Incompressible flow near a solid boundary

by J. A. STRANG

(Department of Mathematics, Ankara University)

Özet : $y = 0$ sabit sınır yüzeyini geçen sıkıştırılamaz bir akışkanın iki boyutlu hareketini inceleyen Goldstein(*) malûm sınır tabakası (boundary layer) denklemlerini bir ayrılma noktası civarındaki kararlı (steady) harekete tatbik ederek negatif neticeler elde etti ve buradan akışkanın gerek ayrılma noktasından evvelki ve gerek daha sonraki hareketinde bu denklemlerin muteber olamıyacağı kanaatine vardı.

Bu yazıda Goldstein'in başarısızlığının sebepleri incelenmiş ve şu neticelere varılmıştır :

(i) Sınır tabakasına ait kararlı hareket denklemlerini elde etmek için genel hareket denklemlerinin sadeleştirilmesinde yapılan kabuller bir ayrılma noktası civarında doğru değildirler.

(ii) Sınır tabakası denklemlerini elde ederken yapılan $\dfrac{\partial p}{\partial y} = 0$ kabulü sınır üzerinde bir ayrılma noktasının teşekkülünü imkânsızlaştırır.

(iii) Sınır tabakası denklemlerinin çözümünde Blasius ve daha sonraki yazarların kullandığı metot, sadeleştirilmiş denklemlerin sınır şartlarını sağlayan yaklaşık bir çözümünün aynı zamanda esas denklemlerin sınır şartlarını sağlayan yaklaşık bir çözümü olacağı kabulüne dayanır ki bu kabul de doğru değildir.

Keza başarısızlığa başka bir sebep olarak çözümün tamamiyle belirtilmesi için gereken şartların yeter derecede incelenmemiş olması gösterilebilir.

Daha sonraki paragraflarda akım fonksiyonunun, y nin yeter derecede küçük olması halinde, y nin pozitif (tam olması şart değil) kuvvetlerine göre bir seriye açılabileceği kabul edilmiş ve bu çözüm şekli, $y = 0$ sınırı üzerinde verilmiş şartları sağlayan çözümlerin şekli hakkında mümkün olan bilgiyi elde etmek gayesiyle, hem lüzucî ve hem de gayri lüzucî akışkanın hareket denklemlerine uygulanmıştır.

Akışkanın lüzucî olması halinde üs'lerin pozitif tam sayılar olması gerektiği ve

$$\psi = \sum_{n=2}^{\infty} c_n \, y^n / n \, !$$

serisindeki c_2 ve c_3 katsayılarının x ve t nin verilmiş fonksiyonları olması

(*) S. Goldstein, 'On laminar boundary flow near a position of separation' Q. J Pure and App. Math. I. 1948.

halinde çözümün tamamiyle belirli olduğu gösterilmiştir. Çeşitli özel haller meyanında

(i) $\nabla_1^2 \psi = f(\psi)$ denklemini sağlayan çözümler;

(ii) akışkanın fazla lüzucî olması hali;

(iii) enerji kaybının c_n katsayılarındaki T zaman çarpanından ileri gelmesi hali (d_n yalnız x in bir fonksiyonu olmak üzere $c_n = d_n \mathrm{T}^n$ veya $c_n = d_n \mathrm{T}$ gibi) de incelenmiştir.

Lüzucî olmayan akışkan halinde iki cins çözüm elde edilmiş olup bunlardan birisinde tam kuvvetlere münhasır kalınmamıştır Tam kuvvetlere tabi olan

$$\psi = \sum_{n=1}^{\infty} c_n y^n / n!$$

çözümünde c_1 in keyfî olduğu ve daha sonra gelen katsayılardan herbirinin birinci mertebeden lineer bir kısmî diferensiyel denklem ile tâyin edildiği, $c_1 \neq 0$ ise bu diferensiyel denklemin çözümünün genel olarak $\varphi(x, t)$ gibi bir keyfî fonksiyon ihtiva edeceği, $\varphi =$ sabit'in $c_1\, dt = dx$ denkleminin çözümü olduğu, $c_1 \equiv 0$ ise keyfî fonksiyonun yalnız x e tabi olacağı ve hareketin kararlı olması halinde ise onun keyfî bir sabite eşit olacağı gösterilmiştir. Keza $c_1 \equiv 0$ olması halinde c_n katsayısının t nin derecesi en fazla $n-2$ olabilen bir polinom olacağı da gösterilmiştir.

Bir ayrılma noktası civarında yapılan basit yaklaşıklıklarla böyle bir nokta civarında, ayrılma tabakası içindeki çevrilerin karakteri de dahil, hareketin yeter derecede sıhhatle temsil edilebildiği gösterilmiştir.

Nihayet, çözümün şekli hakkında yapılan ilk kabullere tabi olarak akım çizgilerinin bir ayrılma noktası civarındaki şeklinin, hattâ ayrılma noktası yüksek mertebeden bir aykırı nokta olsa bile, approximation metotları ile kolayca incelenebileceği fakat bu noktanın en fazla kolları reel olan çok katlı bir düğüm veya köşe noktası olduğu gösterilmiştir.

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Summary: In an investigation of the two dimensional flow of an incompressible fluid past the fixed boundary $y = 0$ Goldstein obtained negative results by applying the usual boundary layer equations to steady flow near a point of separation, and drew the conclusion that these equations appear to be invalid both upstream and downstream of such a point.

In the present paper the reasons for the failure are examined, and it is shown

(i) that the assumptions made in simplifying the general equations of motion to form the boundary layer equations of steady motion are invalid near a point of separation;

(ii) that the assumption $\partial p / \partial y = 0$ used in forming the boundary layer equations makes a separation point on the boundary impossible;

(iii) that the method used in the solution of the boundary layer equations by Blasius and subsequent writers rests on the assumption that an approximate solution of the simplified equations satisfying the boundary con-

ditions will also be an approximation to a similar solution of the exact equations; and that this assumption is invalid.

It is also suggested that an additional reason for the failure is that the conditions necessary for the complete determination of the solution have been inadequately studied.

In subsequent paragraphs it is assumed that the stream function can be expanded in positive (not cecessarily integral) powers of y when y is sufficiently small, and this form of solution is applied to the general equations of motion both for viscous and for nonviscous fluid, with the object of ascertaining as musch as possible about the form of those solutions which satisfy the given conditions on the boundary $y = 0$.

It is shown that in the case of a viscous fluid the indices must be positive integers, and that the solution is completely determined when the coefficients c_2 and c_3 in the series

$$\psi = \sum_{n=1}^{\infty} c_n \, y^n/n!$$

are known functions of x and t. Various particular types are examined, including

(i) solutions for which $\nabla_1^2 \psi = f(\psi)$;

(ii) highly viscous fluid;

(iii) solutions in which the dissipation of energy is due to a time factor T in the coefficient $c_n = d_n \, T^n$, or $c_n = d_n \, T$, where d_n depends only on x.

In the case of nonviscous fluid two types of solution are obtained, one of which is not restricted to integral powers; it is shown that in the solution depending on integrpl powers,

$$\psi = \sum_{n=1}^{\infty} c_n \, y^n/n!$$

the coefficient c_1 is arbitrary, but that each of the succeeding coefficients is determined by a linear partial differential equation of the first order whose solution when $c_1 \neq 0$ will in general contain an arbitrary function of $\varphi(x, t)$, where $\varphi = a$ constant is the solution of $c_1 \, dt = dx$; that if $c_1 \equiv 0$ the arbitrary function depends only on x; and that if the motion is steady it is an arbitrary constant. It is also shown that wher $c_1 \equiv 0$ coefficient c_n is a polynomial in t of degree $n-2$ at most.

By means of simple approximations near a point of separation it is shown that the motion in the vicinity of such a point can be adequately represented, including the behaviour of vortices in the separation layer.

Finally it is shown that subject to the initial assumptions regarding the form of the solution, the form of the streem lines near a point of separation can be easily examined by methods of approximation even when the point of separation is a singularity of higher order; and that it is at most a multiple point with real nodal or cuspidal branches.

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1. The equations of two dimensional flow of a viscous incompressible fluid are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\nabla_1^2 u, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,\nabla_1^2 v, \\ u = \frac{\partial \psi}{\partial y}, \quad v &= -\frac{\partial \psi}{\partial x}. \end{aligned}\right\} \qquad 1.1$$

External forces are presumed to be derivable from a potential function and included in p.

The usual procedure in dealing with a boundary layer is to limit the discussion to steady flow and to omit certain terms with a view to simplification of the solution. Thus Prandtl (1904) and Blasius (1907) reduced the first two equations to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

by regarding derivations of p as negliable near the boundary $y = 0$, and $\partial u/\partial x$ as small compared with $\partial u/\partial y$ within the boundary layer.

More recently S. Goldstein ('On laminar boundary flow near a position of separation.' Q. J. Pure and App. Math. I. 1948) assumes that the equations 1.1 may be replaced by

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \\ u = \frac{\partial \psi}{\partial y}, \quad v &= -\frac{\partial \psi}{\partial x}, \\ -\frac{1}{\rho}\frac{\partial p}{\partial x} &= U\frac{\partial U}{\partial x}, \end{aligned}$$

where U is the main stream velocity outside the boundary layer, and seeks to investigate the existence or non-existence of a singularity at a point of separation on the boundary $y = 0$. But his conclusion (p. 51, end para. 1) is

"The work described may be summed up by saying that it throws doubt on the validity of the boundary layer equations at and near separation on the upstream side, and also downstream of separation; inferences from these equations in these regions, which are fairly common in the literature, are therefore also in doubt; . . . . . „

In these circumstances it is evidently advisable either to amend the approximate boundary layer equations or to discard them altogether and return to the exact equations. The object of this paper is to elicit information from the exact equations on the problem of flow near a straight solid boundary, in particular near a point of separation.

2. The reasons for the failure of the approximate equations to provide the required solution lie partly in the assumptions on which the simplification of the equations of motion is based, and partly in the usual method of solution

It has been assumed that $\frac{\partial u}{\partial x} \Big/ \frac{\partial u}{\partial y}$ is small in the boundary layer.

This assumption is certainly questinable on the downstream side of a separation point when a slow back flow occurs near the boundary, for if separation occurs at S the conditions imply the existence of a locus SB on which $u = 0$; and since $u = 0$ both on SA and on SB, is small and negative in the included region and may be assumed to have a continuous derivative, it is evident that $\partial u / \partial y = 0$ on some locus situated between SA and SB and passing through S. It is curious that although the condition $\partial u / \partial y = 0$ has been recognised as a condition to be satisfied at a point of separation, the existence of the locus seems to have escaped notice, since it renders the assumption invalid in the region ASB.

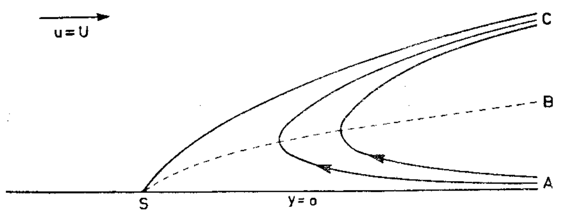

Fig. 1

There are however at least two other features of the usual method of attack which appear to have escaped notice, each of which is in itself sufficient to account for the failure, One of these is the fact, which will be established in para. 6, that if $\partial p/\partial y = 0$ everywhere, as is assumed in the approximate equations, and if the solution sought is of the form

$$\psi = \sum_{n=2}^{\infty} c_n y^n / n! ,$$

then *there can be no point of separation.* The condition $\partial p/\partial y = 0$ excludes it. This is a consequence of the exact equations. So the omission of $\partial p/\partial y$ from the approximate equations destrorys the basis of the problem.

The second point is that the method of Blasius, which appears to have been generally followed, depends on the tacit assumption that an approximate solution of the approximate equations, if it satisfies the given boundary conditions, will also be an approximation to a similar solution of the exact equations of motion. The assumption is unjustified.

If with Blasius we write

$$\eta = \tfrac{1}{2}(u_1/\nu x)^{1/2} y \ , \quad \psi = (\nu u_1 x)^{1/2} f(\eta) ,$$

where $f(\eta)$ depends only on η, and insert in the exact equations 1.1 for the case of steady motion, the resulting equation to determine f can be satisfied only if f satisfies simultaneously the three equations

$$f'''' + ff''' + f'f'' = 0 ,$$

$$\eta^4 f'''' + 14\eta^3 f''' + 45\eta^2 f'' + 15\eta f' - 15f = 0 ,$$

and $$2\eta^3 f'''' + 10\eta f''' + 6f' - 11\eta^2 f'f'' + 5\eta ff'' + \eta^2 ff''' - 2\eta^3 f'f''' - 2\eta^3 f''^2 - 3\eta f'^2 + 3ff' = 0 ,$$

where primes denote differentiation with respect to η.

The first of these corresponds to the equation

$$f''' + ff'' = 0$$

which is solved by Blasius in a infinite series of positive integral powers on η, beginning with η^2. His series is obviously not a solution of the second equation, whose general solution is

$$f(\eta) = c_1\eta + c_2\eta^{-1} + c_3\eta^{-3} + c_4\eta^{-5} ,$$

where the coefficients are arbitrary. The first equation requires either

$$f = c_1\eta \quad \text{or} \quad f = 3\eta^{-1} ,$$

and each of these satisfies the third equation.

There are therefore just two exact solutions common to all three equations, giving respectively

$$\psi = c_1 u_1 y/2 \quad \text{and} \quad \psi = 6\nu x y^{-1},$$

neither of which is relevant to the flow of a viscous fluid along the boundary $y = 0$. The form of solution employed by Blasius is therefore in no sense an approximation to an exact solution of similar type, and it is not surprising that difficullty was experienced near the point of separation, or for that matter elsewhere.

It is probable also that some of the complexity encountered is due to the fact that the information required to define the solution has not been sufficiently studied.

The following paragraphs are concerned chiefly with formal expression which satisfy the exact equations of motion. The question of convergence for instance requires further investigation and offers considerable difficulty. But various types of solution are examined both for viscous and nonviscous motion in presence of the boundary $y = 0$, and it is shown that approximate solutions can be obtained which suggest adequately the nature of the flow near a point of separation, and the existence and behaviour of vortices near the boundary. The solutions are obtained directly from the general equations.

3. The general equation.

The equation to determine ψ obtained by eliminating u, v, p from 1.1 is

$$\left(\frac{\partial}{\partial t}\zeta - \nu \nabla_1^2\right)\zeta + \psi_y \zeta_x - \psi_x \zeta_y = 0 \qquad 3.1$$

where the vorticity $\zeta = \nabla_1^2 \psi$; it is assumed that when y is sufficiently small ψ is expressible as a convergent series of positive powers of y (not necessarily integral) whose coefficients are functions of x and t, and in steady motion depend only on x. If the fluid is viscous it is assumed that when $y = 0$, $\psi = \psi_x = \psi_y = 0$. If it is nonviscous the last condition is omitted, but may nevertheless be satisfied.

We assume therefore

$$\psi = \sum_{r=1}^{\infty} c_r y^{m_r} \qquad 3.2$$

where if $\nu = 0$, $1 \leq m_1 < m_2 < \ldots\ldots$, and if $\nu \neq 0$, $1 < m_1 < m_2 < \ldots\ldots$, since $u = \psi_y$ must be finite when $y = 0$.

When $\nu \neq 0$ there will occur on substitution in the equation 3.1 terms of the form

$$-\nu m_r(m_r - 1)(m_2 - 2)(m_r - 3)c_r y^{m_r - 4},$$

and the lowest power of y occurring is $y^{m_1 - 4}$. It occurs only once, so we must have

$$m_1(m_1 - 1)(m_1 - 2)(m_1 - 3)c_1 = 0.$$

Since $c_1 \neq 0$ and $m_1 > 1$, we must have either $m_1 = 2$ or $m_1 = 3$. By choosing $m_1 = 2$ and $m_2 = 3$ we can dispose of the corresponding terms without imposing conditions on the coefficients c_2 and c_3. But the term of index $m_3 - 4$ cannot be eliminated in this way It must be combined with the term containing y^0, so that m_3 must be 4, and similarly for succeeding indices It follows that when $\nu \neq 0$ the only possible form of the expansion is

$$\psi = \sum_{n=2}^{\infty} c_n y^n / n! \qquad 3{\cdot}3$$

where the coefficients c_n are functions of x and t, and all indices are positive integers.

The series 3.3 is convergent if

$$\lim_{n \to \infty} \left| \frac{c_n y}{n c_{n-1}} \right| < 1,$$

i. e. if

$$y < \lim_{n \to \infty} \left| n c_{n-1} / c_n \right|.$$

This defines the region of convergence at any time t, and its character, as will be seen in para. 4, depends only on the values of c_2 and c_3; but the dependence is not simple.

When $\nu = 0$ the circumstances are different. The lowest occurring indices after substitution are $m_1 - 2$ and $2m_1 - 3$, where $m_1 \nleq 1$.

If $m_1 > 1$ the former is the lower, since

$$2m_1 - 3 - (m_1 - 2) = m_1 - 1 > 0,$$

so the coefficient of $y^{m_1 - 2}$ must vanish, i. e. $m_1(m_2 - 1)c_1 = 0$, contradiction since $c_1 \neq 0$. Hence $m_1 = 1$, and

$$\psi = c_1 y + \sum_{r=2}^{\infty} c_r y^{m_r} .$$

We now find on substitution that there are two alternatives. The first leads to a solution in which $m_r = r$, that is we obtain the form

$$\psi = \sum_{n=1}^{\infty} c_n y^n / n! \qquad 3.4$$

where the indices are the positive integers. But the equation of motion is also satisfied if

$$c''_1 = 0; \text{ and } c'_r = 0, \quad \frac{1}{m_r - 2} \frac{1}{c_r} \frac{\partial c_r}{\partial t} = c'_1, \quad (r > 1);$$

so that if we take

$$c_1 = \frac{xf'}{f} + g, \quad c_r = d_r f^{m_r - 2}, \quad (r > 1)$$

where f and g are arbitrary functions of t, and d_r is a constant, there results the solution

$$\psi = (xf'/f + g)y + f^{-2} \sum_{r=2}^{\infty} d_r (yf)^{m_r}, \qquad 3.5$$

where the indices m_r are still arbitrary.

The series 3.5 is convergent if

$$| yf | < \lim_{\nu \to \infty} \left| \left(\frac{d_{r-1}}{d_r} \right)^{1/m} \right|, \quad \text{where } m = m_r - m_{r-1},$$

so that on the boundary of the region of convergence y is independent of x, but depends in general on t.

The solution just obtained is evidently a particular case of

$$\psi = (xf'/f + g)y + f^{-2} h(z), \qquad 3.6$$

where $z = yf(t)$, f and g are arbitrary functions of t, and $h(z)$ an arbitrary function of z subject to the conditions $h(0) = h'(0) = 0$; and it may be immediately verified that this expression satisfies the equation 3.1 when $\nu = 0$.

It furnishes

$$u = \psi_y = \frac{xf'}{f} + g + f^{-1} h'(z), \quad v = -\psi_x = -\frac{yf'}{f}, \quad \zeta = h(z),$$

and

$$2f^3(p_0 - p)/\rho = ff'' x^2 + (2f'^2 - ff'')y^2 + 2f(fg)' x ,$$

where p_0 is the pressure at the origin. The pressure does not depend on h.

There is a point of separation at $x = -fg/f'$ which is fixed at the origin if $g = 0$. Whether there is or is not a singularity at the point of separation depends entirely on the character of the arbitrary function $h(z)$. No vortex can occur in the flow since $\psi_{xx} = 0$, while at a vortex the condition

$$\Delta = \psi_{xy}^2 - \psi_{xx}\psi_{yy} < 0$$

must be satisfied. If $f' = 0$ there is no point of separation.

Since $\zeta = h''(z)$ the flow is irrotational if $h'' = 0$, otherwise rotational.

If $$2f'^2 = ff'', \quad f = (at + b)^{-1},$$
the equipressure lines are straight lines parallel to the y axis, otherwise they are concentric and coaxial conics with axes parallel to the axes of coordinates, the common centre being at $[-(fg)'/f'', 0]$. They are circles if $f'^2 = ff''$.

From the point of view of vortices another solution of some interest which still contains an arbitrary function is

$$\psi = y[a(x-f)^2 + b(x-f) + f' + c] + dy^2 - ay^3/3, \quad 3.7$$

where f is an arbitrary function of t, and a, b, c, d are constans.

This solution is of course included id 3.4, but is of interest as a polynomial in y obtained directly from the general form. Since

$$u = \psi_y = a(x-f)^2 + b(x-f) + f' + c + 2dy - ay^2,$$
$$v = -\psi_x = -y[2a(x-f) + b],$$
$$\zeta = 2d, \quad \text{and} \quad \Delta = 4a^2(x-f)^2 - 4ay(d - ay);$$

it is clear that there can be no vortex if $ad \leq 0$. At a vortex $\psi_x = \psi_y = 0$, hence either $y = 0$, which makes $\Delta \geq 0$, or $x - f = -b/2a$. Substituting this in $\psi_y = 0$ we find

$$4a^2(y - d/a)^2 = 4af' - b^2 + 4d^2$$
and
$$\Delta = 4af'.$$

If therefore $$b^2 - 4d^2 < 4af' < 0$$
there will be two real vortices, otherwise not.

There are two points of separation, given by $y = 0$ and

$$a(x-f)^2 + b(x-f) + f' + c = 0,$$

which are real if $\quad b^2 - 4ac - 4af' > 0.$

The two conditions may be written in the form

$$4ac - b^2 < -4af' < 4d^2 - b^2,$$

in which the middle term must be positive Both conditions, or one, or neither may be satisfied. If $4ac$ is sufficiently large and positive the points of separation are not real and if $4d^2 < b^2$ the vortices are not real.

4. The recurrence relations. Viscous flow.

On substituting in 3.1 the formal solution 3.3 it is found that in viscous motion

$$c_4 = \frac{1}{\nu}\frac{\partial c_2}{\partial t} - 2c_2'',$$

$$c_5 = \frac{1}{\nu}\left(\frac{\partial c_3}{\partial t} + c_2 c_2'\right) - 2c_3'',$$

$$c_6 = \frac{1}{\nu^2}\frac{\partial^2 c_2}{\partial t^2} - \frac{1}{\nu}\left(3\frac{\partial c_2''}{\partial t} - 2c_2 c_3'\right) + 3c_2'''',$$

$$c_7 = \frac{1}{\nu^2}\left(\frac{\partial^2 c_3}{\partial t^2} + 4c_2\frac{\partial c_2'}{\partial t} - c_2'\frac{\partial c_2}{\partial t}\right) - \frac{1}{\nu}\left(3\frac{\partial c_3''}{\partial t} + 5c_2 c_2''' \right.$$

$$\left. + 5c_2' c_2'' - 2c_3 c_3'\right) + 3c_3'''',$$

. . . .

the general recurrence relation after the first two terms being

$$\frac{\partial}{\partial t}(c_n'' + c_{n+2}) - \nu(c_n'''' + 2c_{n+2}'' + c_{n+4}) + [c_{n+1}c_2' + \tbinom{n}{1}c_n c_3'$$

$$+ \tbinom{n}{2}c_{n-1}(c_2''' + c_4') + \ldots + \tbinom{n}{n-1}c_2(c_{n-1}''' + c_{n+1}')]$$

$$- [c_n' c_3 + \tbinom{n}{1}c_{n-1}'(c_2'' + c_4) + \ldots + \tbinom{n}{n-2}c_2'(c_{n-1}'' + c_{n+1})] = 0; \quad 4.1$$

where primes denote differentiation with respect to x, and $\binom{n}{r}$ is a binomial coefficient.

The solution is determinate when c_2 and c_3 are known, since the recurrence relations then determine the remaining coefficients uniquely; that is, the completion of the solution requires knowledge of the values of $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ when $y = 0$ for all x and t, or the equivalent of this information.

5. Nonviscous flow.

It is evident that after obtaining a solution in the above form we cannot deduce from it a solution for a nonviscous fluid by making ν assume the value zero, or by making it approach zero as a limiting value. The two types of flow are essentially distinct. When the fluid is nonviscous the equations to be satisfied are quite different in character.

Inserting 3.4 in 3.1 with $\nu = 0$ we find that they are

$$\frac{\partial c_2}{\partial t} + c_1 c_2' = 0 \,,$$

$$\frac{\partial}{\partial t}(c_1'' + c_3) + c_1(c_1''' + c_3') + c_2 c_2' - c_1'(c_1'' + c_3) = 0 \,,$$

$$\frac{\partial}{\partial t}(c_2'' + c_4) + c_1(c_2''' + c_4') + 2\,c_2(c_1''' + c_3') + c_3 c_2'$$
$$- 2\,c_1'(c_2'' + c_4) - c_2'(c_1'' + c_3) = 0 \,,$$

. . . .

$$\frac{\partial}{\partial t}(c_n'' + c_{n+2}) + \left[c_1(c_n''' + c_{n+2}') + \ldots + \binom{n}{n-2}c_{n-1}(c_2''' + c_4')\right.$$
$$+ \left.\binom{n}{n-1}c_n(c_1''' + c_3') + c_{n+1}c_2'\right] - \left[\binom{n}{1}c_1'(c_n' + c_{n+2})\right.$$
$$+ \ldots + \left.\binom{n}{n-1}c_{n-1}'(c_2'' + c_4) + c_n'(c_1'' + c_3)\right] = 0 \,. \qquad 5.1$$

The coefficient c_1 is arbitrary, but each of the succeeding coefficients is now determined by means of a linear partial differential equation of the first order whose general form, if we write $z = c_{n+2} + c_n''$, is

$$\frac{\partial z}{\partial t} + c_1 z' - n c_1' z = f(c_1 \,, \, \ldots \,, c_{n+1}) \,,$$

where f is a determinate function of the preceding coefficients and their derivatives. Every coefficient from c_2 onwards therefore involves an arbitrary function except in the case of steady motion, when only an arbitrary constant is involved. If $c_1 = 0$, in which case $u = 0$ when $y = 0$, the arbitrary functions depend only on x and are addivite, and c_2 does not depend on t.

Solutions of this kind, in which a nonviscous fluid has no tangential velocity at the boundary, are quite distinct from similar solutions for a viscous fluid, and their form is of considerable interest. Since $c_1 = 0$ the recurrence relations are now

$$\left.\begin{aligned} &\frac{\partial c_2}{\partial t} = 0 , \\ &\frac{\partial c_3}{\partial t} = - c_2 c_2' , \\ &\frac{\partial}{\partial t}(c_2'' + c_4) = - 2 c_2 c_3' , \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \qquad 5.2$$

so that c_2 is independent of t, and apart from particular cases arising when $c_2' = 0$, etc., c_3 is a linear function of t, c_4 a quadratic function and so on. The general recurrence relation shows that if all coefficients up to c_{n-1} are of this form so is c_n. Hence c_n *is in general a polynomial in* t *of degree* $n-2$ *at most*, and the term which is independent of t is an arbitrary function of x. These arbitrary functions are determined by the initial value of ψ, i.e. they are the initial values of $\partial^r u/\partial y^r$ $(r = 1, 2, \ldots)$ on the boundary $y = 0$. When these are known the solution can be completed by means of the recurrence relations.

When $c_1 = f_1(x, t) \neq 0$ is a known function of x and t the first equation of 5.1 , namely

$$\frac{\partial c_2}{\partial t} + c_1 \frac{\partial c_2}{\partial x} = 0 ,$$

has for its solution $c_2 = f_2(\varphi)$, where $\varphi(x, t) = k$ is an integral of the subsidiary equation

$$\frac{dt}{1} = \frac{dx}{c_1} ,$$

and $f_2(\varphi)$ is an arbitrary function of φ.

Let $z = c_3 + c_1''$. The second equation of 5.1 is

$$\frac{\partial z}{\partial t} + c_1 \frac{\partial z}{\partial x} = c_1' z - c_1 c_2' ,$$

whose subsidiary equations are

$$\frac{dt}{1} = \frac{dx}{c_1} = \frac{dz}{c_1' z - c_2 c_2'} ,$$

the second of which may be written

$$(c_1 \cdot dz \quad c_1' z \cdot dx)/ c_1^2 = - c_1^{-2} c_2 c_2' \cdot dx$$

since $c_1 \neq 0$. If the right hand side contains t it may be eliminated by using the relation $\varphi(x, t) = k$ obtained from the first subsidiary equation. Integrating with respect to x we find

$$z/c_1 = f_3 - \int c_1^{-2} c_2 c_2' \cdot dx$$

where f_3 is a constant of integration. After integratinn k is replaced by $\varphi(x, t)$, and the constant f_3 by an arbitrary function $f_3(\varphi)$. The coefficient c_3 is then given by

$$c_3 = -c_1'' + c_1 f_3(\varphi) - c_1 \int c_1^{-2} c_2 c_2' \cdot dx\,.$$

Succeeding coefficients may be evaluated in the same manner. The arbitrary functions in the solution are therefore arbitrary functions of $\varphi(x, t)$, which is determined by c_1. How these arbitrary elements are to be determined in any particular problem is not obvious. It is clear that the motion described cannot in general be steady, since φ must depend on t even if c_1 does not; but of course in steady motion the time derivatives are absent from the recurrence relations, and no difficulty arises. The great variety of possible motions is sufficiently obvious from the infinite set of arbitrary functions available for determination.

If c_1 is a constant and the motion is steady, the first equation of 5.1 shows that $c_2' = 0$, and the succeeding equations that $c_n' = 0$, so that ψ is a function of y only.

If c_1 is a function of x only and the motion is steady the first equation requires $c_2' = 0$. The second and subsequent equations determine c_3 etc. as functions of x with in each case a constant of integration which multiplies a power of c_1.

6. Pressure distribution.

In some boundary layer discussions the attempt is made to render the solution more precise by imposing conditions on the pressure distribution, for instance by assuming that $p = a + bx$.

If we express the pressure derivatives in terms of the coefficients c_n in the nonviscous case the result is

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \left(\frac{\partial c_1}{\partial t} + c_1 c_1'\right) + \left(\frac{\partial c_2}{\partial t} + c_1 c_2'\right) y + \dots\,,$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\left(\frac{\partial c_1'}{\partial t} + c_1 c_1'' - c_1'^2\right) y + \dots\,,$$

which is better suited to the determination of the pressure when the coefficients are known than to the converse process, except perhaps when the motion is steady, or when $c_1 = 0$.

In viscous motion

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\nu c_3 + \left[\frac{\partial c_2}{\partial t} - \nu(c_2'' + c_4)\right]y + \dots,$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \nu c_2' + \nu c_3' y - \left[\frac{\partial c_2'}{\partial t} - \nu(c_2''' + c_4')\right]y^2/2! + \dots,$$

so that the values of the pressure derivatives on the boundary are sufficient to determine c_2' and c_3. In particular if on the boundary $\partial p/\partial x$ is a function of t only and $\partial p/\partial y = 0$ neither c_2 nor c_3 can depend on x, and there can be no point of separation on the boundary. If the pressure derivatives are functions of t only on the boundary, c_3 depends only on t and c_2 is at most linear in x. There is in general one (mobile) point of separation OX, but $\psi_{xx} = 0$, and there can be no vortex.

Finally, if $\partial p/\partial y = 0$. or is assumed to be zero, everywhere it follows that $c_2' = c_3' = 0$, and no point of separation on the boundary can exist. This establishes the statement made in the second paragraph regarding the omission of $\partial p/\partial y$ from the approximate equations of motion.

There is in this case a simple expression for the stream function which is of interest in itself, and which is also reached from several different points of view. The equation 3.1 admits of solutions in which the quadratic terms as a group vanish separately, in which case the same must be true of the quadratic terms in each equation of 4.1. Using this fact and the linear terms we obtain $c_2' = c_3' = 0$,

$$c_4 = \frac{1}{\nu}\frac{\partial c_2}{\partial t}, \quad c_5 = \frac{1}{\nu}\frac{\partial c_3}{\partial t}, \dots$$

and
$$\psi = \sum_{n=1}^{\infty}\left(\frac{1}{\nu}\frac{\partial}{\partial t}\right)^{n-1}\left[c_2\frac{y^{2n}}{(2n)!} + c_3\frac{y^{2n+1}}{(2n+1)!}\right], \qquad 6.1$$

where c_2 and c_3 depend only on t.

Another instance of the occurrence of this type of solution is met if we inquire whether there exist, in viscous motion, solutions in which the dissipation of energy is associated with the presence of exponential or other factors involving only the time and the viscosity; and in either viscous or nonviscous motion whether the convergence of the solution may not depend on such factors The result of the inquiry is of considerable interest.

Assume $\nu \neq 0$, $c_n = d_n T^n$, where d_n depends only on x and T only on t.

Substituting in 4 1, and determining T so that the first equation is independent of t, we find that if $d_2 \neq 0$ we must have $d_2'' = 0$ and T'/T^3 constant, so that we may take $T = (at+b)^{-1/2}$.

The second equation requires $d_2' = 0$ and $d_3'' = 0$, and the third $d_3' = 0$. But the conditions $d_2' = d_3' = 0$ show that ψ is independent of x, so that the solution is a particular case of 6.1. The general recurrence relation furnishes

$$d_n = -(n-2)\frac{a}{2\nu} d_{n-2},$$

and hence

$$d_{2n} = (-1)^{n-1}(n-1)!\,(a/\nu)^{n-1} d_2,$$
$$d_{2n+1} = (-1)^{n-1}(2n-1)(2n-3)\,\ldots\,5.3\,(a/2\nu)^{n-1} d_3.$$

If we modify d_2 and d_3 slightly and write $z^2 = y^2/\nu(at+b)$ we can without loss take $a = 1$, and the solution may be written

$$\psi = d_2 \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(n-1)!}{(2n)!} z^{2n} + d_3 \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(2n-1)\cdots 3.1}{(2n+1)!\,2^n} z^{2n+1} \qquad 6.2$$

If b is positive the series converge for all positive values of y and t, and the motion dies away as t approaches infinity.

If b is negative the solution ceases to be valid when $t = -b$, but is valid both before and after that instant.

Reference to preceding expressions or to equations of motion shows that $\partial p/\partial y = 0$ everywhere, while p and $\partial p/\partial x$ become infinite when $t = -b$ at all points of the fluid. *The conditions suggest an instantaneous state of shock, forming the transition from an unstable to a stable condition.*

If $\nu \neq 0$ and we assume $c_n = d_n T$ a similar process leads to the solution

$$\psi = \exp(\nu k^2 t)\,[d_2(\cosh ky - 1) + d_3 k^{-1}(\sinh ky - ky)], \qquad 6.3$$

where d_2, d_3 and k are constants, and k may be imaginary.

When $\nu = 0$ it may be shown without difficulty that comparable solutions exist. For instance

(1) if $c_1 = x$, $c_n = 2^{2-n} x \exp[-(n-1)t]$ $(n > 1)$

the recurrence relations are satisfied, and

$$\psi = 4x\, e^t \left[\exp(\tfrac{1}{2}\, ye^{-t}) - 1 - \tfrac{1}{4}\, ye^{-t}\right]; \qquad 6.4$$

(2) if $c_1 = 0$, $c_n = (-1)^n x t^{n-2}$ $(n > 1)$

we obtain $$\psi = xt^{-2}[\exp(-yt) - 1 + yt]; \qquad 6.5$$

both of which are easily verified by direct substitution.

But the presence of arbitrary functions in the coefficients makes it extremely diffucult to define the character of the solution adequately by such a restriction as $c_n = d_n T^n$. The possibilites are somewhat too extensive for simple general conclusions of this kind.

7. Steady nonviscous motion.

If $\nu = 0$ and the quadratic terms are separately equated to zero the solution consists of those motions for which

$$\frac{\partial \zeta}{\partial t} = 0 \quad \text{and} \quad \zeta = \nabla_1^2 \psi = f(\psi),$$

where $f(\psi)$ may be any function of ψ. The first equation requires $\dfrac{\partial f}{\partial \psi}\dfrac{\partial \psi}{\partial t} = 0$, so that either f is constant or the motion is steady, if we assume that f does not contain t explicity.

When f is constant we can solve the equation

$$\zeta = \nabla_1^2 \psi = a \text{ constant} \qquad 7.1$$

generally. Its solution is

$$\psi = F(x + iy) + G(x - iy) + \tfrac{1}{2}\, \zeta y^2,$$

in which F and G are arbitrary functions of the arguments.

The conditions $\psi = \psi_x = 0$ when $y = 0$ are satisfied if

$$\psi = \frac{i}{2}[f(x + iy) - f(x - iy)] + \tfrac{1}{2}\, \zeta\, y^2, \qquad 7.2$$

where the function f is arbitrary, and the multiplier $i/2$ is inserted to keep ψ real.

Assume that $f(x + iy)$ is expansible by Taylor's theorem in powers of y when y is small, and let f_n denote the n^{th} derivative of $f(x)$ with respect to x. Then

$$\psi = yf_1 + \tfrac{1}{2}\zeta y^2 + \sum_{n=1}^{\infty}(-1)^n f_{2n+1}\, y^{2n+1}/(2n+1)! \qquad 7.3$$

$$= \tfrac{1}{2}\zeta y^2 + \sin(y\mathrm{D})\, f(x, t), \qquad 7.4$$

where $f(x,t)$ is an arbitrary function of x and t, and $\mathrm{D} = d/dx$.

This generalises 3.7. It is a polynomial in y if f is a polynomial in x, and can be conveniently used to furnish motions in which points of separation and vortices occur.

Consider as an example $f = -ax + x^{2n}/2n!$. The stream lines are

$$\psi = y(-ax + x^{2n}/(2n)!) + \tfrac{1}{2}\zeta y^2 - \frac{x^{2n-2}y^3}{(2n-2)!\,3!}$$
$$+ \frac{x^{2n-4}y^5}{(2n-4)!\,5!} \cdots + \frac{(-1)^n y^{2n+1}}{0!\,(2n+1)!}.$$

From the second component of the stream line $\psi = 0$, the first being $y = 0$, it appears that there are just two separation points on the boundary, namely at the origin and at

$$x = [(2n)!\, a]^{1/(2n-1)}.$$

The approximate form of the second component near the origin is given by

$$y = 2ax/\zeta - x^{2n}/(2n)!,$$

if a/ζ is small, so that the curve lies below its tangent, which if a and ζ are positive makes a small positive angle with OX.

From 7.2

$$\Delta = \psi_{xy}^2 - \psi_{xx}\psi_{yy} = f''(x+iy) f''(x-iy)$$
$$- \frac{i}{2}\zeta[f''(x+iy) - f''(x-iy)],$$

and using polar coordinates this becomes

$$\Delta = [2n-2)!]^{-2} r^{2n-2}[r^{2n-2} + (2n-2)!\,\zeta \sin(2n-2)\theta].$$

At a vortex $\Delta < 0$, so $\sin(2n-2)\theta$ must be negative, and if vortices exist they must lie inside the circle

$$r = [(2n-2)!\,\zeta]^{1/(2n-2)}.$$

When $n = 1$

$$\psi = y(-ax + x^2/2) + \tfrac{1}{2}\zeta y^2 - y^3/6.$$

The second component of the stream line ψ 0 is the hyperbola

$$y^2 - 3x^2 + 6ax - 3\zeta y = 0.$$

The points of separation are the origin and $(2a, 0)$, and between the hyperbola and the sector of OX terminated by the points of separation there is a closed region entirely bounded by the stream line $\psi = 0$. This closed region contains a vortex, the only vortex in the flow.

A singular point on the stream lines must satisfy the equations $\psi_x = \psi_y = 0$, i.e

$$y(x - a) = 0,$$
$$-ax + x^2/2 + \zeta y - y^2/2 = 0.$$

Of the four points determined by these equations two are the points of separation on the boundary; the other two are

$$x = a, \quad y = \zeta \pm (\zeta^2 - a^2)^{1/2}.$$

If we take the upper sign $\Delta > 0$, and the point is a stagnation point at which the corresponding stream line has a double point with real tangents. But if we take the lower sign we find that $\Delta < 0$, so that the singularity is an isolated singular point, i. e. a vortex, and it is easily shown to be situated in the closed region mentioned above.

Although ζ does not depend on t we may of course regard a as an arbitrary function of t. If a increases with t the separation point at the origin remains fixed, while the vortex and the other separation point move away from it until $a = \zeta$.

When $\partial f/\partial\psi \neq 0$ the motion must be steady, but its form will of course depand on the particular form of $f(\psi)$. One simple solution is obtained if we put

$$c_{2n} = 0, \quad c_{2n+1} = (1 - D^2)^n c_1,$$

where c_1 is an arbitrary function of x only; so

$$\psi = c_1 y + (1 - D^2) c_1 y^3/3! + \cdots + (1 - D^2)^n c_1 y^{2n+1}/(2n+1)! + \cdots \qquad 7.5$$

if convergent furnishes a group of steady motions which can be adjusted by suitable choice of c_1 to give points of separation and vortices, and evidently includes polynomial solutions in y. For the members of this group

$$\zeta = \nabla_1^2 \psi = \psi.$$

Solutions can be constructed for other forms of $f(\psi)$, but are of less interest.

8. Highly viscous fluid.

A particular case of some interest occurs when the kinematic viscosity is relatively large, since the equation of motion becomes linear if regard ν as infinite, namely

$$\nabla_1^2 \zeta = \nabla_1^2 \cdot \nabla_1^2 \psi = 0 \qquad 8.1$$

If as usual we assume the solution to be of the form

$$\psi = \sum_{n=2}^{\infty} c_n y^n / n!$$

we find on substitution

$$(2c_2'' + c_4) + (2c_3'' + c_5) y + \sum_{n=2}^{\infty} (c_n'''' + 2c_{n+2}'' + c_{n+4}) y^n / n! = 0 .$$

Equating the coefficients to zero and solving we find

$$c_{2n} = (-1)^{n-1} n \, D^{2n-2} c_2 ,$$
$$c_{2n+1} = (-1)^{n-1} n \, D^{2n-2} c_3 ,$$

where $D = d/dx$ as usual, and

$$\psi = (y^2/2! - 2y^4 D^2/4! + 3y^6 D^4/6! \ldots) c_2 + (y^3/3! - 2y^5 D^2/5! \ldots) c_3 \qquad 8.2$$

in which c_2 and c_3 are arbitrary functions of x and t. The solutions are polynomials in y if c_2 and c_3 are polynomials in x.

A finite symbolic form can be given to the result by writing $c_2 = Da$, $c_3 = Db$, when it may be written

$$\psi = \tfrac{1}{2} y \sin(yD) a + \tfrac{1}{2} \int_0^y y \sin(yD) b . dy \qquad 8.3$$

This of course assumes that the expansion is in integral ascending powers of y, and it is possible to solve without this assumption by starting from the general solution of 8.1, namely

$$\psi = f_1(x+iy) + x f_2(x+iy) + g_1(x-iy) + x g_2(x-iy) ,$$

where f_1, f_2, g_1, g_2 are arbitrary functions of the arguments. If primes denote differentiation with respect to the argument

$$\psi_x = f_1' + f_2 + x f_2' + g_1' + x g_2' ,$$
$$\psi_y = i \, [f_1' + x f_2' - g_1' - x g_2'] ,$$

and the boundary conditions $\psi = \psi_x = \psi_y = 0$ require

$$f_1 + g_1 + x(f_2 + g_2) = 0 \ ,$$
$$f_1' + g_1' + f_2 + g_2 + x(f_2' + g_2') = 0 \ ,$$
$$f_1' - g_1' + x(f_2' - g_2') = 0 \ ,$$

where the argument is now x in every case. All three equations are satisfied if

$$g_1 = 2x(f_1' + xf_2') - f_1 \ ,$$
$$g_2 = -2(f_1' + xf_2') - f_2 \ .$$

The required solution is therefore

$$\psi = f_1(x+iy) - f_1(x-iy) + x\,[f_2(x+iy) - f_2(x-iy)] - 2iy\,[f_1'(x-iy) + (x-iy)\,f_2'(x-iy)] \qquad 8.4$$

This includes 8.2 if we take

$$c_2 = -4D\,(f_1' + xf_2') \ ,$$
$$c_3 = 4iD^3(f_1 + xf_2) \ .$$

The singularities of the solution are determined by those of the functions f_1 and f_2, which in general depend on t. The real and imaginary parts of 8.4 furnish separate solutions.

9. Some approximations.

In the equations 5.2 for nonviscous motion with $c_1 = 0$ the first three equations are satisfied by

$$c_2 = -ax, \ c_3 = 1 + bx - a^2xt, \ c_4 = (2abt - a^3t^2)\,x, \ \ldots ,$$

where we assume a to be very small and positive, and b relatively large and positive; and we omit higher powers of y as negligible when y is small. Then

$$\psi = -axy^2/2! + (1 + bx - a^2xt)y^3/3! + (2abt - a^3t^2)xy^4/4! \ \ldots$$

The stream line $\psi = 0$ consists of two components

$$y^2 = 0 \text{ and } -ax/2 + (1 + bx - a^2xt)\,y/3! + (2abt - a^3t^2)xy^2/4! = 0$$

The second approximation to the shape of the latter component near the origin is

$$y = 3ax - 3a(b - a^2t)\,x^2 \ \ldots ,$$

which shows that if $t < b/a^2$, a considerable time, the curve lies below its tangent, which makes a small positive angle with OX The velocity components are

$$u = -axy + (1 + bx - a^2xt)y^2/2 \ldots,$$
$$v = ay^2/2! + (a^2t - b)y^3/3! \ldots,$$

so that near the boundary when y is positive u is opposite in sign to x, while v is always positive. There is a locus passing through the origin on which $u = 0$, namely

$$0 = -ax + (1 + bx - a^2xt)y/2 \ldots,$$

above which both u and $\partial u/\partial y$ are positive, so that while $t < b/a^2$ the approximation gives a clear idea of the conditions near a point of separation at the origin. As the terms comprising c_3 may be regarded as merely the first terms in the expansion of c_3 when x and t are sufficiently small, the illustration is less particular than it looks at first sight.

If we retain only the first two terms in the second component of $\psi = 0$ the shape of the curve for large values of x is easily obtained, for the component is a hyperbola whose asymptote parallel to OX is

$$y = 3a/(b - a^2t),$$

so that when t is small the breadth of the separated layer is approximately $3a/b$, a small length; but it increases indefinitely as t approaches b/a^2.

Now let us modify slightly by adding a term to the arbitrary function contained in c_3, and taking

$$c_1 = 0, \quad c_2 = -ax, \quad c_3 = 1 + bx + cx^2 - a^2xt,$$
$$c_4 = (2abx + 4acx^2)t - a^3xt^2;$$

and for simplicity let $b = 2$, $c = 1$. Then

$$u = -axy + [(1 + x)^2 - a^2xt]y^2/2! + [4a(x + x^2) - a^3xt^2]y^3/3!,$$
$$v = ay^2/2! - (2 + 2x - a^2t)y^3/3! - [4a(1 + 2x)t - a^3t^2]y^4/4!.$$

An isolated singular point not lying on $y = 0$ must satisfy the equations

$$[4a(x + x^2)t - a^3xt^2]y^2 + 3[(1 + x)^2 - a^2xt]y - 6ax = 0,$$
$$[4a(1 + 2x)t - a^3t^2]y^2 + 4[2(1 + x) - a^2t]y - 12a = 0.$$

Each of these equations to determine y has when $t = 0$ one infinite root with which we are not concerned. Equating the finite roots we find

$$y = 2ax/(1 + x)^2 = 3a/2(1 + x),$$

and discarding $x = -1$, which is before separation and makes y infinite,

$$x = 3, \quad y = 3a/8.$$

The second approximation when t is small is

$$x = 3 + 33a^2t/16, \quad y = 3a/8 + 3a^3t/64,$$

which represent a point below the second component of $\psi = 0$ and to the right of the separation point, i. e. in the separated area.

Since

$$\Delta = -9a^4 (1 - 87a^2t/4)/512$$

when t^2 is negligible, the singularity is an isolated singular point, i. e. a vortex, provided that $t < 4/87a^2$; it moves slowly away form the point of separation with component velocities

$$u = 33a^2/16 \quad \text{and} \quad v = 3a^3/64.$$

It requires no great stretch of the imagination to see that similar phenomena are likely to be encountered whenever c_2, c_3, etc are analytic functions of x and t in the neighbourhood of $x = 0$, $t = 0$, provided that the values of the coefficients are suitably adjusted. Viscous flow can be dealt with similarly.

Now when $\nu \neq 0$ the method applied to show that if ψ can be expanded in positive powers of y the powers must be positive integers will apply also to an expansion in powers of x, or to an expansion in powers of t. Either therefore ψ is not expansible in positive powers of the variables, or it is expansible only in positive integral powers, i. e. it is an analytic function of x, y and t within its domain of convergence, say D. It follows at once that all the coefficients are analytic in D.

In nonviscous motion the conditions are less simple. The outsanding factor is that even when an expansion exists in positive integral powers of y, whereas in viscous motion the formation of the coefficients involves successive differentiations, in the nonviscous case it involves a succession of integrations, in each of which an arbitrary element, a constant, a function of x, or a function of x and t may be introduced. It is no longer true to say that the solution is 'determined' when c_1 is known, for the complete solution requires the determination of the arbitrary functions which appear during the integration, and there is no guarantee that these are analytic, unless the boundary and other conditions supply such a guarantee. The boundary conditions hitherto considered certainly do not, by themselves, nor does the equation of motion, for these only imply the existence of certain derivatives and a relation connecting them. Nothing whatever is necessarily implied regarding higher derivatives, for example, and in fact the form of some of the solutions obtained in the preceding paragraphs makes it clear that the arbitrary functions involved can be so chosen that all derivatives of ψ with respect to y will become infinite when $y=0$, when the order of the derivative exceeds a specified finite number. This is true for instance of the solution 3.6 if we take

$$h(z) = z^m \sin z^{-1},$$

and 7.2 and 8.4 offer similar possibilities. These are for nonviscous or highly viscous motion. But it would be very rash to assume that similar possibilities are excluded in the general case, if the initial restriction on the form of the solution is withdrawn.

10. The stream lines at a point of separation.

In view of the negative result of Goldstein's paper referred to in paragraph 1 it is perhaps worth while to inquire briefly into the character of the stream lines at a point of separation, assuming the solution to be of the form

$$\sum_{n=1}^{\infty} c_n y^n / n!,$$

where $c_1 \neq 0$ occurs only in nonviscous motion, while $c_1 \equiv 0$ may occur in either viscous or nonviscous motion.

In the former case let $\psi = y\chi$, where

$$\chi = c_1 + c_2 y/2! + c_3 y^2/3! \ldots,$$

$\chi = 0$ being the second component of $\psi = 0$ at the point of separation $x = y = 0$.

If in addition to the simple node due to the intersection of $y = 0$ and $\chi = 0$ there is a singularity at the origin it must be a singularity of $\chi = 0$. Hence

$$\chi = \chi_x = \chi_y = 0 \text{ when } x = y = 0,$$

i. e. $c_1 = c_1' = c_2 = 0$ when $x = 0$.

We cannot have $c_1'' \neq 0$ when $x = 0$, for in that case c_1 and therefore u would not change sign with x when y is small. Hence for some positive integer r we must have

$$c_1 = c_1' = c_1'' = \cdots = c_1^{(2r)} = 0, \quad c_1^{(2r+1)} \neq 0 \quad \text{vhen } x = 0$$

if there is a point of separation at the origin. Now

$$\Delta = \chi_{xy}^2 - \chi_{xx}\chi_{yy} = (\tfrac{1}{4}c_2'^2 - \tfrac{1}{8}c_1''c_3) + \cdots$$
$$= \tfrac{1}{4}c_2'^2 \text{ when } x = y = 0 \text{ since } c_1'' = 0,$$
$$\leq 0.$$

Hence if $c_2' \neq 0$ when $x = 0$ the point of separation will be a real nodal point on $\chi = 0$. It cannot be an isolated singularity. It will be cuspidal if $c_2' = 0$, in general, or else a triple or multiple point of higher order

Approximations to the shapes of the branches of the curve are easily obtained. Let

$$c_1 = ax^{2r+1} + \cdots, \quad c_2 = 2(b_1 x + \cdots), \quad c_3 = 6d + \cdots,$$

so that the equation of the second component is

$$(ax^{2r+1} + \cdots) + (b_1 x + \cdots)y + dy^2 + \cdots = 0.$$

There are two approximations near the origin, namely

$$b_1 y + ax^{2r} = 0 \quad \text{and} \quad dy + b_1 x = 0,$$

so that if $b_1 \neq 0$ the origin is a nodal point as stated above. $(c_2' \neq 0.)$

If $c_2' = 0$ when $x = 0$ we may assume $c_2 = 2(b_2 x^s + \cdots)$, where $s > 1$. We have then

$$(ax^{2r+1} + \cdots) + (b_2 x^s + \cdots)y + dy^2 + \cdots = 0.$$

If $r \geq s$ this furnishes two approximations at the origin,

$$b_2 y + ax^{2r+1-s} = 0 \quad \text{and} \quad dy + b_2 x^s = 0,$$

and if $r < s$ $\qquad dy^2 + ax^{2r+1} = 0.$

This completes the discussion so long as $d \neq 0$.

When $c_1 \equiv 0$ we take $\psi = y^2\chi$, where

$$\chi = c_2/2! + c_3y/3! + c_4y^2/4! + \cdots$$

As before, when $x = y = 0$ we must have

$$\chi = \chi_x = \chi_y = 0,$$

i. e. $$c_2 = c_2' = c_3 = 0 \quad \text{when} \quad x = 0;$$

and since u must change sign with x when y is small,

$$c_2 = c_2' = \cdots = c_2^{(2r)} = 0, \quad c_2^{(2r+1)} \neq 0 \quad \text{when} \quad x = 0;$$

and the remaining steps are exactly as before, except that $c_1/1!$, $c_2/2!$ and $c_3/3!$ are replaced by $c_2/2!$, $c_3/3!$, and $c_4/4!$ respectively.

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