi(u)}du\bigg|,$$

and for any $x,t\in(0,\infty)$,

$$\bigg|\int_x^t\frac{1}{\phi(u)}du\bigg|=\bigg|\int_x^t\frac{1}{\sqrt{u}}du\bigg|=2|(\sqrt{t}-\sqrt{x})|=2\frac{|t-x|}{\sqrt{t}+\sqrt{x}}\leq2\frac{|t-x|}{\phi(x)},$$

we have

$$\bigg|\int_x^t g'(u)du\bigg|\leq2\|\phi g'\|\frac{|t-x|}{\phi(x)}. \tag{2.7}$$

Now, combining equations (2.4) and (2.7) and using Cauchy-Schwarz inequality, for any $x\in(1/n,\infty)$, we have

$$\begin{aligned} |M^{\beta}_{n,\rho}(g;x)-g(x)| &\leq 2\|\phi g'\|\phi^{-1}(x)M^{\beta}_{n,\rho}\bigg(|t-x|;x\bigg) \\ &\leq 2\|\phi g'\|\phi^{-1}(x)\bigg(M^{\beta}_{n,\rho}((t-x)^2;x)\bigg)^{1/2} \\ &\leq 2\|\phi g'\|\phi^{-1}(x)\bigg(\frac{k_2\beta x}{n}\bigg(1+\frac{1}{\rho}+D''(1)\bigg)\bigg)^{1/2} \\ &= \Delta_2\frac{\|\phi g'\|}{\sqrt{n}}, \end{aligned} \tag{2.8}$$

where $\Delta_2 = \left(k_2\beta\left(1+\frac{1}{\rho}+D''(1)\right)\right)^{1/2}$.

Again, combining equations (2.4), (2.6) and (2.8), for $x \in [0,\infty)$ we have

$$\begin{aligned} |M^{\beta}_{n,\rho}(g;x)-g(x)| &\leq \Delta_2\frac{\|\phi g'\|}{\sqrt{n}}+\frac{\Delta_1}{n}\|g'\| \\ &\leq \Delta\left(\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right), \quad \text{where} \quad \Delta=\max(\Delta_1,\Delta_2). \end{aligned}$$

Hence, using Lemma 1.4 and above equation, we get

$$\begin{aligned} |M^{\beta}_{n,\rho}(f;x)-f(x)| &\leq |M^{\beta}_{n,\rho}(g;x)-g(x)|+|f(x)-g(x)|+|M^{\beta}_{n,\rho}(f-g;x)| \\ &\leq 2\|f-g\|+\Delta\left(\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right) \\ &\leq \Delta'\left(\|f-g\|+\frac{\|\phi g'\|}{\sqrt{n}}+\frac{1}{n}\|g'\|\right), \quad \Delta'=\max(2,\Delta). \end{aligned}$$

Finally, taking the infimum on the right side of the above equation over all $g \in W_\phi$,

$$|M^{\beta}_{n,\rho}(f;x)-f(x)| \leq \Delta' K_\phi\left(f;\frac{1}{\sqrt{n}}\right),$$

and using the relation (2.3), we get

$$|M^{\beta}_{n,\rho}(f;x)-f(x)| \leq \Delta'\gamma\ \omega_\phi\left(f;\frac{1}{\sqrt{n}}\right).$$

Now taking $C=\Delta'\gamma$, the proof of the theorem is completed. □

3. Functions with Derivatives of Bounded Variation

Let $DBV_2[0,\infty)$, be the class of all functions f defined on $[0,\infty)$ with $|f(t)| \leq C(1+t^2),\ C>0$ and having a derivative f' equivalent to a function of bounded variation on every finite subinterval of $[0,\infty)$. Then we observe that for all functions $f \in DBV_2[0,\infty)$, there holds the following representation

$$f(x)=\int_0^x g(t)dt+f(0),$$

where g is a function of bounded variation on every finite subinterval of $(0,\infty)$. In view of the Dirac-delta function, the alternate form of the operator $M^{\beta}_{n,\rho}(f;x)$ can be written as

$$M^{\beta}_{n,\rho}(f;x)=\int_0^\infty F^{\beta}_{n,\rho}(x,t)f(t)dt, \quad \rho>0, \tag{3.1}$$

where $F^{\beta}_{n,\rho}(x,t)=\sum_{k=1}^{\infty}N^{(\beta)}_{n,k}(x)Q^{\rho}_{n,k}(t)+N^{(\beta)}_{n,0}(x)\delta(t)$ and $\delta(t)$ is a Dirac-delta function.

To establish the rate of convergence of the operators given by (3.1) for $f \in DBV_2[0,\infty)$, the following lemma is needed:

Lemma 3.1. *Let $x \in (0,\infty)$ and $\mu > 2$. Then for sufficiently large n, we have*

(i) $\Phi^{\beta}_{n,\rho}(x,x_1) = \int_0^{x_1} F^{\beta}_{n,\rho}(x,t)dt \leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_1)^2}, \quad 0 \leq x_1 < x,$

(ii) $1-\Phi^{\beta}_{n,\rho}(x,x_2) = \int_{x_2}^{\infty} F^{\beta}_{n,\rho}(x,t)dt \leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_2)^2}, \quad x < x_2 < \infty.$

Proof. (i) Using (3.1) and Remark 1.3, we have

$$\begin{aligned}\Phi^{\beta}_{n,\rho}(x,x_1) &= \int_0^{x_1} F^{\beta}_{n,\rho}(x,t)dt \leq \int_0^{x_1}\left(\frac{x-t}{x-x_1}\right)^2 F^{\beta}_{n,\rho}(x,t)dt \\ &= (x-x_1)^{-2}M^{\beta}_{n,\rho}((t-x)^2;x) \leq \beta(x-x_1)^{-2}M_{n,\rho}((t-x)^2;x) \\ &\leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\frac{1}{(x-x_1)^2}.\end{aligned}$$

In the same way, assertion (ii) can be easily proved. □

Theorem 3.2. *Let $f \in DBV_2[0,\infty)$ and $\mu > 2$. Then for each $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}|M^{\beta}_{n,\rho}(f;x)-f(x)| &\leq \frac{\beta^{1/2}}{\beta+1}|f'(x+)+\beta f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\ &+ \frac{\beta^{3/2}}{\beta+1}|f'(x+)-f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\ &+ \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=0}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^{x+\frac{x}{k}}(f'_x) + \frac{x}{\sqrt{n}} V_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}}(f'_x) \\ &+ \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)\{|f(2x)-f(x)-xf'(x+)|\} \\ &+\left\{\frac{|f(x)|}{x^2}+C\left(4+\frac{1}{x^2}\right)\right\}\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right) \\ &+|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)},\end{aligned}$$

where $V_c^d(f'_x)$ denotes the total variation of f'_x on $[c,d]$ and f'_x is defined by

$$f'_x(t) = \begin{cases} f'(t)-f'(x-), & 0 \leq t < x, \\ 0, & x = t, \\ f'(t)-f'(x+), & x < t < \infty. \end{cases} \tag{3.2}$$

Proof. In view of the fact that $M^{\beta}_{n,\rho}(1;x) = 1$, and the alternate form (3.1) of the operators given by (0.8), for every $x \in (0,\infty)$ we have

$$M^{\beta}_{n,\rho}(f(t);x)-f(x) = M^{\beta}_{n,\rho}(f(t);x)-M^{\beta}_{n,\rho}(f(x);x) = M^{\beta}_{n,\rho}(f(t)-f(x);x)$$

$$= \int_0^\infty F_{n,\rho}^{\beta}(x,t)(f(t)-f(x))dt$$
$$= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\bigg(\int_x^t f'(\nu)d\nu\bigg)dt. \qquad (3.3)$$

For any $f \in DBV_2[0,\infty)$, and using (3.2), we can write

$$\begin{aligned} f'(\nu) &= \delta_x(\nu)\bigg[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\bigg]+\bigg[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\bigg] \\ &+ \frac{f'(x+)-f'(x-)}{2}\bigg[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\bigg]+f'_x(\nu), \end{aligned} \qquad (3.4)$$

where

$$\delta_x(\nu)=\begin{cases} 1, & x=\nu \\ 0, & x\neq\nu. \end{cases}$$

Combining equations (3.3) and (3.4), we get

$$\begin{aligned} M_{n,\rho}^{\beta}(f(t);x)-f(x) &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\bigg(\int_x^t \bigg\{\delta_x(\nu)\bigg[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\bigg] \\ &\quad +\bigg[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\bigg] \\ &+ \frac{f'(x+)-f'(x-)}{2}\bigg[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\bigg] \\ &\quad +f'_x(\nu)\bigg\}d\nu\bigg)dt \\ &= \Psi_1+\Psi_2+\Psi_3+\Psi_4, \end{aligned} \qquad (3.5)$$

where

$$\begin{aligned} \Psi_1 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \delta_x(\nu)\bigg[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\bigg]d\nu dt \\ \Psi_2 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \bigg[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\bigg]d\nu dt \\ \Psi_3 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \frac{f'(x+)-f'(x-)}{2}\bigg[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\bigg]d\nu dt \\ \Psi_4 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t f'_x(\nu)d\nu dt. \end{aligned}$$

We can easily see from the definition of $\delta_x(t)$ that

$$\Psi_1=\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \delta_x(\nu)\bigg[f'(\nu)-\frac{f'(x+)+f'(x-)}{2}\bigg]d\nu dt=0 \qquad (3.6)$$

and

$$\begin{aligned}
\Psi_2 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]d\nu dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t d\nu dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]\int_0^\infty F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&= \left[\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right]M_{n,\rho}^{\beta}(t-x;x). \qquad (3.7)
\end{aligned}$$

Now, we evaluate Ψ_3,

$$\begin{aligned}
\Psi_3 &= \int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t \frac{f'(x+)-f'(x-)}{2}\left[sgn(\nu-x)+\frac{\beta-1}{\beta+1}\right]d\nu dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2}M_{n,\rho}^{\beta}((t-x),x) \\
&+ \frac{f'(x+)-f'(x-)}{2}\int_0^\infty F_{n,\rho}^{\beta}(x,t)\int_x^t sgn(\nu-x)d\nu dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2}M_{n,\rho}^{\beta}((t-x),x) \\
&- \frac{f'(x+)-f'(x-)}{2}\int_0^x F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&\quad +\frac{f'(x+)-f'(x-)}{2}\int_x^\infty F_{n,\rho}^{\beta}(x,t)(t-x)dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2}M_{n,\rho}^{\beta}((t-x),x) \\
&\quad +\frac{f'(x+)-f'(x-)}{2}\int_0^\infty F_{n,\rho}^{\beta}(x,t)|x-t|dt \\
&= \left(\frac{\beta-1}{\beta+1}\right)\frac{f'(x+)-f'(x-)}{2}M_{n,\rho}^{\beta}((t-x),x) \\
&\quad +\frac{f'(x+)-f'(x-)}{2}M_{n,\rho}^{\beta}(|t-x|,x). \qquad (3.8)
\end{aligned}$$

Combining equations (3.5)-(3.8), we have

$$\begin{aligned}
|M_{n,\rho}^{\beta}(f(t);x)-f(x)| &\leq \left|\frac{f'(x+)+\beta f'(x-)}{1+\beta}\right||M_{n,\rho}^{\beta}(t-x;x)| \\
&\quad +\frac{\beta-1}{\beta+1}\left|\frac{f'(x+)-f'(x-)}{2}\right||M_{n,\rho}^{\beta}((t-x),x)|
\end{aligned}$$

$$+ \quad \left|\frac{f'(x+)-f'(x-)}{2}\right| M^{\beta}_{n,\rho}(|t-x|,x)+|\Psi_4|. \quad (3.9)$$

Now, applying Lemma 1.5 and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|M^{\beta}_{n,\rho}(f(t);x)-f(x)| &\leq \frac{1}{\beta+1}|f'(x+)+\beta f'(x-)|(\beta M_{n,\rho}((t-x)^2;x))^{1/2} \\
&+ \frac{\beta}{\beta+1}|f'(x+)-f'(x-)|(\beta M_{n,\rho}((t-x)^2;x))^{1/2}+|\Psi_4| \\
&\leq \frac{\beta^{1/2}}{\beta+1}|f'(x+)+\beta f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\
&\quad +\frac{\beta^{3/2}}{\beta+1}|f'(x+) \\
&\quad -f'(x-)|\sqrt{\frac{\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}+|\Psi_4|. \qquad (3.10)
\end{aligned}$$

We now estimate $|\Psi_4|$. We may write

$$\Psi_4=\int_0^{\infty}F^{\beta}_{n,\rho}(x,t)\int_x^t f'_x(\nu)d\nu dt=\Psi_5+\Psi_6,$$

where

$$\Psi_5=\int_0^{x}F^{\beta}_{n,\rho}(x,t)\int_x^t f'_x(\nu)d\nu dt$$

$$\Psi_6=\int_x^{\infty}F^{\beta}_{n,\rho}(x,t)\int_x^t f'_x(\nu)d\nu dt.$$

Since $\int_c^d d_t\Phi^{\beta}_{n,\rho}(x,t)\leq 1$, for each $[c,d]\subset[0,\infty)$ and $f'_x(x)=0$, using Lemma 3.1 and integration by parts with $x_1=x-\frac{x}{\sqrt{n}}$, we have

$$\begin{aligned}
|\Psi_5| &= \left|\int_0^{x}F^{\beta}_{n,\rho}(x,t)\int_x^t f'_x(\nu)d\nu dt\right| \\
&= \left|\int_0^{x}\left(\int_x^t f'_x(\nu)d\nu\right)d_t\Phi^{\beta}_{n,\rho}(x,t)\right| = \left|\int_0^{x}f'_x(t)\Phi^{\beta}_{n,\rho}(x,t)dt\right| \\
&\leq \int_0^{x_1}|f'_x(t)-f'_x(x)||\Phi^{\beta}_{n,\rho}(x,t)|dt+\int_{x_1}^{x}|f'_x(t)-f'_x(x)||\Phi^{\beta}_{n,\rho}(x,t)|dt \\
&\leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x_1}V^x_t(f'_x)\frac{1}{(x-t)^2}dt+\int_{x_1}^{x}V^x_t(f'_x)dt \\
&\leq \frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x_1}V^x_t(f'_x)\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}V^x_{x-\frac{x}{\sqrt{n}}}(f'_x). \qquad (3.11)
\end{aligned}$$

Substituting $t = x - \frac{x}{u}$, we obtain

$$\begin{aligned}
&\frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_0^{x-\frac{x}{\sqrt{n}}} V_t^x(f_x')\frac{1}{(x-t)^2}dt\\
&= \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_1^{\sqrt{n}} V_{x-\frac{x}{u}}^x(f_x')du\\
&\le \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1} V_{x-\frac{x}{k}}^x(f_x')du\\
&\le \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^x(f_x'). \qquad (3.12)
\end{aligned}$$

Combining (3.11) and (3.12), we have

$$\Psi_5 \le \frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_{x-\frac{x}{k}}^x(f_x') + \frac{x}{\sqrt{n}}V_{x-\frac{x}{\sqrt{n}}}^x(f_x'). \qquad (3.13)$$

From Lemma 3.1(ii), $F_{n,\rho}^{\beta}(x,t) = -d_t(1-\Phi_{n,\rho}^{\beta}(x,t)),\ t > x$, hence we may write

$$\begin{aligned}
|\Psi_6| &\le \left|\int_x^{2x}\left(\int_x^t f_x'(\nu)d\nu\right)d_t(1-\Phi_{n,\rho}^{\beta}(x,t))\right| + \left|\int_{2x}^{\infty}\left(\int_x^t f_x'(\nu)d\nu\right)d_tF_{n,\rho}^{\beta}(x,t)\right|\\
&= \Psi_7+\Psi_8, \text{say.}
\end{aligned}$$

First estimate Ψ_7,

$$\begin{aligned}
\Psi_7 &= \left|\int_x^{2x}\left(\int_x^t f_x'(\nu)d\nu\right)d_t(1-\Phi_{n,\rho}^{\beta}(x,t))\right|\\
&\le \left|\int_x^{2x} f_x'(\nu)d\nu\right||(1-\Phi_{n,\rho}^{\beta}(x,2x))| + \left|\int_x^{2x} f_x'(t)(1-\Phi_{n,\rho}^{\beta}(x,t))dt\right|\\
&\le \left|\int_x^{2x} (f'(\nu)-f'(x+))d\nu\right||(1-\Phi_{n,\rho}^{\beta}(x,2x))| + \left|\int_x^{2x} f_x'(t)(1-\Phi_{n,\rho}^{\beta}(x,t))dt\right|\\
&\le \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)|\\
&\quad+\frac{\beta\mu x}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_{x+x/\sqrt{n}}^{2x}\frac{V_x^t(f_x')}{(x-t)^2}dt + \int_x^{x+x/\sqrt{n}} V_x^t(f_x')dt.
\end{aligned}$$

Now, substituting $t = x + \frac{x}{u}$, we have

$$\begin{aligned}
&\le \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)|\\
&\quad+\frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\int_1^{\sqrt{n}} V_x^{x+\frac{x}{u}}(f_x')du + \int_x^{x+x/\sqrt{n}} V_x^t(f_x')dt
\end{aligned}$$

$$\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(2x)-f(x)-xf'(x+)|$$
$$+\frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_x^{x+\frac{x}{k}}(f'_x)+\frac{x}{\sqrt{n}}V_x^{x+\frac{x}{\sqrt{n}}}(f'_x). \tag{3.14}$$

Using Cauchy-Schwarz inequality

$$\begin{aligned}
\Psi_8 &= \left|\int_{2x}^{\infty}\left(\int_x^t f'_x(\nu)d\nu\right)F^{\beta}_{n,\rho}(t,x)dt\right| \\
&= \left|\int_{2x}^{\infty}\left(\int_x^t (f'(\nu)-f'(x+))d\nu\right)F^{\beta}_{n,\rho}(x,t)dt\right| \\
&\leq \left|\int_{2x}^{\infty}(f(t)-f(x))F^{\beta}_{n,\rho}(t,x)dt\right|+\int_{2x}^{\infty}|t-x||f'(x+)|F^{\beta}_{n,\rho}(t,x)dt \\
&\leq \left|\int_{2x}^{\infty}f(t)F^{\beta}_{n,\rho}(t,x)dt\right|+|f(x)|\left|\int_{2x}^{\infty}F^{\beta}_{n,\rho}(t,x)dt\right| \\
&\quad +|f'(x+)|\left(\int_{2x}^{\infty}(t-x)^2F^{\beta}_{n,\rho}(t,x)dt\right)^{1/2} \\
&\leq C\left|\int_{2x}^{\infty}(1+t^2)F^{\beta}_{n,\rho}(t,x)dt\right|+|f(x)|\left|\int_{2x}^{\infty}F^{\beta}_{n,\rho}(t,x)dt\right| \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}
\end{aligned}$$

Since $t\geq 2x$, we have $t\leq 2(t-x)$, hence

$$\begin{aligned}
\Psi_8 &\leq C\left(4+\frac{1}{x^2}\right)\left(\int_{2x}^{\infty}(t-x)^2F^{\beta}_{n,\rho}(t,x)dt\right)+\frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)|f(x)| \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \\
&= \left\{\frac{|f(x)|}{x^2}+C\left(4+\frac{1}{x^2}\right)\right\}\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right) \\
&\quad +|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)} \qquad (3.15)
\end{aligned}$$

From (3.14) and (3.15), we obtain

$$\begin{aligned}
|\Psi_6| &\leq \frac{\beta\mu}{nx}\left(1+\frac{1}{\rho}+D''(1)\right)\{|f(2x)-f(x)-xf'(x+)|\} \\
&\quad +\frac{\beta\mu}{n}\left(1+\frac{1}{\rho}+D''(1)\right)\sum_{k=1}^{[\sqrt{n}]} V_x^{x+\frac{x}{k}}(f'_x)+\frac{x}{\sqrt{n}}V_x^{x+\frac{x}{\sqrt{n}}}(f'_x)
\end{aligned}$$

$$+ \quad \left\{\frac{|f(x)|}{x^2}+C\left(4+\frac{1}{x^2}\right)\right\}\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}$$

$$+|f'(x+)|\sqrt{\frac{\mu x\beta}{n}\left(1+\frac{1}{\rho}+D''(1)\right)}. \tag{3.16}$$

Combining the estimates (3.10), (3.13) and (3.16), we obtain the desired result. □

Acknowledgement. The second author is extremely grateful to the Ministry of Human Resource Development of India (Grant code:- MHR-01-23-200-428) to carry out his research work.

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