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PAGES: 320-330

ORIGINAL PDF URL: https://dergipark.org.tr/tr/download/article-file/1078917

A METRIC FORMULA ON A QUOTIENT SPACE WHICH IS RELATED TO THE SEQUENCE SPACE Σ_2

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ABSTRACT. In this paper, we first define an equivalence relation on the sequence space Σ_2. Then we equip the quotient set $\Sigma_2/_\sim$ with a metric d_1. We also determine an isometry map between the metric spaces $(\Sigma_2/_\sim, d_1)$ and $([0,1], d_{eucl})$. Finally, we investigate the symmetry conditions with respect to some points on the metric space $(\Sigma_2/_\sim, d_1)$ and we compare truncation errors for the computations which is obtained by the metrics d_{eucl} and d_1.

1. INTRODUCTION

Let Σ_n denote the set of all infinite sequences of 0's, 1's, ..., $n-1$'s. That is,

$$\Sigma_n = \{s_1 s_2 s_3 \ldots | s_i \in \{0, 1, \ldots, n-1\}\}$$

(see [1,2,8]). This set is called the sequence space on the n symbols $0, 1, \ldots, n-1$. (Σ_n, d) is the code space on n symbols with the metric

$$d(s,t) = \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{n^i} \quad \text{for all} \quad s, t \in \Sigma_n. \tag{1}$$

A semi-metric on Σ_n is also defined as follows:

$$d'(s,t) = \left| \sum_{i=1}^{\infty} \frac{s_i - t_i}{n^i} \right| \quad \text{for all} \quad s, t \in \Sigma_n. \tag{2}$$

It is well-known that there is the surjective mapping h between Σ_2 and $[0,1]$ such that

$$h : \Sigma_2 \to [0,1], \ h(s_1 s_2 \ldots s_n \ldots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots + \frac{s_n}{2^n} + \ldots.$$

But, the mapping h is not injective. To exemplify, let us consider $01111\ldots$ and $1000\ldots$ of Σ_2. It is clear that

$$h(01111\ldots) = h(1000\ldots) = \frac{1}{2}.$$

Generally, any rational number $\frac{p}{2^k} \in [0,1]$ (p, k are integers) has exactly two preimages s', s'' under h; they have forms $s' = s_1 s_2 \ldots s_n \alpha \overline{\beta}$ and $s'' = s_1 s_2 \ldots s_n \beta \overline{\alpha}$ for some $n \in \mathbb{N}, s_i, \alpha, \beta \in \{0,1\}$ and $\overline{a}$ stands for the sequence in Σ_2 with all the items equal to a where $a \in \{0,1\}$. To obtain an injective mapping, an equivalence relation $\sim$ on Σ_2 is defined by

$$\begin{aligned} s' \sim s'' \quad \Leftrightarrow \quad & s' = s'' \text{ or there are } s_i, \alpha, \beta \in \{0,1\} \text{ such that} \\ & s' = s_1 s_2 \ldots s_n \alpha \overline{\beta}, \ s'' = s_1 s_2 \ldots s_n \beta \overline{\alpha} \text{ for an integer } n. \end{aligned} \qquad (3)$$

Therefore, the mapping h' defined as

$$h' : \Sigma_2/_\sim \ \to [0,1] \qquad (4)$$

$$h'(s_1 s_2 \ldots s_n \ldots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots + \frac{s_n}{2^n} + \ldots.$$

is bijective.

From now on, one can identify each element of $\Sigma_2/_\sim$ with an appropriate element of $[0,1]$. That is, $\Sigma_2/_\sim$ can be considered as the code set of the points on $[0,1]$. Note that, d is not well defined on $\Sigma_2/_\sim$. For example, for the elements $s = 00\overline{1}$, $s' = 0\overline{1}$ and $s'' = 1\overline{0}$ ($s' \sim s''$) of $\Sigma_2/_\sim$, we compute $d(s, s') \neq d(s, s'')$. Therefore, d is not a metric on $\Sigma_2/_\sim$, but $(\Sigma_2/_\sim, d')$ is a metric space. Obviously, the metric spaces $(\Sigma_2/_\sim, d')$ and $([0,1], d_{eucl})$ are isometric since h' is an isometry. On the other hand, different metrics can be defined on the same set X. The interesting question is whether these metrics are equivalent or not. Because equivalent metrics on the same set X generate the same topology (see [3]). It is a remarkable problem, how to define a metric d_1, which is formulated in a different way from the metric d' given in (2) on $\Sigma_2/_\sim$ such that $(\Sigma_2/_\sim, d_1)$ and $([0,1], d_{eucl})$ are isometric.

Any set can be expressed as a quotient space which is related to a sequence space. Using the metrics defined on code sets, many geometric features of the corresponding space can be proved. Moreover, unusual metric formulas can be obtained on different quotient spaces. For example, the Sierpinski Gasket which is one of the fundamental models of fractals can be defined as the quotient space $\Sigma_3/_{\sim'}$ where

$$\begin{aligned} c' \sim' c'' \quad \Leftrightarrow \quad & c' = c'' \text{ or there are } c_i, \alpha, \beta \in \{0,1,2\} \text{ such that} \\ & c' = c_1 c_2 \ldots c_n \alpha \overline{\beta}, \ c'' = c_1 c_2 \ldots c_n \beta \overline{\alpha} \text{ for an integer } n. \end{aligned} \qquad (5)$$

(see Figure 1).

In [6], the geodesic metric on the code set of the Sierpinski Gasket is defined as follows:

Definition 1. *Let $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ be two representations of the points a and b respectively on the Sierpinski Gasket such that $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$. The distance $d_{geo}(a, b)$ between a and b is determined with the following formula:*

$$d_{geo}(a, b) = \min\left\{ \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}, \; \frac{1}{2^k} + \sum_{i=k+1}^{\infty} \frac{\gamma_i + \delta_i}{2^i} \right\} \tag{6}$$

where

$$\alpha_i = \begin{cases} 0, & a_i = b_k \\ 1, & a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0, & b_i = a_k \\ 1, & b_i \neq a_k \end{cases},$$

$$\gamma_i = \begin{cases} 0, & a_i \neq a_k \;\; and \; a_i \neq b_k \\ 1, & otherwise \end{cases}, \qquad \delta_i = \begin{cases} 0, & b_i \neq b_k \;\; and \; b_i \neq a_k \\ 1, & otherwise \end{cases}.$$
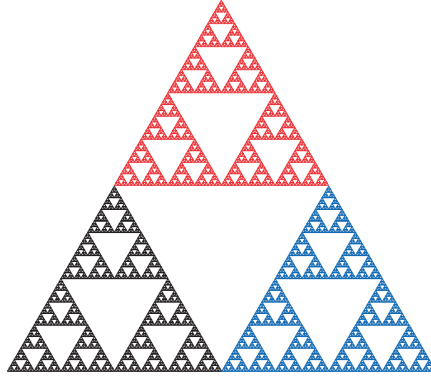

FIGURE 1. The Sierpinski Gasket

This metric formula describes the shortest distance between any two points on the code sets of the Sierpinski Gasket. Thanks to the metric formula d_{geo}, many different and interesting properties of the Sierpinski Gasket can be proved (for details see [6, 7]).

As seen in (6), this metric is actually expressed on the quotient space $\Sigma_3/_{\sim'}$. In a similar way, a metric on the quotient space $\Sigma_2/_{\sim}$ can be defined. Note that, although relations $\sim$ and $\sim'$ defined in (3) and (5) are quite similar, the corresponding spaces are different.

In this paper, to define the metric d_1 we take into account the construction of the intrinsic metric on the Sierpinski Gasket (for details see proof of the Theorem 1 in the paper [5]). We use the quotient space $\Sigma_2/_{\sim}$ given in (3) instead of the quotient space $\Sigma_3/_{\sim'}$ given in (5) which is the code set of the Sierpinski Gasket. We also take the first sum, the same α_i and β_i given in the metric formula (6). Thus we obtain the metric space $(\Sigma_2/_{\sim}, d_1)$ and show that this metric space is isometric to

the metric space $([0,1], d_{eucl})$ in Proposition 1. Since d_1 has a different construction than the metric d_{eucl}, we compare some properties of these two metrics. So, we get some interesting results in Proposition 6 and Proposition 8.

2. A DIFFERENTLY FORMULATED METRIC ON THE QUOTIENT SET $\Sigma_2/\sim$

We first define a distance on the quotient set $\Sigma_2/\sim$ and then we show that this distance function is a metric on $\Sigma_2/\sim$.

Suppose that $a = a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $b = b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ are two elements of $\Sigma_2/\sim$. If $a_i = b_i$ for all $i = 1, 2, 3, \ldots$, then $d_1(a, b) = 0$. If a_i does not equal to b_i for any $i = 1, 2, 3, \ldots$, then there is at least one k such that $a_k \neq b_k$. k is chosen as the smallest index $i = 1, 2, \ldots$ such that $a_i \neq b_i$. That is, let $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$. We define the distance $d_1(a, b)$ between a and b as

$$d_1(a, b) = \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i}$$

where

$$\alpha_i = \begin{cases} 0 & \text{if } a_i = b_k \\ 1 & \text{if } a_i \neq b_k \end{cases}, \qquad \beta_i = \begin{cases} 0 & \text{if } b_i = a_k \\ 1 & \text{if } b_i \neq a_k \end{cases}. \tag{7}$$

Since there are many stages to prove that $(\Sigma_2/\sim, d_1)$ is a metric space, we give it as a result of the following proposition:

Proposition 2. $d_{eucl}(h'(a), h'(b)) = d_1(a, b)$ *for all* $a, b \in \Sigma_2/\sim$. *Thus,* $(\Sigma_2/\sim, d_1)$ *is a metric space.*

Proof. Let us first consider the bijective mapping h' defined in (4) as follows:

$$h' : \Sigma_2/\sim \to [0, 1]$$

$$h'(s_1 s_2 \ldots s_n \ldots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots + \frac{s_n}{2^n} + \ldots.$$

We now show that

$$d_{eucl}(h'(a), h'(b)) = d_1(a, b)$$

for all $a, b \in \Sigma_2/\sim$. Suppose that

$$\begin{aligned} a &= a_1 & a_2 & \ldots & a_{k-1} & a_k & a_{k+1} & a_{k+2} & \ldots \\ b &= b_1 & b_2 & \ldots & b_{k-1} & b_k & b_{k+1} & b_{k+2} & \ldots. \end{aligned}$$

where $a_i = b_i$ for $i = 1, 2, \ldots, k-1$ and $a_k \neq b_k$. So, we have either $a_k > b_k$ or $a_k < b_k$. Assume that $a_k < b_k$ (that is, chosen as $a_k = 0$ and $b_k = 1$). Then,

$|a_{i+1} - b_k| = b_k - a_{i+1}$ and $|b_{i+1} - a_k| = b_{i+1} - a_k$ for $i = k, k+1, k+2, \ldots$. The other case is done similarly. Observe that

$$
\begin{aligned}
d_1(a, b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
&= \frac{|a_{k+1} - b_k| + |b_{k+1} - a_k|}{2^{k+1}} + \frac{|a_{k+2} - b_k| + |b_{k+2} - a_k|}{2^{k+2}} + \cdots \\
&= \frac{(b_k - a_{k+1}) + (b_{k+1} - a_k)}{2^{k+1}} + \frac{(b_k - a_{k+2}) + (b_{k+2} - a_k)}{2^{k+2}} + \cdots \\
&= \frac{(b_k - a_k) + (b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_k - a_k) + (b_{k+2} - a_{k+2})}{2^{k+2}} + \cdots \\
&= (b_k - a_k)\left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots\right) + \frac{(b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_{k+2} - a_{k+2})}{2^{k+2}} + \cdots \\
&= \frac{(b_k - a_k)}{2^k} + \frac{(b_{k+1} - a_{k+1})}{2^{k+1}} + \frac{(b_{k+2} - a_{k+2})}{2^{k+2}} + \cdots \\
&= \left(\frac{b_k}{2^k} + \frac{b_{k+1}}{2^{k+1}} + \frac{b_{k+2}}{2^{k+2}} + \cdots\right) - \left(\frac{a_k}{2^k} + \frac{a_{k+1}}{2^{k+1}} + \frac{a_{k+2}}{2^{k+2}} + \cdots\right) \\
&= h'(b) - h'(a) \\
&= |h'(a) - h'(b)| \\
&= d_{eucl}(h'(a), h'(b))
\end{aligned}
$$

So, $(\Sigma_2/_\sim, d_1)$ is a metric space and it is also isometric to the metric space $([0, 1], d_{eucl})$. This concludes the proof. $\square$

Observe that if we define the mapping h'' as below, we have the following result:

$$h'' : (\Sigma_2/_\sim, d_2) \to ([m, n], d_{eucl})$$

$$h''(s_1 s_2 \ldots s_n \ldots) = m + (n - m)\left(\frac{s_1}{2} + \frac{s_2}{2^2} + \ldots + \frac{s_n}{2^n} + \ldots\right)$$

where

$$d_2(a, b) = \sum_{i=k+1}^{\infty} \frac{\alpha_i' + \beta_i'}{2^i}$$

such that

$$
\alpha_i' = \begin{cases} 0 & \text{if } a_i = b_k \\ n - m & \text{if } a_i \neq b_k \end{cases}, \qquad
\beta_i' = \begin{cases} 0 & \text{if } b_i = a_k \\ n - m & \text{if } b_i \neq a_k \end{cases} \quad \text{where } i \geq k + 1.
$$

Corollary 3. *The metric space $(\Sigma_2/_\sim, d_2)$ is isometric to the metric space $([m, n], d_{eucl})$ for any $m, n \in \mathbb{R}$ and $n > m$.*

Remark 4. *One can also use the metric d_2 defined on $\Sigma_2/_\sim$ instead of the metric d_{eucl} defined on $[m, n]$ because of Corollary (3).*

Using the metric d_1, we can make some calculations easier than using Euclidean metric. To better understand the importance of this metric, we will give the following example:

Example 5. *Consider the points $a = \overline{010100}$ and $b = \overline{101011}$ of $\Sigma_2/_\sim$. Let us first compute $d_1(a, b)$ and then let us find the point c of $\Sigma_2/_\sim$ such that $d_1(a, b) = d_1(\overline{0}, c)$.*

$$
\begin{aligned}
d_1(a, b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
&= \left(\frac{0+0}{2^2} + \frac{1+1}{2^3} + \frac{0+0}{2^4} + \frac{1+1}{2^5} + \frac{1+1}{2^6} + \frac{1+1}{2^7}\right) + \\
&\quad \left(\frac{0+0}{2^8} + \frac{1+1}{2^9} + \frac{0+0}{2^{10}} + \frac{1+1}{2^{11}} + \frac{1+1}{2^{12}} + \frac{1+1}{2^{13}}\right) + \cdots \\
&= \left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}\right) + \left(\frac{1}{2^8} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{12}}\right) + \cdots \\
&= \frac{1}{2^2}\left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\right) + \frac{1}{2^4}\left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\right) \\
&\quad + \frac{1}{2^5}\left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\right) + \frac{1}{2^6}\left(1 + \frac{1}{2^6} + \frac{1}{2^{12}} + \cdots\right) \\
&= \frac{64}{63}\left(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}\right) \\
&= \frac{64}{63} \cdot \frac{23}{64} = \frac{23}{63}
\end{aligned}
$$

Note that the calculation above gives us the point c of $\Sigma_2/_\sim$ such that $d_1(a, b) = d_1(\overline{0}, c)$. That is, the point c equals to

$$
\left(\frac{0}{2^1} + \frac{1}{2^2} + \frac{0}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6}\right) + \left(\frac{0}{2^7} + \frac{1}{2^8} + \frac{0}{2^9} + \frac{1}{2^{10}} + \frac{1}{2^{11}} + \frac{1}{2^{12}}\right) + \cdots
$$

or equals to $\overline{010111}$ as an element of $\Sigma_2/_\sim$. If we try to calculate this distance and to find the point c with the Euclidean metric, then we will have to deal with a bit more processing. Because we do not deal with subtraction and absolute value in calculations made with the metric d_1. Moreover, we directly obtain this distance as a binary number.

In the following proposition, we emphasize a geometrical property of the metric d_1 different form the metric d_{eucl}.

Proposition 6. *Let* $a = a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ *and* $b = b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ *be arbitrary elements of* $\Sigma_2/\sim$. *If* $a_1 \neq b_1$ *and* $\alpha_i = \beta_i$ *for all* $i = 2, 3, \ldots$, *then these points are symmetric with respect to the point* $0\bar{1}$ *(or equivalently* $1\bar{0}$).

Proof. We must show that

$$d_1(a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots, 0\bar{1}) = d_1(b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots, 0\bar{1}).$$

Then

$$d_1(a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots, 0\bar{1}) = \sum_{i=k+1}^{\infty} \frac{\alpha_i' + \beta_i'}{2^i}$$

and

$$d_1(b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots, 0\bar{1}) = \sum_{i=k+1}^{\infty} \frac{\alpha_i'' + \beta_i''}{2^i}$$

where $\alpha_i', \alpha_i'', \beta_i'$ and β_i'' are determined in accordance with (7). Since $a_1 \neq b_1$, we have

$$\alpha_i = \begin{cases} 0 & \text{if } a_i = b_1 \\ 1 & \text{if } a_i \neq b_1 \end{cases} \quad , \quad \beta_i = \begin{cases} 0 & \text{if } b_i = a_1 \\ 1 & \text{if } b_i \neq a_1 \end{cases}$$

and $\alpha_i = \beta_i$ for all $i = 2, 3, \ldots$. In both cases, we get $b_i \neq a_i$ for all $i = 2, 3, \ldots$ because $a_1 = 0, b_1 = 1$ or $a_1 = 1, b_1 = 0$. Let us compute and compare the two distances between $a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $0\bar{1}$ and the distance between $b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ and $0\bar{1}$. Without loss of generality, suppose that $a_1 = 0$ and $b_1 = 1$. For all $i = 2, 3, \ldots$ if $a_i = 1$ then $b_i = 0$ and $a_i = 1$ for all $i > 1$ yields

$$d_1(a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots, 0\bar{1}) = d_1(b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots, 0\bar{1}) = 0.$$

Otherwise, there is at least one s such that $a_s = 0$ and $a_i = 1$ for $i = 2, 3, \ldots, s-1$ and then we have $b_s = 1$ and $b_i = 0$ for $i = 2, 3, \ldots, s - 1$. We obtain $\alpha_i'' = 0$ for all $i = 2, 3, \ldots, s - 1$ and $\alpha_s'' = 1$, and also $\beta_i' = 1$ for all $i = s + 1, s + 2, s + 3, \ldots$ and $\beta_i'' = 0$ for all $i = 2, 3, 4, \ldots$. Furthermore, for any $t \in \{s + 1, s + 2, s + 3, \ldots\}$, if $a_t = 1$, then we get $b_t = 0$ and thus $\alpha_t' = 0$ and $\alpha_t'' = 0$. Similarly, for any $t \in \{s + 1, s + 2, s + 3, \ldots\}$, if $a_t = 0$, then we obtain $b_t = 1$ and thus $\alpha_t' = 1$ and $\alpha_t'' = 1$. Therefore

$$\begin{aligned} d_1(a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots, 0\bar{1}) &= \sum_{i=s+1}^{\infty} \frac{\alpha_i' + \beta_i'}{2^i} \\ &= \frac{x_{s+1} + 1}{2^{s+1}} + \frac{x_{s+2} + 1}{2^{s+2}} + \frac{x_{s+3} + 1}{2^{s+3}} + \cdots \\ &= \frac{1}{2^s} + \frac{x_{s+1}}{2^{s+1}} + \frac{x_{s+2}}{2^{s+2}} + \frac{x_{s+3}}{2^{s+3}} + \cdots \end{aligned}$$

$$d_1(b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \cdots, 0\bar{1}) = \sum_{i=2}^{\infty} \frac{\alpha_i'' + \beta_i''}{2^i}$$

$$= \frac{0+0}{2^2} + \frac{0+0}{2^3} + \ldots + \frac{0+0}{2^{s-1}} + \frac{1+0}{2^s}$$

$$+ \frac{x_{s+1}+0}{2^{s+1}} + \frac{x_{s+2}+0}{2^{s+2}} + \cdots$$

$$= \frac{1}{2^s} + \frac{x_{s+1}}{2^{s+1}} + \frac{x_{s+2}}{2^{s+2}} + \frac{x_{s+3}}{2^{s+3}} + \cdots$$

where $x_i \in \{0, 1\}$ for $i = s+1, s+2, s+3, \ldots$. This completes the proof. □

In a similar way, we can generalize Proposition 6 as follows:

Corollary 7. *Let $a = a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots$ and $b = b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots$ be arbitrary elements of $\Sigma_2/\sim$. If $a_i = b_i = \sigma_i$ for all $i = 1, 2, \ldots, k-1$, $a_k \neq b_k$ and $\alpha_i = \beta_i$ for all $i = k+1, k+2, \ldots$, then these points are symmetric with respect to the point $\sigma_1 \sigma_2 \ldots \sigma_{k-1} 0\bar{1}$ (or equivalently $\sigma_1 \sigma_2 \ldots \sigma_{k-1} 1\bar{0}$) where $x_i \in \{0, 1\}$ for $i = 1, 2, \ldots, k-1$.*

Proof. The proof will be omitted. □

The following proposition gives us the relationship between truncation errors for the computations which is obtained by the metrics d_{eucl} and d_1. In Proposition 8, we choose a and b as

$$a_1 a_2 \ldots a_{k-1} a_k a_{k+1} \ldots a_n a_{n+1} a_{n+1} \ldots$$

and

$$b_1 b_2 \ldots b_{k-1} b_k b_{k+1} \ldots b_n b_{n+1} b_{n+1} \ldots$$

respectively where $a_i = b_i$ for $i = 1, 2, \ldots k-1$ and $a_k \neq b_k$.

Proposition 8. *The truncation errors E_n and E'_{n+1} are determined such that*

$$d_1(a, b) = \sum_{i=k+1}^{n} \frac{\alpha_i + \beta_i}{2^i} + E_n$$

and

$$d_{eucl}(h'(a), h'(b)) = \left| \sum_{i=k}^{n} \frac{a_i - b_i}{2^i} + E'_{n+1} \right|.$$

Then we have

$$E_n = 2^{-n} + E'_{n+1}. \tag{8}$$

Proof. It must be either $b_k > a_k$ or $a_k > b_k$ since $a_k \neq b_k$. Without loss of generality, let us take $a_k > b_k$ and then we have $a_k = 1$ and $b_k = 0$. With simple calculations, we obtain

$$
\begin{aligned}
d_1(a,b) &= \sum_{i=k+1}^{\infty} \frac{\alpha_i + \beta_i}{2^i} \\
&= \sum_{i=k+1}^{n} \frac{\alpha_i + \beta_i}{2^i} + E_n \\
&= \frac{|a_{k+1} - b_k| + |b_{k+1} - a_k|}{2^{k+1}} + \frac{|a_{k+2} - b_k| + |b_{k+2} - a_k|}{2^{k+2}} \\
&\quad + \cdots + \frac{|a_n - b_k| + |b_n - a_k|}{2^n} + E_n \\
&= \frac{(a_{k+1} - b_k) + (a_k - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_k) + (a_k - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_n - b_k) + (a_k - b_n)}{2^n} + E_n \\
&= \frac{(a_k - b_k) + (a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_k - b_k) + (a_{k+2} - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_k - b_k) + (a_n - b_n)}{2^n} + E_n \\
&= \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots + \frac{1}{2^n} \right) + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} \\
&\quad + \cdots + \frac{(a_n - b_n)}{2^n} + E_n \\
&= \frac{1 - 2^{k-n}}{2^k} + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} + \cdots + \frac{(a_n - b_n)}{2^n} + E_n
\end{aligned}
$$

Moreover, we compute

$$
\begin{aligned}
d_{eucl}(h'(a), h'(b)) &= \left| \sum_{i=k}^{n} \frac{a_i - b_i}{2^i} + E'_{n+1} \right| \\
&= \sum_{i=k}^{n} \frac{a_i - b_i}{2^i} + E'_{n+1} \\
&= \frac{1}{2^k} + \frac{(a_{k+1} - b_{k+1})}{2^{k+1}} + \frac{(a_{k+2} - b_{k+2})}{2^{k+2}} + \cdots + \frac{(a_n - b_n)}{2^n} \\
&\quad + E'_{n+1}
\end{aligned}
$$

We know that $d_1(a,b) = d_{eucl}(h'(a), h'(b))$ because of Proposition 2. Thus,

$$E_n = 2^{-n} + E'_{n+1}$$

is obtained if the required simplifications are made. Observe that, this equation does not depend on the choice of k and the error E_n is always positive while the error E'_{n+1} can be positive or negative. $\qquad\square$

As seen in the formula (8), the metric d_1 works better in some computations. An example which shows that the absolute value of truncation error E_n is less than the absolute value of truncation error E'_{n+1} is given as follows:

Example 9. *Consider the points $a_1 a_2 \ldots a_{k-1} 0111 \ldots$ and $a_1 a_2 \ldots a_{k-1} 1000 \ldots$. It is obvious that truncation error E_n is always zero for every natural number n. But, truncation error E'_{n+1} equals to -2^{-n}.*

3. CONCLUSION

We give a metric formula on the quotient space $\Sigma_2/\sim$. Using a similar method, metric formulas on quotient spaces, which are related to the sequence spaces Σ_n for every natural n, can be defined (for an example see ([6]). Moreover, the metric spaces which are isometric to these quotient spaces equipped with these metrics can be investigated. In this paper, we show the metric space $(\Sigma_2/\sim, d_1)$ is isometric to the $([0,1], d_{eucl})$. The metric d_1 provides some facilities since there is no absolute value and subtraction in the metric formula (for details see Example 5, 9 and Proposition 8). The distance is also obtained as a binary number. As the metric formula d_1 is simple and understandable, it can also be used in the computer science applications (that is, the metric formula (7) is quite suitable for the basic coding commands).

The metric formula d_1 also has a different interpretation from the Euclidean metric defined on $[0,1]$: The distance between two points in the Euclidean metric is formulated as the absolute value of the difference of the distances of these two points to the point 0 while the distance between two points in the metric d_1 is formulated as the sum of the distances of these points to the point $\frac{1}{2^k}$ where k is the smallest index such that $a_i \neq b_i$ for $i = 1, 2, 3, \ldots$ (for details see Proposition 6 and Corollary 7).

Authors Contribution Statement The authors contributed equally to all parts of the paper.

Declaration of Competing Interest The authors have no potential conflict of interest.

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