Some Cases Of Superposable Fluid Motions *

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PREFACE

An outstanding difficulty in the theory of fluid motions is the fact that the differential equations of motion are not linear. Their solutions are difficult and sometimes impossible. Workers therefore are frequently forced to use approximate methods of solution, or to make assumptions which are not always sound. For example, in a number of cases it is assumed that two distinct solutions of the hydrodynamical equations of motion are linearly superposable, i.e. their sum is also a solution. The same assumption is used again and again, for instance, in Lamb's hydrodynamics.

But have wethe right to do so, since the equations of motion are not linear? Under what conditions is the sum of two distinct solutions again a solution.?

Professor J. A. Strang in Ankara University asked these questions. He studied the subject fully and published his results in «Comm. de la Faculté des Sciences de L'Universite D'Ankara, Tome I. pp. 1-32. 1948.» He says that any two

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solutions of the equations of motion are not, in general, linearly superposable. If $U_1 = (u_1, v_1, w_1)$ and $U_2 = (u_2, v_2, w_2)$ are two solutions of the equations of motion of a viscous incompressible fluid corresponding to given external forces, initial and boundary conditions, not necessarily the same in both cases, they are superposable on each other, if and only if

$$U_1 \times (\nabla \times U_2) + U_2 \times (\nabla \times U_1) = \nabla\chi,$$

where χ is an arbitrary scalar function of x, y, z and t. This is the superposability condition. If $U_1 = U_2 = U$ we obtain the condition for self-superposability,

$$U \times (\nabla \times U) = \nabla\chi,$$

where as before χ means any scalar.

When U_1 is given the superposability condition defines an infinite class of vector functions U_2 which forms an additive set with U_1. This therefore seems to be a powerful means of constructing new motions. That is precisely the object of this memoir.

It is divided into four paragraphs. The first deals with the motions of a given type superposable on a purely rotatory motion with a constant angular velocity. The solutions of the steady and non-steady cases have been obtained.

In the second paragraph a more general type of the motion U_2 is used and the corresponding results for the same U_1 are obtained.

In the third, U_1 is the same again, but U_2 represents the type of motion which is constituted by a rotation about the z-axis plus a parallel flow along the- axis. It is found that motions of this type, if they are superposable on U_1, are also self-superposable. The solution for the steady motion of a fluid about a circular cylinder, rotating with constant angular velocity, is obtained. Also the steady motion between two circular cylinders, rotating with different constant angular velocities, is obtained.

In the last paragraph U_1 is a two dimensional radial motion, and U_2 represents a radial flow plus an axial motion. The forms of the arbitrary functions in U_2 are determined, both in steady and nonsteady cases. The function h which defines the axial motion is obtained from a non-linear differential equation of the 4^{th} order in series form, and the radius of convergence of the series is obtained by a method found by Prof. J. A. Strang.

I am very glad to express my hearty thanks to my respected Professor J. A. Strang for his constant interest and valuable suggestions in preparing this memoir.

1. Motions superposable on

$$U_1: \quad \begin{aligned} u_1 &= -y\omega \\ v_1 &= x\omega \\ w_1 &= 0, \end{aligned} \tag{1}$$

where ω is a constant. This amounts to the motion of a fluid about a solid of revolution, together with which it rotates as a solid about the z-axis with the constant angular velocity ω. Now we inquire what motions U_2 are superposable on U_1. We are assuming that the motion (1) is taking place in a viscous, incompressible fluid filling infinite space. Since in the planes perpendicular to z-axis there will be a radial velocity outwards from the axis, as a result of centrifugal effect, there will also be an axial flow towards the rotating body. Hence we may take

$$U_2: \quad \begin{aligned} u_2 &= xf \\ v_2 &= yf \\ w_2 &= h \end{aligned} \tag{2}$$

as the simplest possible form of fluid motion, which is superposable on U_1; where f, g are functions of $r = (x^2 + y^2)^{1/2}$, z, and t only.

The continuity equation is

$$2f + rf_r + h_z = 0. \tag{3}$$

The vorticity components of each motion are respectively

$$\xi_1 = 0, \qquad \xi_2 = -y\,(f_z - \frac{1}{r} h_r),$$

$$\eta_1 = 0, \qquad \eta_2 = \;\; x\,(f_z - \frac{1}{r} h_r),$$

$$\zeta_1 = 2\omega, \qquad \zeta_2 = \;\; 0.$$

The superposability condition requires [1]

$$\chi_x = \;\; 2\omega \cdot yf$$
$$\chi_y = -2\omega \cdot xf$$
$$\chi_z = 0,$$

where χ is an arbitrary harmonic function of x, y, z and t. The consistency conditions require

$$f_z = 0, \quad \text{and} \quad 2f + rf_r = 0.$$

The first shows that f is a function of r and t only; and the second gives

$$f = a\,(t) \cdot r^{-2} \tag{4}$$

where $a\,(t)$ is an arbitrary function of t only. Now the equation of continuity (3) furnishes

$$h_z = 0, \tag{5}$$

i. e. h is also a function of r and t only.

Therefore ξ_2, η_2, ζ_2 and χ_x, χ_y, χ_z become

$$\xi_2 = \;\; \frac{y}{r} h_r, \qquad \chi_x = 2\,\omega\, a \cdot \frac{y}{r^2},$$

$$\eta_2 = -\frac{x}{r} h_r, \qquad \chi_y = -2\omega a \cdot \frac{x}{r^2},$$

$$\zeta_2 = \;\; 0, \qquad \chi_z = 0;$$

and
$$\chi = 2\omega a. \text{ arc tg } \frac{x}{y} + k\,(t),$$

(1) See J. A. Strang, Superposable Fluid Motions, Comm. de la Faculté des Sciences, Ankara. vol. I, 1948, p. 4.

where $k(t)$ is an arbitrary function of t, which may be omitted since only space derivatives of χ are used. Hence the motion

$$u_2 = a(t) \cdot xr^{-2},$$
$$v_2 = a(t) \cdot yr^{-2}, \tag{6}$$
$$w_2 = h(r)$$

is superposable on the motion U_1, $h(r)$ being an arbitrary function of r and t only.

It remains to satisfy the equations of motion. These require

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{a\,x}{r^2} = \frac{a^2 x}{r^4} - \frac{\partial}{\partial x}\left(\frac{p}{\rho} + \Omega\right),$$
$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{a\,y}{r^2} = \frac{a^2 y}{r^4} - \frac{\partial}{\partial y}\left(\frac{p}{\rho} + \Omega\right),$$
$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)h = -\frac{a}{r}\; h_r - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \Omega\right),$$

where p is the pressure, ρ density, and Ω the force potential. But since xr^{-2}, yr^{-2} are harmonics these can be written as

$$\frac{x}{r^2}a_t - \frac{a^2 x}{r^4} = -\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \Omega\right),$$
$$\frac{y}{r^2}a_t - \frac{a^2 y}{r^4} = -\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \Omega\right),$$
$$h_t - \nu\nabla^2 h + \frac{a}{r}h_r = -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \Omega\right)$$

The consistency conditions $\chi_{xy} = \chi_{yx}$, etc. are satisfied if

$$h_t + a(t) \cdot r^{-1}\, h_r - \nu\nabla^2\, h = b(t),$$
$$h_t + a(t) \cdot r^{-1}\, h_r - \nu\,(h_{rr} + r^{-1}\, h_r) = b(t), \tag{7}$$

where $b(t)$ is an arbitrary function of t only.

When the motion is steady (7) reduces to an ordinary differential equation, whose solution is easily seen to be

$$h = \frac{br^2}{2(a - 2\nu)} + c_1 r^{a/\nu} + c_2, \tag{8}$$

the constants of integration being c_1 and c_2.

When the motion is not steady the equation (7) can still be solved by separating the variables. This will be done in two ways.

(i) Let $h = RT$, $a(t) = 2A\nu$, $b(t) = -B\nu T$,

where A, B are independent of t. It then becomes

$$\frac{T_t}{T} = \nu\left(\frac{R_{rr}}{R} + \frac{1}{r}\frac{R_r}{R}\right) - \frac{2A\nu}{r}\frac{R_r}{R} - \frac{B\nu}{R}.$$

The left hand side depends only on t, and the right hand side is a function of r only. Hence each must be a constant, say $-\nu C^2$. Therefore we have to solve the equations

$$T_t/T = -\nu C^2 \tag{9}$$

$$R_{rr} + (1 - 2A)r^{-1}R_r + C^2 R = B. \tag{10}$$

The equation (9) shows that $T \backsim e^{-\nu C^2 t}$, and the solution of (10) is

$$R = r^A . [c_1 J_A(Cr) + c_2 J_{-A}(Cr)] + BC^{-2},$$

provided $2A$ is not an integer. Hence

$$h = \sum \mu R e^{-\nu C^2 t},$$

and (6) becomes

$$\begin{aligned} u_2 &= 2A\nu x . r^{-2} \\ v_2 &= 2A\nu y . r^{-2} \\ w_2 &= \sum \mu R . e^{-\nu C^2 t}. \end{aligned} \tag{11}$$

The pressure equation is

$$p/\rho + \Omega = -2A^2\nu^2 . r^{-2} + \nu Bz . e^{-\nu C^2 t} + \text{an arbitrary function of } t.$$

(ii) In equation (7) the effect of $b(t)$ is simply to add an arbitrary function of t to the value of h given by

$$h_t + a(t)r^{-1} h_r - \nu (h_{rr} + r^{-1}h_r) = 0.$$

Let $\alpha = r^2/4\nu t$, $h = T.g(\alpha)$,

where T is a function of t only, and $g(\alpha)$ is a function of α only.

Since $h_t = T'g - \frac{\alpha}{t} Tg'$, $h_r = \frac{r}{2\nu t} \cdot Tg'$,

$$h_{rr} = \frac{T}{2\nu t} (g' + 2\alpha g''),$$

(7) can be written in the form

$$\alpha g'' + [1 - \frac{a(t)}{2\nu} + \alpha] g' - \frac{tT'}{T} g = 0,$$

and the variables are separable if $a(t) = a$, $tT'/T = c_1$, *i.e.*

$$T = k\, t^{c_1},$$

where a, c_1, k are constants; and g must satisfy

$$\alpha g'' + (1 - \frac{a}{2\nu} + \alpha) g' - c_1 g = 0. \qquad (12)$$

This equation reduces to Kummer's confluent hypergeometric equation on writing $\beta = -\alpha$

$$\beta \frac{d^2g}{d\beta^2} + \left(1 - \frac{a}{2\nu} - \beta\right)\frac{dg}{d\beta} + c_1 g = 0, \qquad (13)$$

so that two solutions are in general given by

$$G_1 = {}_1F_1(-c_1, 1 - \frac{a}{2\nu}; \beta) = {}_1F_1(a, c; \beta) \text{ say,}$$

$$= 1 + \frac{a}{c} \frac{\beta}{1!} + \frac{a(a+1)}{c(c+1)} \frac{\beta^2}{2!} + \ldots ,$$

and $G_2 = \beta^{a/2\nu} \, {}_1F_1 \left(\frac{a}{2\nu} - c_1 \, , \frac{a}{2\nu} + 1 \, ; \beta \right)$,

provided that $c = 1 - a/2\nu$ is not an integer or zero.

Hence $g = AG_1 + BG_2$, provided that $a/2\nu$ is not an integer;

and $h = kt^{c_1}(AG_1 + BG_2)$ + an arbitrary function of t. Also if a is fixed we can sum any set of such solutions, since (7) is linear in h.

If $a/2\nu$ is zero or an integer, one or other of the series G_1, G_2 ceases to have a meaning because of zero factors in the denominator. We can then apply Frobenius's method to obtain the general solution, which will then contain log β. But $\log \beta = \infty$ if $\beta = 0$ or $\beta = \infty$, and so is excluded from the solution of the hydrodynamical problem; so there remains only that one of the series G_1 , G_2 in which no zero factors occur in the denominator.

If $c_1 = 0$

$$\alpha g'' + \left(\alpha + 1 - \frac{a}{2\nu} \right) g' = 0$$

and

$$g = p \, . \int_0^{\alpha} \alpha^{a/2\nu - 1} \, e^{-\alpha} \, d\alpha + q \, , \qquad (14)$$

where p , q are constants.

This makes:

When $t = 0$, $g = p \int_0^{\infty} \alpha^{a/2\nu - 1} \, . \, e^{-\alpha} \, d\alpha + q = pI + q$ say,

it is finite everywhere except on the axis $r = 0$.

When $t > 0$, $g = q$ on the axis,

$$= p \int_0^{\alpha} \alpha^{a/2\nu - 1} \, . \, e^{-\alpha} \, d\alpha + q \text{ at all other points.}$$

$$= pl + q \quad \text{at infinity.}$$

When $t = \infty$, $g = q$ everywhere, since $\alpha = 0$.

So the value of g represents an initial steady state, plus a disturbance which spreads out from the axis to infinity = a final steady state after the disturbance has died away.

But the initial and final steady states are not the same. It is assumed above that $a > 0$, so that $\frac{a}{2\nu} - 1 > -1$. If $a \leq 0$ we cannot take zero as the lower limit of the integral, since the integral is then infinite.

2. It has been shown (J. A. Strang, «Superposable Fluid Motions» Com. de la Facultè des Sciences, Ankara. I. 1948, p. 31) that if a solid of revolution rotates about its axis OZ with constant angular velocity ω, fluid motion of type

$$u = -yg \; ,$$
$$v = xg \; ,$$
$$w = 0$$

is possible only if $g = \omega$, i. e. the fluid moves as a solid with the rotating body, except only when the rotating solid and fluid boundary are circular cylinders whose axis is the axis of rotation.

This suggests the question what motions are superposable on U_1 , where

$$u_1 = -y\omega \; ,$$
$$v_1 = x\omega \; , \qquad (\omega = \text{constant})$$
$$w_1 = 0 \; .$$

In particular do there exist U_2 superposable on U_1 of the type

$$u_2 = xf - yg \; ,$$
$$v_2 = yf + xg \; ,$$
$$w_2 = h \; ;$$

where f, g, h depend only $r = (x^2 + y^2)^{\frac{1}{2}}$, z , and t?

Since
$$\xi_2 = - xg_z - y\left(f_z - \frac{1}{r}h_r\right),$$
$$\eta_2 = - yg_z + x\left(f_z - \frac{1}{r}h_r\right),$$
$$\zeta_2 = 2g + rg_r \; ;$$

the superposability condition (loc. cit, p. 49) furnishes if we replace the function χ by $\omega\chi$

$$\chi_x = x\,(4g + rg_r) + 2yf\,,$$
$$\chi_y = y\,(4g + rg_r) - 2xf$$
$$\chi_z = r^2 g_z\,.$$

The superposability condition does not involve h.

The consistency conditions $\chi_{xy} = \chi_{yx}$ etc. now require

$$f_z = g_z = 0, \quad \text{and} \quad f = a\,(t)\,.\,r^{-2}\,. \tag{1}$$

The equation of continuity for U_2 is

$$2f + rf_r + h_z = 0\,,$$
i. e.
$$h_z = 0\,, \tag{2}$$

and it remains only to satisfy the Navier-Stokes equations of motion. If we denote by Ω the force potential of the external forces these may be written

$$-\frac{\partial}{\partial x}\left(\Omega + \frac{p}{\rho}\right) = xP - yQ\,,$$

$$-\frac{\partial}{\partial y}\left(\Omega + \frac{p}{\rho}\right) = yP + xQ\,,$$

$$-\frac{\partial}{\partial z}\left(\Omega + \frac{p}{\rho}\right) = h_t + \frac{a}{r}h_r - \nu\left(h_{rr} + \frac{1}{r}h_r\right);$$

where $P = a_t r^{-2} - a^2 r^{-4} - g^2\,,$

$$Q = g_t + ar^{-2}\,(2g + rg_r) - \nu\left(g_{rr} + \frac{3}{r}g_r\right).$$

The consistency conditions show that

$$Q = b\,(t)\;.\;r^{-2}\quad ,$$

and that since P and Q do not involve z the expression $-\frac{\partial}{\partial z}(\Omega + \frac{p}{\rho})$ cannot involve x or y, i. e. it is independent of r, and therefore is a function of t only, say $c(t)$.

We have therefore to solve the equations

$$g_t + ar^{-2}(2g + rg_r) - \nu\,(g_{rr} + \frac{3}{r}\,g_r) = b\,(t)\,r^{-2} \tag{3}$$

$$h_t + ar^{-1}h_r - \nu\,(h_{rr} + \frac{1}{r}\,h_r) = c\,(t) \tag{4}$$

where $a\,(t)$, $b\,(t)$ and $c\,(t)$ are arbitrary functions of t.

When the motion is steady these reduce to ordinary differential equations, whose solutions are easily seen to be

$$g = c_1 r^{-2} + c_2 r^{a/\nu} + \frac{b}{2\,a}\,,$$

$$h = \frac{cr^2}{2(a-2\nu)} + c_3 r^{a/\nu} + c_4\,;$$

the constants of integration being c_1 , c_2 , c_3 and c_4.

When the motion is not steady the variables r and t can still be separated by suitable choice of $a\,(t)$, $b\,(t)$, $c\,(t)$.

If we write $g = RT$ as usual, and if

$$a = 2A\nu\;,\qquad b = B\nu T\;,\qquad T_t = -\,\nu C^2 t$$

where A, B, C are real and independent of t, we obtain

$$R_{rr} + (3 - 2A)\,r^{-1}\,R_r + (C^2 - 4Ar^{-2})\,R = -\,Br^{-2}. \tag{5}$$

The solution of the homogeneous equation is given by

$$R_{B=0} = r^{A-1} \,.\; J_{\pm(A+1)}(C\nu)\,, \quad (C \neq 0)$$

and the solution of (5) follows by variation of parameters, after which that of (3) takes the form

$$g = \sum \lambda R e^{-\nu C^2 t} \,. \tag{6}$$

Similarly if $h = ST$, where S depends only on r, and if $c(t) = -D\nu T$, equation (4) becomes

$$S_{rr} + (1 - 2A)\, r^{-1} S + C^2 S = D \tag{7}$$

so that if $C \neq 0$

$$S = r^A \,.\, [c_1 J_A (Cr) + c_2 J_{-A} (Cr)] + \frac{D}{C^2}$$

and $$h = \sum \mu S e^{-\nu C^2 t}. \tag{8}$$

These equations, along with

$$f = 2A\nu \,.\, r^{-2} \tag{9}$$

complete the solution apart from cases arising from particular values of A, B, and C; and those in which the variables are not separable.

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Let us try to solve the equations (3) and (4) in a different way.

Take the equation (3) first. It can be written as

$$-\nu r^2 g_{rr} + (a - 3\nu)\, r g_r + 2ag + r^2 \,.\, g_t = b(t)\,. \tag{10}$$

The solution of the homogeneous equation, i. e.

$$-\nu r^2 g_{rr} + (a - 3\nu)\, r g_r + 2ag + r^2 g_t = 0 \tag{11}$$

can be obtained by the substitution

$$g = T \,.\, G(\alpha)\,, \quad \text{where} \quad \alpha = \frac{-r^2}{4\nu t}\,.$$

After the substitution dividing both sides by $2T$, rearranging terms, we obtain

$$-\nu\,[2\alpha^2 G_{\alpha\alpha} + 2\,(2-\alpha)\,\alpha G_\alpha] + a\alpha G_\alpha + aG - 2\nu\alpha G \,.\, \frac{tT'}{T} = 0\,.$$

Now if $a = 2A\nu$, $\frac{tT'}{T} = c_1$, $\therefore$ $T = kt^{c_1}$,

the equation becomes

$$\alpha^2 G_{\alpha\alpha} + (2 - A - \alpha)\,\alpha G_\alpha - (A - c_1\alpha)\,G = 0\,, \tag{12}$$

where A, c_1, k are constants.

Change the dependent variable from G to F by the substitution

$$G = \alpha^m \,.\, F(\alpha)\,,$$

where m is a constant, and $F(\alpha)$ is a function of α only. The equation (12) becomes

$$\alpha^2 F'' + [(2m + 2 - A) - \alpha]\,\alpha F' + [(m+1)(m-A) + (c_1 - m)\,\alpha]F = 0,$$

which reduces to a confluent hypergeometric equation if

$$(m+1)\,(m-A) = 0\,,$$

i. e. if $m = -1$, or $m = A$.

If A is not zero or an integer we obtain two independent series solutions, which can be expressed as

$${}_1F_1(-c_1 - 1\,,\,-A\,;\,\alpha)\,, \quad \text{and} \quad {}_1F_1(A - c_1\,,\,A + 2\,;\,\alpha)\,.$$

Hence the general solution of (12) is

$$G = C_1\alpha^{-1}\,.\,{}_1F_1(-c_1 - 1\,,\,-A\,;\,\alpha)$$
$$+ C_2\alpha^A\,.\,{}_1F_1(A - c_1\,,\,A + 2\,;\,\alpha)\,,$$

and the general solution of (11) is

$$g = kt^{c_1} \cdot \sum G , \qquad \text{since (11) is linear.}$$

Now the solution of the homogeneous equation is complete. It can easily be seen that a particular integral of (10) is

$$g_1 = r^{-2} \cdot \int b(t)\, dt \;;$$

hence the general solution of (10) is

$$g = kt^{c_1} \cdot \sum G + r^{-2} \cdot \int b(t)\, dt.$$

For the solution of (4) see page 54.

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3. Determination of motions U_2 of the kind

$$U_2: \begin{array}{l} u_2 = -\,yf \\ v_2 = \quad xf \\ w_2 = \quad h \end{array} \quad \text{which are superposable on } U_1: \begin{array}{l} u_1 = -\,y\omega \\ v_1 \quad\;\; x\omega \\ w_1 = \quad 0, \end{array}$$

where f and h are arbitrary functions of $r = (x^2 + y^2)^{\frac{1}{2}}$, z and t.

U_2 represents a rotatary motion about the z-axis, plus a parallel flow along the axis. U_1 is a purely rotary motion with constant angular velocity ω.

The continuity equation for U_2 is

$$h_z = 0\,. \tag{1}$$

If we calculate the vorticity components we obtain

$$\xi_1 = 0 \quad , \qquad \xi_2 = \frac{y}{r} h_r - x f_z \quad ,$$

$$\eta_1 = 0 \quad , \qquad \eta_2 = -\frac{x}{r} h_r - y f_z \quad ,$$

$$\zeta_1 = 2\omega \;; \qquad \zeta_2 = 2f + r f_r \quad .$$

Superposability condition requires

$$\chi_x = \omega x \ . \ (4f + rf_r) \quad ,$$
$$\chi_y = \omega y \ . \ (4f + rf_r) \quad ,$$
$$\chi_z = \omega r^2 \ . \ f_z \quad .$$

The consistency conditions $\chi_{xy} = \chi_{yx}$, etc. are all satisfied if

$$f_z = 0 \, ; \tag{2}$$

and so

$$\chi = \omega \ . \int \frac{1}{r^2} \frac{d}{dr} (r^4 f) \, dr + \text{an arbitrary function of t.}$$

Hence if $f_z = h_z = 0$, the motion U_2 is always superposable on U_1 ; and it is also selfsuperposable (See, J. A Strang, loc. cit. p. 51). For if we put

$$U_2 \times (\nabla \times U_2) = \nabla \psi \quad ,$$

where ψ is an arbitrary scalar function of x, y, z and t; this requires

$$\psi_x = 2xf^2 + xrff_r + xr^{-1} hh_r \quad ,$$
$$\psi_y = 2yf^2 + yrff_r + yr^{-1} hh_r \quad ,$$
$$\psi_z = \quad 0 \quad .$$

The consistency conditions are satisfied identically, and

$$\psi = \int r^{-2} \frac{d}{dr} \left(\frac{1}{2} r^4 f^2 \right) dr + \frac{1}{2} h^2 + C, \tag{3}$$

where C may be an arbitrary function of t or a constant.

Now let us satisfy the equation of motion.
Since the motion is self-superposable the equation of motion may be written

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) U_2 = \nabla (\psi - \chi') \tag{4}$$

where ψ is given in (3), and

$$\chi' = \frac{p}{\rho} + \Omega + \frac{1}{2} U^2$$

$$= \frac{p}{\rho} + \Omega + \frac{1}{2} r^2 f^2 + \frac{1}{2} h^2$$

$$\therefore \ \psi - \chi' = \int \frac{1}{r^2} \frac{d}{dr} \left(\frac{1}{2} r^4 f^2 \right) dr - \left(\frac{p}{\rho} + \Omega \right) - \frac{1}{2} r^2 f^2 + C.$$

The three components of (4) can now be written

$$-\frac{\partial}{\partial x}\left(\frac{p}{\rho} + \Omega \right) = -y f_t + \nu y \left(f_{rr} + 3r^{-1} f_r \right) - x f^2 ,$$

$$-\frac{\partial}{\partial y}\left(\frac{p}{\rho} + \Omega \right) = x f_t - \nu x \left(f_{rr} + 3r^{-1} f_r \right) - y f^2 ,$$

$$-\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \Omega \right) = h_t - \nu \left(h_{rr} + r^{-1} h_r \right) .$$

These are consistent if

$$2 f_t + r f_{rt} = \nu \left(r f_{rrr} + 5 f_{rr} + 3 r^{-1} f_r \right) , \tag{5}$$

and

$$h_t = \nu \left(h_{rr} + r^{-1} h_r \right) + c(t) . \tag{6}$$

When the motion U_2 is steady the solution of (5) and (6) are respectively

$$f = A_1 \log r + A_2 r^{-2} + A_3 ,$$

$$h = B_1 r^2 + B_2 \log r + B_3 ,$$

where A_i , B_i are arbitrary constants.

Hence the required solution for steady motion is

$$\begin{aligned} u_2 &= -y \left(A_1 \log r + A_2 r^{-2} + A_3 \right) , \\ v_2 &= x \left(A_1 \log r + A_2 r^{-2} + A_3 \right) , \\ w_2 &= B_1 r^2 + B_2 \log r + B_3 . \end{aligned} \tag{7}$$

If we regard U_1 as the velocity of a fluid rotating as a solid about a circular cylinder of cross section a which rotates uniformly with a constant angular velocity ω about its axis (z-axis), and if the fluid extends to infinity and is at rest there we must have

$$U_2 = 0 \quad \text{when} \quad r \to \infty$$
$$U_2 = U_1 \quad \text{when} \quad r = a.$$

When these boundary conditions are satisfied we obtain

$$\begin{aligned} u_2 &= -\,\omega\, a^2 y r^{-2}\,, \\ v_2 &= \omega\, a^2 x r^{-2}\,, \\ w_2 &= 0\,. \end{aligned} \tag{8}$$

This is the steady motion which is both self-superposable and superposable on U_1. It satisfies the Navier-Stokes equations and boundary conditions. It is an irrotational motion about the cylinder.

If the fluid is bounded externally by a coaxal circular cylindrical surface of radius $b > a$ which rotates also with a constant angular velocity ω, then we can retain all the coefficients in the solution (7).

The boundary conditions are

$$U_2 = U_1 \quad \text{when} \quad r = a\,,$$
$$\left.\begin{aligned} f(r) &= \omega_1 \\ h(r) &= 0 \end{aligned}\right\} \quad \text{when} \quad r = b\,.$$

There are six coefficients to be determined, but there are only four boundary conditions. Hence two of the coefficients will remain arbitrary. The solution between the two cylinders is

$$u_2 = -y\left[\frac{\omega - \omega_1 + A_2(b^{-2} - a^{-2})}{\log(a/b)} \log\frac{r}{a} + A_2\left(\frac{1}{r^2} - \frac{1}{a^2}\right) + \omega\right],$$

$$v_2 = x\left[\frac{\omega - \omega_1 + A_2(b^{-2} - a^{-2})}{\log(a/b)} \log\frac{r}{a} + A_2\left(\frac{1}{r^2} - \frac{1}{a^2}\right) + \omega\right], \tag{9}$$

$$w_2 = B_2 \cdot \left[\frac{\log(a/b)}{b^2 - a^2}(r^2 - a^2) + \log\frac{r}{a}\right].$$

When the motion is not steady the equation (5) and (6) can still be solved by separating the variables. Thus if we put

$$\varphi = 2f + rf_r$$

the equation (5) raduces to

$$\varphi_t = (\varphi_{rr} + r^{-1}\varphi_r) .$$

Let $\varphi = RT$, then this becomes

$$\frac{T_t}{T} = \nu\left(\frac{R_{rr}}{R} + \frac{1}{r}\ \frac{R_r}{R}\right) = -\nu\lambda^2 \quad \text{say,}$$

where λ is a constant. Hence $T \sim e^{-\nu\lambda^2 t}$, and

$$R_{rr} + r^{-1} R_r + \lambda^2 R = 0 \quad ;$$

$$\therefore \quad R = C_1 J_o(\lambda r) .$$

Therefore

$$2f + rf_r = C_1 e^{-\nu\lambda^2 t} . J_o(\lambda r) ,$$

$$\therefore \quad f(r, t) = C_1 e^{-\nu\lambda^2 t} . r^{-2}\int rJ_o(\lambda r)\, dr + C_2 r^{-2} , \qquad (10)$$

where C_1 , C_2 are constants of integration.

Similarly if we put $h = ST$, where S is a function of r only, the equation (6) may be written as

$$S . \frac{T_t}{T} = \nu\,(S_{rr} + \frac{1}{r} S_r) + \frac{c(t)}{T} .$$

But we may choose $c(t)$ and $T_t|T$ suitably. Let

$$c(t) = -\nu BT \quad \text{and} \quad T_t = -\nu\lambda^2 T$$

as before, where B is an arbitrary constant. Hence

$$S_{rr} + r^{-1} S_r + \lambda^2 S = B ,$$

$$\therefore \quad S = d_1 J_o(\lambda r) + B\lambda^{-2} .$$

Then $$h = [d_1 J_o(\lambda r) + B\lambda^{-2}] . e^{-\nu\lambda^2 t} , \qquad (11)$$

and the final solution for variable motion is

$$u_2 = - yr^{-2} \cdot [C_1 e^{-\nu\lambda^2 t} \cdot \int r J_o (\lambda r)\, dr + C_2] \, ,$$

$$v_2 = xr^{-2} \cdot [C_1 e^{-\nu\lambda^2 t} \cdot \int r J_o (\lambda r)\, dr + C_2] \, , \qquad (12)$$

$$w_2 = [d_1 J_o (\lambda r) + B\lambda^{-2}] \cdot e^{-\nu\lambda^2 t} \, .$$

If we choose $C_2 = \omega a^2$, solution (12) reduces to (8) after an infinite time, for the term containing the integral vanishes when $t = \infty$.

Solution of (5) in an other way.

If we assume that $2f + rf_r = \varphi$, the equation (5) reduces to

$$\varphi_t = \nu \left(\varphi_{rr} + \frac{1}{r} \varphi_r \right) \qquad (13)$$

Hence

$$r^2 f = \int r\varphi dr + b(t)$$

$$\therefore \qquad f = r^{-2} \cdot \int r\varphi dr + r^{-2} \cdot b(t) \qquad (14)$$

where φ is the general solution of (13) , and $b(t)$ is an arbitrary function of t.

In order to solve (13) let $\alpha = - r^2/4\nu t$, and $\varphi = T . F(\alpha)$, where T is a function of t only, and $F(\alpha)$ is a function of α only. Then

$$\frac{\alpha F''}{F} + (1 - \alpha) \frac{F'}{F} = - \frac{tT'}{T} \, .$$

The variables are separated. Each side must be a constant.

If $\quad tT'/T = - c$, i. e. $T = kt^{-c}$

where c , k are constants, then

$$\alpha F'' + (1 - \alpha) F' = cF \ , \tag{15}$$

which is Kummer's confluent hypergeometric equation. One particular solution is

$${}_1F_1 (c \ , 1 \ ; \alpha) = 1 + \frac{c}{1!} \frac{\alpha}{1!} + \frac{c(c+1)}{2!} \frac{\alpha^2}{2!} + \dots ,$$

the other is not independent from the first, since $\gamma = 1$. But Frobenius's method of solution furnishes the other, which then contains log α . It is

$${}_1F_1 (c \ , 1 \ ; \alpha) \log |\alpha| + \sum A_m \alpha^m \ ,$$

where $A_m = \left\{ \frac{1}{c} + \frac{1}{c+1} + \dots + \frac{1}{c+m-1} - \frac{2}{1} - \frac{2}{2} - \dots \frac{2}{m} \right\}$

$$\cdot \frac{c(c+1)\dots(c+m-1)}{(m!)^2} , \ (m \geq 1).$$

If c is zero or a negative integer the first solution terminates, otherwise it is an infinite series which is convergent for all values of α .

Hence

$$\varphi = kt^{-c} \left\{ {}_1F_1 (c \ , 1 \ ; \alpha) \ (A + B \log |\alpha|) + \sum A_m \alpha^m \right\} ,$$

provided c is not zero or a negative integer, and f is given by (14).

The equation (6) is just the same as (13) plus an arbitrary function of t. The second part will add only an arbitrary function of t to the solution of

$$h_t = \nu (h_{rr} + r^{-1} h_r),$$

and this can be solved as exactly the same as in page 66. Hence

$$h = k_1 t^{-c} . \left\{ {}_1F_1 (c \ , 1 ; \alpha) (A + B \log |\alpha|) + \sum A_m \alpha^m \right\}$$

+ an arbitrary function, of t.

4. To determine the motions of type

$$\left.\begin{aligned} u_2 &= xf \\ v_2 &= yf \\ w_2 &= h \end{aligned}\right\} \quad \text{which are superposable on } U_1 : \quad \left\{\begin{aligned} u_1 &= xr^{-2} \\ v_1 &= yr^{-2} \\ w_1 &= 0\ , \end{aligned}\right.$$

where f and h are functions of $r = (x^2 + y^2)^{\frac{1}{2}}$, z , and t only.

The superposability condition requires

$$\chi_x = \chi_y = 0, \quad \text{and} \quad \chi_z = f_z - r^{-1} h_r \quad , \tag{1}$$

so that χ can depend only on z and t.

Since

$$\begin{aligned} \xi_2 &= -y\,(f_z - r^{-1} h_r)\ , \\ \eta_2 &= x\,(f_z - r^{-1} h_r)\ , \\ \zeta_2 &= 0 \quad , \end{aligned}$$

it is clear that U_2 is irrotational or rotational according as χ_z is or is not zero.

The continuity equation is

$$2f + rf_r + h_z = 0. \tag{2}$$

Eliminating h from (1) and (2) we find

$$f_{rr} + 3r^{-1} f_r + f_{zz} = \chi_{zz}\ ,$$

which is reduced by the substitution $f = F + \chi$ to

$$F_{rr} + 3r^{-1} F_r + F_{zz} = 0\ , \tag{3}$$

which is equivalent to $\nabla^2 (xF) = \nabla^2 (yF) = 0$, since F is a function of r, z and t only.

The same substitution reduces (1) to

$$F_z - r^{-1} h_r = 0 \quad , \tag{4}$$

so that

$$\begin{aligned} u_2 &= xF \quad , \\ v_2 &= yF \quad , \\ w_2 &= h \quad , \end{aligned}$$

is an irrotational motion satisfying the required conditions. Since U_1 is itself irrotational the existence of such solutions is to be expected, for any two irrotational motions are superposable.

In general therefore when the motion is rotational we must look for solutions of the form

$$u = x(F + \chi) ,$$
$$v = y(F + \chi) ,$$
$$w = h \qquad ,$$

where $\chi_z \neq 0$, and F is a solution of (3) ; h is a solution of (4) , χ satisfies (1), and the Navier-Stokes equations are also to be satisfied.

The continuity equation (2) may now be written

$$2F + rF_r + 2\chi + h_z = 0 . \tag{5}$$

Eliminating F by means of (4) we have

$$\chi_z = -\frac{1}{2} \nabla^2 h , \tag{6}$$

so that $\nabla^2 h$ is a function of z and t only. Let

$$A = \frac{p}{\rho} + \Omega + \frac{1}{2} r^2 (F + \chi)^2 + \frac{1}{2} h^2 .$$

The Navier - Stokes equations can be expressed in the form

$$\frac{\partial A}{\partial r} = r(\nu\chi_{zz} - h\chi_z - F_t - \chi_t) ,$$

$$\frac{\partial A}{\partial z} = \nu\nabla^2 h + r^2 \chi_z (F + \chi) - h_t ,$$

and the consistency condition is reduced on using (5) and (6) to

$$\nu\chi_{zzz} - h\chi_{zz} - \chi_{zt} = 0 \quad , \tag{7}$$

which shows that h is a function of z, t only; so (6) becomes

$$\chi_z = -\frac{1}{2} h_{zz} ,$$

and (7) becomes

$$\nu h_{zzzz} - hh_{zzz} - h_{zzt} = 0 \quad . \tag{8}$$

If the motion is steady we have to solve

$$\nu h_{zzzz} = hh_{zzz} \quad . \tag{9}$$

Since h depends only on z, (4) becomes

$$F_z = 0 ,$$

so that F depends only on r, and hence from (5)

$$2F + rF_r = 2a = -2\chi - h_z , \tag{10}$$

where a is a constant, and so

$$F = a + br^{\,2} , \tag{11}$$

where a , b are constants.

The solution of the steady motion problem is now complete, h being the solution of (9) , F is given by (11) , and χ by (10).

The equation (9) is of cansiderable interest. Its. form shows that if h and its first three derivatives are finite and definite when $z = 0$, h can be expressed in the form of a Taylor series whose coefficients can be calculated from the equation itself. But the region of convergence of the series is not at all evident, nor indeed is it clear whether the series converges for any values of z other than zero. This requires investigation.

There is obviously a solution

$$h = a + bz + cz^2 ,$$

where a , b , c are arbitrary constants. This makes $h_{zzz} = 0$ for all values of z. It may be shown that this derivative is a factor in every later derivative, for if we denote by h_n the n^{th} derivative, and if

$$h_n = h_3 \, . \, \mathrm{f}\,(h\ ,\ h_1\ ,\ h_2)$$

then
$$h_{n+1} = h_4 f + h_3 \frac{df}{dz}\ ,$$

$$= h_3\left(\frac{1}{\nu} hf + \frac{df}{dz}\right) .$$

Hence if h_3 is a factor of h_n, it is also a factor of h_{n+1}. But the differential equation itself shows it to be a factor of h_4, so it is a factor of h_n for $n \geq 4$.

This explains the significance of the presence of the quadratic solution. It is the most general solution for which $h_3 = 0$ when $z = 0$, and h, h_1 and h_2 are finite.

If we write $h = \nu H$ the equation (9) becomes

$$HH_3 = H_4\ . \tag{12}$$

Since the equation is invariant when we replace z by $z + c$, where c is an arbitrary constant, any solution is still a solution when we replace z by $z + c$.

The equation is homogeneous if H is taken to be of degree -1 in z. This furnishes the particular solution

$$H = -\,4z^{-1}\ ,$$

so that
$$H = -\,4\,(z + c)^{-1}$$

is also a solution. This shows that the Taylor series solution does in fact converge under certain circumstances, in this case for $\left|\frac{z}{c}\right| < 1$, where c is arbitrary.

The equation (12) is also exact, and on integration furnishes

$$H_3 = HH_2 - \frac{1}{2} H_1^2 + C_1\ .$$

where C_1 is the constant of integration. This equation has the solution $H = a + bz + cz^2$ if $C_1 = \frac{1}{2}(b^2 - 4ac)$, and $H = -4z^{-1}$ if $C_1 = 0$.

In the particular case when $H = -4z^{-1}$ the motion U_2 becomes

$$u_2 = x\left(\frac{b}{r^2} - \frac{2\nu}{z^2}\right)$$

$$v_2 = y\left(\frac{b}{r^2} - \frac{2\nu}{z^2}\right)$$

$$w_2 = -\frac{4\nu}{z} ,$$

where b is a constant.

The streamlines of the motion are given by the simultaneous equations

$$\frac{dx}{x\left(\frac{b}{r^2} - \frac{2\nu}{z^2}\right)} = \frac{dy}{y\left(\frac{b}{r^2} - \frac{2\nu}{z^2}\right)} = \frac{dz}{-\frac{4\nu}{z}} ,$$

$$\therefore \quad \frac{y}{x} = c_1 , \text{ and } \quad bz^2 + 2\nu(x^2 + y^2) - c_2 z = 0 .$$

The streamlines are the intersections of the surfaces

$$\frac{y}{x} = c_1 , \quad bz^2 + 2\nu(x^2 + y^2) = c_2 z .$$

If $b > 0$, these are similar ellipses touching XOY plane at the origin. One of the principal axis is the z-axis always. The distance of the centre of the ellipse is given by $c_2/2b$. They are similar, because the ratio of the principal axes is $(b/2\nu)^{\frac{1}{2}}$, i. e. it is a constant. The streamlines in the xz — plane are shown in fig. 1.

If $b < 0$ the streamlines are similar hyperbolae, one of the branches of which touches XOY plane at the origin. The fig. 2

shows the stream lines in the xz — plane. The inclination of the asymptotes is constant and is the same as the ratios of the principal axes of ellipses. When $c_2 = 0$ the streamlines re-

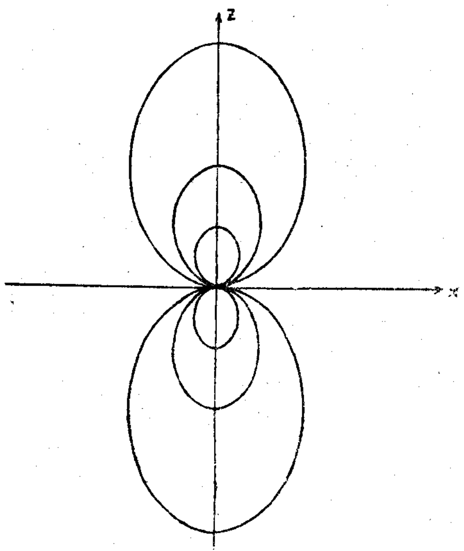

Fig. 1

duce to a pair of straight lines passing through the origin, which are parallel to the asymptotes.

The solution of equation (12) by Taylor series.

$$HH_3 = H_4 \quad ,$$

where the suffixes denote differentiation with respect to z. Let

$$H = c_0 + c_1 z + c_2 z^2 + \dots + c_n z^n + \dots$$

Substituting in the differential equation and equating the coefficients of like terms we obtain

$$g_4 c_4 = f_3 c_3 c_0$$
$$g_5 c_5 = f_3 c_3 c_1 + f_4 c_4 c_0$$

. . . .

$$g_n c_n = f_3 c_3 c_{n-4} + f_4 c_4 c_{n-5} + \dots + f_{n-1} c_{n-1} c_0 \,,$$

where $\quad f_n = \frac{4}{3} n\,(n-1)\,(n-2)\,,$ and

$$g_n = n\,(n-1)\,(n-2)\,(n-3)\,.$$

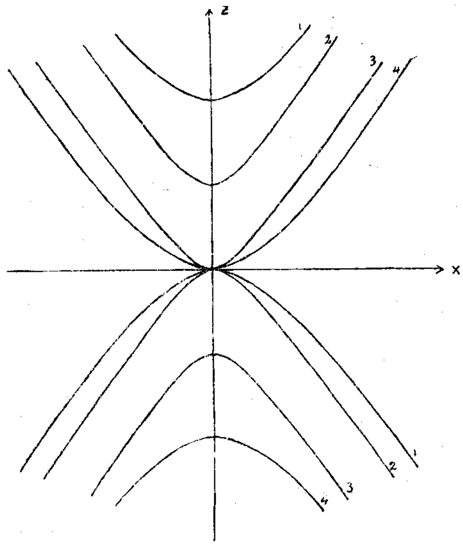

Fig. 2

Using these recurrence relations we can obtain a direct proof of the convergence of the Taylor series *. They do not give any information about the first four coefficients, i. e. c_0 , c_1 , c_2 , c_3 ; they are quite arbitrary. Replace all the coefficients by their moduli, which for brevity are denoted by the same letters, and let

$$c \geqq \max.\ (c_0\,,\ c_1\,,\ c_2\,,\ c_3). \tag{13}$$

We want to determine two constants $K > 0$ and $\lambda \geqq 1$, such that

$$c_n \leqq K\lambda^n \tag{14}$$

(*) The method of proof of the convergence is due to Prof. J. A. Strang.

for all values of n. If this is true we shall then have a dominant series $K\sum \lambda^n z^n$, of wahich the radius of convergence is λ^{-1} .

According to (13) and (14) we have

$$c_0 \leqq c \leqq K \quad ,$$
$$c_1 \leqq c \leqq K\lambda \quad ,$$
$$c_2 \leqq c \leqq K\lambda^2 \quad ,$$
$$c_3 \leqq c \leqq K\lambda^3 \quad .$$

Hence if $K = c$ and $\lambda \geqq 1$ these are all satisfied.

After replacing the coefficients by their moduli the general recurrence relation becomes

$$g_n c_n \leqq f_3 c_3 c_{n-4} + f_4 c_4 c_{n-5} + \ldots\ldots + f_{n-1} c_{n-1} c_0$$

$$\leqq K^2 \lambda^{n-1} . \sum_{s=3}^{n-1} f_s \quad , \quad \text{by (14)}$$

Dividing both sides by g_n , since it is positive for $n > 3$, and using (14) again we obtain

$$c_n \leqq \frac{\Sigma f_s}{g_n} K^2 \lambda^{n-1} \quad \leqq K\lambda^n \; .$$

But $$f_3 + f_4 + f_5 + \ldots\ldots + f_{n-1} = \frac{1}{4} g_n \; ,$$

and so $$\lambda \geqq \frac{1}{4} K.$$

Now if $K = c$ we require

$$\lambda \geqq \frac{c}{4} \; , \quad \text{and} \quad \lambda \geqq 1$$

$$\therefore \quad \lambda \geqq \max. \left(\frac{c}{4} \; , \; 1\right) \; ;$$

and the least permissible value of λ is $\max. \left(\frac{c}{4} , 1\right)$. Therefore the series converges inside the circle

$$| z | = \frac{1}{\max. \left(\frac{c}{4} , 1\right)}$$

and the dominant function is $c\,(1 - \lambda z)^{-1}$.

One-way power series solutions other than Taylor series.

Let
$$H = \sum_{r=1}^{\infty} c_r z^{m_r} \quad , \quad (c_1 \neq 0)$$

so that if $m_r \neq 0,\ 1,\ 2,\ 3$

$$H_3 = \sum_{r=1}^{\infty} f(m_r)\, c_r\ z^{m_r - 3} \quad ,$$

$$H_4 = \sum_{r=1}^{\infty} g(m_r)\, c_r\ z^{m_r - 4} \quad ,$$

where
$$f(m_r) = m_r (m_r - 1)(m_r - 2) , \quad \text{and}$$
$$g(m_r) = m_r (m_r - 1)(m_r - 2)(m_r - 3) \ .$$

The leading terms in the differential equation are

$$g(m_1)\, c_1 z^{m_1 - 4} + \ldots\ldots = f(m_1)\, c_1^2\ z^{2m_1 - 3} + \ldots\ldots$$

Since $f(m_1) \neq 0$, $g(m_1) \neq 0$, we must have

$$m_1 - 4 = 2m_1 - 3 \quad \text{and} \quad f(m_1)\, c_1^2 = g(m_1)\, c_1 \ ;$$

i. e. $m_1 = -1$ and $c_1 = -4$, since $c_1 \neq 0$.

Hence apart from $m_1 = 0\ ,\ 1\ ,\ 2\ ,\ 3$ the only possible solutions in one - way accending or descending powers of z must begin with $-4z^{-1}$.

It is evident that the case $m_1 = 0$ includes the Taylor series in ascending integral powers of z, and that with suitable values of the initial coefficients c_0, c_1, c_2 the Taylor series will also include the cases $m_1 = 0, 1, 2, 3$, provided the series are series of ascending integral powers. It only remains to inquire whether these values of m_1 can furnish other types of series. It is easily shown that they cannot. For instance if we assume that $m_1 = 0$, and that the solution is a descending power series, so that

$$H = c_1 + \sum_{r=2}^{\infty} c_r z^{-m_r}, \quad (c_1 \neq 0,\ c_2 \neq 0,\ 0 < m_2 < m_3 < \dots)$$

we find that the highest index after substitution is $-m_2 - 3$, and it occurs only once. Hence the corresponding coefficient must vanish. i. e.

$$f(-m_2)\, c_1 c_2 = 0,$$

but this is a contradiction, since none of the factors is zero. Hence $m_1 = 0$ does not furnish a solution in descending powers of z, whether integral powers or not; and the same is true of $m_1 = 1, 2, 3$. Similarly with positive nonintegral powers. The values of $0, 1, 2, 3$ lead only to series included in the Taylor series.

Let $m_1 = -1$, $c_1 = -4$, so that

$$H = -4z^{-1} + \sum_{r=2}^{\infty} c_r z^{m_r}. \tag{16}$$

On substituting in the differential equation and rearranging terms we find

$$\sum_{r=2}^{\infty} g_r c_r z^{m_r - 4} = \left(\sum_{r=2}^{\infty} c_r z^{m_r} \right) \left(\sum_{r=2}^{\infty} f_r c_r z^{m_r - 3} \right),$$

where $\quad f_r = m_r (m_r - 1)(m_r - 2)$, and

$$g_r = (m_r + 1) m_r (m_r - 1)(m_r - 2) - 24,$$

and c_2, being merely the coefficient of the first term after z^{-1}, is not zero. The leading terms cannot contain the same power of z, for this would require

$$2m_2 - 3 = m_2 - 4 \quad \text{i. e.} \quad m_2 = -1 = m_1 \ ,$$

hence at least the first term on one sids or the other must disappear, because its coefficient is zero. Hence

$$\text{either} \quad f_2 c_2^2 = 0 \quad \text{or} \quad g_2 c_2 = 0 .$$

Assume that $f_2 = 0$, and that f_s is the first non-vanishing coefficient in the corresponding series. Then comparing the indices we must have

$$m_2 + m_s - 3 = m_2 - 4,$$

that is $m_s = -1 = m_1$, which can not be true.

Hence $f_2 \neq 0$, and we must have $g_2 = 0$, or

$$(m_2 + 1)\, m_2\, (m_2 - 1)\, (m_2 - 2) - 24 = 0 ,$$

$$\text{i. e.} \qquad m_2 = -2 \ , 3 \ , \frac{1}{2}\,(1 \mp i\,\sqrt{15}) \ . \tag{17}$$

The other indices and coefficients now furnish

$$m_3 = 2m_2 + 1 \qquad g_3 c_3 = f_2 c_2 c_2$$

$$m_4 = 3m_2 + 2 \qquad g_4 c_4 = f_2 c_2 c_3 + f_3 c_3 c_2$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$m_n = (m_2 + 1)\,n - (m_2 + 2) \qquad g_n c_n = \sum_{s=2}^{n-1} f_s c_s c_{n+1-s}$$

$$\cdots\cdots \qquad \cdots\cdots$$

When $m_2 = -2$ we obtain the expansion of $-4\left(z + \frac{c_2}{4}\right)^{-1}$ in descending powers of z. The other series are new, and evidently two of them are conjugate complex numbers, with terms such as

$$cz^p \, . \, [\, \cos \, (q \, \log \, z) \pm i \, \sin \, (q \, \log \, z)] \, .$$

The discussion of convergence proceeds on lines similar to those employed in dealing with the Taylor series, except since m_2 may have complex values we must substitute for f_r and g_r their moduli.

Let us try to determine the constants $K > 0$ and $\lambda \geqq 1$, such that

$$c_n \leqq K\lambda^n \qquad \text{for all values of n.} \qquad (18)$$

This is true for $\qquad n \leqq \nu - 1 , \quad$ if

$$K \geqq c = \text{max.} \, (c_2 \, , \, c_3 \, . \, . \, . \, . \, . \, c_{\nu-1}) \, , \quad \lambda \geqq 1 \, .$$

In order to satisfy (18) for $n \geqq \nu$ take the general recurrence relation, which when all the coefficients are replaced by their moduli becomes

$$g_n c_n \leqq \sum_{s=2}^{n-1} f_s c_s c_{n+1-s} \, ,$$

and assume that (18) is true, then

$$g_n c_n \leqq \left(\sum_{s=2}^{n-1} f_s \right) . \, K^2 \lambda^{n+1} \, .$$

After a sufficiently large value of n, say N, g_n becomes positive, and

$$\lim_{n \to \infty} \frac{\Sigma f}{g_n} = \frac{1}{4 \, | \, m_2 + 1 \, |} \, .$$

Hence we can find a sufficiently large positive number N, so that

$$c_n \leqq \frac{1}{4 \, | \, m_2 + 1 \, |} K^2 \lambda^{\, n+1} \, , \quad \text{for } n \geqq N,$$

and the condition that (18) will also be true for $n \geqq \nu \; (=N)$ is

$$\frac{1}{4\,|\,m_2+1\,|}\,K^2\lambda^{n+1} \leqq K\lambda^n$$

$$\therefore \qquad K\lambda \leqq 4\,|\,m_2+1\,|.$$

If $K = c$, we have to satisfy the simultaneous equations

$$1 \leqq \lambda \leqq \frac{4\,|\,m_2+1\,|}{c}\,.$$

Hence if $c \leqq 4\,|\,m_2+1\,|$ the series converges inside the circle λ^{-1}. The maximum radius of convergence is 1.

Solution of the non-steady case.

When the motion is variable we have to solve the equation (8) instead of (9). F again has the form given in (11), but a and b are now arbitrary funtions of time; χ is given by (10), a being an arbitrary function of t.

It remains therefore to solve the equation (8). First of all observe that it is exact, and can be integrated once. We have then

$$\nu\,h_{zzz} - h\,h_{zz} + \frac{1}{2}\,h_z^2 - h_{zt} = C_1;$$

but this does not help us very much.

Now we wish to obtain the solution of the original equation

$$\nu\,h_{zzzz} - h_{zzt} = h\,h_{zzz}$$

approximately in series form, and we shall do this in two different ways.

(i) Let $$h = \nu\,f(z)\cdot g(\alpha),$$

where $\alpha = \dfrac{z^2}{4\nu t}$, $f(z)$ is a function of z only, and $g(\alpha)$ is a function of α only; ν is constant, it is kinematic coefficient of viscosity. By denoting all the derivatives by suffixes the equation becomes

$$f_4 g + \frac{8\alpha}{z} f_3 g_1 + \frac{24\alpha^2}{z^2} f_2 g_2 + \frac{32\alpha^3}{z^3} f_1 g_3 + \frac{16\alpha^4}{z^4} f g_4 +$$

$$(\alpha + 3) \left(\frac{4\alpha}{z^2} f_2 g_1 + \frac{16\alpha^2}{z^3} f_1 g_2 + \frac{16\alpha^3}{z^4} f g_3 \right) + \frac{16\alpha^2}{z^3} f_1 g_1$$

$$+ (10\,\alpha + 3) \frac{4\alpha^2}{z^4} f g_2 + \frac{8\alpha^2}{z^4} f g_1$$

$$= fg \cdot \left[f_3 g + \frac{6\alpha}{z} f_2 g_1 + \frac{12\alpha^2}{z^2} f_1 g_2 + \frac{8\alpha^3}{z^3} f g_3 + \frac{6\alpha}{z^2} f_1 g_1 + \frac{12\alpha^2}{z^3} f g_2 \right] .$$

Multiply both sides by z^4 / f. Then z disappears from the equation if $z^4 f_4 / f$, $z^3 f_3 / f$, $z^2 f_2 / f$, $z f_1 / f$, $z^4 f_3$, $z^3 f_2$, $z^2 f_1$, zf are all constants, which is true if $f = kz^{-1}$, where k is any constant.

The remaining equation is

$$12 g - 12\,\alpha\, g_1 + 6\,\alpha^2 g_2 + 8\,\alpha^3 g_3 + 8\,\alpha^4 g_4 + 12\alpha^3 g_2 + 8\,\alpha^4 g_3$$

$$= kg\,(-\,3\,g + 3\,\alpha\, g_1 + 4\,\alpha^3 g_3). \qquad (19)$$

We note that $g = -4\,k^{-1}$, i.e. $h = -4\,\nu\, z^{-1}$ is a solution, that $\alpha = 0$ is a singularity, so that negative powers of α must be expected in the general solution.

If $\Sigma\, c_n\, \alpha^n$ is a solution of (19) we have the following recurrance relations.

$$3k\, c_0^2 = -\,12\, c_0 \quad \text{i.e.} \quad c_0 = -\,4\,k^{-1}. \qquad (20)$$

$$3k c_0 c_1 = 0 \qquad \text{i.e.} \quad c_1 = 0\,;$$

$3k\,(c_0\, c_2 - c_2\, c_0) = 0$ identically $\therefore$ c_2 is arbitrary,

$$3k(10c_0c_3 + c_1c_2 - c_0c_3) = 60c_3 + 24c_2 \quad \text{i. e.} \quad c_3 = -\frac{1}{7}c_2$$

. . . .

$$A_n c_n + B_n c_{n-1} = k \cdot \sum_{m=0}^{n} \varphi_m c_m c_{n-m} \tag{21}$$

where

$$A_n = 12 - 12n + 6n(n-1) + 8n(n-1)(n-2) +$$
$$8n(n-1)(n-2)(n-3) = 2(n-1)(n-2)(2n-1)(2n-3).$$

$$B_n = 12(n-1)(n-2) + 8(n-1)(n-2)(n-3)$$
$$= 4(n-1)(n-2)(2n-3),$$

and

$$\varphi_m = -3 + 3m + 4m(m-1)(m-2) = (m-1)(2m-1)(2m-3).$$

By using the recurrence relation (21) we can prove the convergence of the Taylor series. Rearranging the terms and picking out those containing c_n the farmula (21) can be written as

$$[A_n - kc_0(\varphi_0 + \varphi_n)]\, c_n = k \sum_{m=1}^{n-1} \varphi_m c_m c_{n-m} - B_n c_{n-1}.$$

On using (20) and replacing the terms on the right by their moduli this can certainly be written in the form

$$[A_n + 4(\varphi_0 + \varphi_n)]\, c_n \leq k \cdot \sum_{m=1}^{n-1} \varphi_m c_m c_{n-m} + B_n c_{n-1}.$$

Now if we assume, as in p. 79, that

$$c_n \leq K\lambda^n, \qquad \text{for all values of } n$$

we must have

$$L_n c_n \leq K^2 \lambda^n \cdot k \sum_{m=1}^{n-1} \varphi_m + B_n K \lambda^{n-1}, \tag{22}$$

where $$L_n = A_n + 4(\varphi_0 + \varphi_n)$$

$$= 2n(n-1)(2n-1)(2n-3) - 12,$$

and $$\sum_{m=1}^{n-1} \varphi_m = (n-1)\left(n^3 - 5n^2 + \frac{15}{2}n - 3\right).$$

Since $L_n > 0$ when $n > 2$ we can divide both sides of (22) by L_n and obtain

$$c_n \leq K^2 \lambda^n \cdot \frac{k\,\Sigma\varphi_m}{L_n} + \frac{B_n}{L_n} K\lambda^{n-1}.$$

$\frac{\Sigma\varphi_m}{L_n}$ is an increasiug function of n, and when $n \geq 3$ it is positive and remains less than $\frac{1}{8}$. B_n/L_n diminishes rapidly as n increases, its maximum value when $n \geq 3$ is $\frac{1}{7}$. Hence if we replace both the ratios by $\frac{1}{7}$ it will be true that

$$c_n \leq \frac{k}{7} K^2 \lambda^n + \frac{1}{7} K\lambda^{n-1}, \text{ when } n \geq 3.$$

Now the condition $c_n \leq K\lambda^n$ requires

$$\frac{k}{7} K^2\lambda^n + \frac{1}{7} K\lambda^{n-1} \leq K\lambda^n \quad \therefore \quad \lambda \geq \frac{1}{7 - kK}.$$

Hence $$\lambda = \max.\left(\frac{1}{7 - kK}, 1\right),$$

and $K = \max.\ (c_0, c_1, c_2)$, since $\nu = N = 3$.

So the Taylor series for $g(x)$ converges for

$$|\alpha| < \frac{1}{\max.\left(\frac{1}{7-kK}, 1\right)} = l \quad \text{say,}$$

i.e. $$|\alpha| = \frac{z^2}{4\nu t} < l$$

or $$-\sqrt{4\nu l}\cdot t^{1/2} < z < \sqrt{4\nu l}\cdot t^{1/2}.$$

Hence, since $h = \frac{\nu k}{z}\cdot g(\alpha)$

(a) $h = \infty$ always when $z = 0$, because of the z in the denominator ($c_0 \neq 0$).

(b) When $t = 0$, $g(\alpha)$ converges only on $z = 0$, and diverges everywhere else. Hence $h = \infty$ everywhere.

(c) When $t > 0$, $g(\alpha)$ converges in

$$-\sqrt{4\nu l}\cdot t^{1/2} < z < \sqrt{4\nu l}\cdot t^{1/2},$$

so that the convergence extends through an expanding region. Hence h converges throughout this region except on $z = 0$, where $h = \infty$, ($c_0 \neq 0$).

(d) When $t \to \infty$, z finite, $z^2/4\nu t \to 0$, $\therefore g(\alpha) \to c_0$, a finite value.

Hence $h \to \frac{\nu c_0 k}{z}$.

. . . .

(ii) Solution of

$$\nu h_{zzzz} - h_{zzt} = h h_{zzz}$$

by a different substitution.

The varibles can be separated and ν cancels out if we put

$$h = \nu^{1/2} \cdot f(t) \cdot g(\alpha),$$

where $\alpha = z^2 / 4\nu t$. If we denote again all derivatives by suffixes the equation becomes

$$\frac{\alpha^2}{t^2} f g_4 + \frac{\alpha^2 + 3\alpha}{t^2} f g_3 + \frac{10\alpha + 3}{4t^2} f g_2 + \frac{1}{2t^2} f g_1 - \frac{\alpha}{t} f_1 g_2 - \frac{1}{2t} f_1 g_1$$

$$= fg \left(\frac{\alpha^{3/2}}{t^{3/2}} f g_3 + \frac{3\alpha^{1/2}}{2t^{3/2}} f g_2 \right) \cdot$$

Multiply both sides by t^2/f. Then t disappears from the equation if tf_1/f and $t^{1/2}f$ are constants, which is true if

$$f = bt^{-\frac{1}{2}}, \tag{23}$$

where b is any constant.

Now the equation becomes

$$\alpha^2 g_4 + (\alpha^3 + 3\alpha) g_3 + \left(3\alpha + \frac{3}{4}\right) g_2 + \frac{3}{4} g_1 = bg \left(a^{3/2} g_3 + \frac{3}{2} \alpha^{1/2} g_2 \right) \cdot \tag{24}$$

By the substitution $u = \alpha^{1/2}$ this reduces to

$$g_4 + 2ug_3 + 6g_2 = 2bgg_3, \tag{25}$$

where g is now a function of u, and the suffixes denote differentiation with respect to u.

The equation (25) has a particular solution obtained from $g_2 = 0$, which is

$$g = Au \quad + B,$$

$$g = A\alpha^{1/2} + B,$$

or

$$g = \frac{Az}{2\sqrt{\nu t}} + B,$$

where A, B are constants of integration. This makes

$$h = \frac{A b z}{2 t} + \frac{B b \nu^{1/2}}{t^{1/2}},$$

which corresponds to an irrotational motion.

The equation (25) is exact, and can be integrated once, which furnishes

$$g_3 + 2\, u g_2 + 4\, g_1 - 2\, b g g_2 + b g^2_1 = C_1 \qquad (26)$$

If we assume that $\sum c_n u^n$ is a solution of the equation (25), on substituting in the differential equation we obtain a recurrence formula of the form

$$A_n c_n + B_n c_{n-2} = 2b \sum_{m=4}^{n} \varphi_m c_{m-1} c_{n-m}, \qquad (27)$$

where

$$A_n = n(n-1)(n-2)(n-3), \quad \sum_{m=4}^{n} \varphi_m = \frac{1}{4} n(n-1)(n-2)(n-3),$$

$$B_n = 2(n-1)(n-2)(n-3),$$

$$\varphi_m = (m-1)(m-2)(m-3), \quad \frac{\Sigma \varphi_m}{A_n} = \frac{1}{4} \text{ for all values of } n.$$

Let us apply the preceding method to find the interval of convergence of the Taylor series for $g(u)$. The recurrence relation can be written as

$$A_n c_n \leq 2b \sum_{m=4}^{n} \varphi_m c_{m-1} c_{n-m} + B_n c_{n-2}.$$

Dividing both sides by A_n which is positive when $n \geq 4$, we obtain

$$c_n \leq \frac{2b}{A_n} \sum_{m=4}^{n} \varphi_m c_{m-1} c_{n-m} + (B_n / A_n) c_{n-2},$$

and the same assumptian $c_n \leq K \lambda^n$ furnishes

$$c_n \leq \frac{2b}{A_n} \sum_{m=4}^{n} \varphi_m \cdot K^2 \lambda^{n-1} + \frac{B_n}{A_n} K \lambda^{n-2} \leq K \lambda^n.$$

But $\dfrac{\Sigma\, \varphi_m}{A_n} = \dfrac{1}{4}$ for all values of n, and $\dfrac{B_n}{A_n} = \dfrac{2}{n} \leq \dfrac{1}{2}$ for $n \geq 4$.

Hence $$\frac{b}{2} K\lambda + \frac{1}{2} \leq \lambda^2,$$

or $$2\lambda^2 - bK\lambda - 1 \geq 0.$$

This is true if

$$\lambda \geq \frac{bK + \sqrt{b^2K^2+8}}{4},$$

therefore the best value for λ is given by

$$\lambda = \max. \left(1 ; \frac{bK + \sqrt{b^2K^2+8}}{4}\right),$$

and $K = \max. (c_0, c_1, c_2, c_3)$, since $\nu = N = 4$.

The series converges in the interval of length $2\lambda^{-1}$. i. e. for

$$|u| = |\alpha^{1/2}| < \lambda^{-1}$$

or $$|z| < \frac{\sqrt{4\nu}}{\lambda} t^{1/2}.$$

Therefore, since $h = b\nu^{1/2} \cdot t^{-\frac{1}{2}} \cdot g(\alpha)$

(a) When $t = 0$, $h = \infty$ everywhere,

(b) When $t > 0$, $g(\alpha)$ converges in

$$-\sqrt{4\nu} \cdot \lambda^{-1} t^{1/2} < z < \sqrt{4\nu}\, \lambda^{-1} t^{1/2},$$

so that the convergence extends through an expanding region. Hence, h converges throughout this region.

(c) When $t \to \infty$, z finite, $z^2/4\nu t \to 0$ $\therefore$ $g(\alpha) \to c_0$, a finite value.

Hence $h \to 0$ because of the factor $t^{-\frac{1}{2}}$.

Hence, h represents initially an infinite velocity everywhere, but a region of finite velocity spreads away from the plane $z=0$ as t increases. The boundary of the region of finite velocity is at a distance $\sqrt{4\nu}\,\lambda^{-1} \cdot t^{1/2}$ at time t. Its velocity of propagation is $\sqrt{\nu} \cdot \lambda^{-1} t^{-\frac{1}{2}}$. At time $t = \infty$ the whole fluid has no velocity parallel to OZ.

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