Linear Differential Difference Equations with Constant Coefficients

by

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Özet: Mühendislikte karşılaşılan birçok problemler sonlu fark denklemleri ile çözülürler. Mesela merdiven şeklindeki bir elektrik devresindeki akım ve potansiyel dağılımı hesabı veya eşit aralıklı noktalarında yüklenmiş gergin bir sicimin enine titreşimlerinin tabii frekanslarının hesabı en iyi şekilde fark denklemleri ile çözülür. Bu tip mühendislik problemlerinin çözümüne götüren fark denklemleri ekseriya ikinci mertebeden türevleri de ihtiva ederler. İşte bu yazıda bilinmiyen bir fonksiyonun hem türevlerini ve hem de farklarını ihtiva eden sabit katsayılı lineer bir denklemin en genel şekli ele alınmış ve çözüm yolları incelenmiştir.

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1. Introduction. A great many problems encountered in mechanical and electrical engineering can be solved by the use of finite difference equations. Such problems, for example, as the determination of the current distribution along an electrical network of ladder type, or the determination of the natural frequencies of the transversal vibrations of a stretched string loaded at equidistant points are easily solved by difference equations, which mostly include the differential coefficients of the second order. This suggests the introduction of a new type of equation which may conveniently be called "differential difference equations".

2. Consider the homogeneous lineer equation

$$p_0(\mathrm{D})\, u(x) + p_1(\mathrm{D})\, u(x+1) + \cdots + p_n(\mathrm{D})\, u(x+n) = 0$$

$$\text{i. e.} \quad \sum_{r=0}^{n} p_r(\mathrm{D})\, u(x+r) = 0, \tag{2.1}$$

where $p_r(\mathrm{D})$ is a polinomial in $\mathrm{D} = d/dx$ with constant coefficients of degree ν. It is called linear, since it is linear with respect to u and its derivatives.

The equation (2.1) is satisfied by

$$u(x) = Ce^{ax} \tag{2.2}$$

if

$$\sum_{r=0}^{n} e^{ar} p_r(a) = 0, \tag{2.3}$$

where C is an arbitrary constant and a is a parameter to be determined by the characteristic equation (2.3). For

$$\sum p_r(D)\, u(x+r) = \sum p_r(D) E^r u(x),$$

and

$$\sum p_r(D) E^r Ce^{ax} = C \sum p_r(D) e^{ar} \cdot e^{ax},$$

where E is the operator which, when applied to $u(x)$, gives

$$E\, u(x) = u(x+1).$$

The characteristic equation (2·3) is a transcendental equation and generally it has an infinite number of roots, real or complex. If there is k distinct roots, the general solution becomes

$$u(x) = \sum_{r=1}^{k} C_r e^{a_r x}, \tag{2.4}$$

since the functions $e^{a_r x}$ are linearly independent when a_r are distinct. This follows directly from the linear property of the operator.

If there is a multiple root ot multiplicity s of the equation (2.3), the corresponding terms in the general solution are represented by

$$e^{ax} \cdot \sum_{r=1}^{s} C_i x^{i-1}, \tag{2.5}$$

where C_i are arbitrary constants.

To prove this, first notice that the operator in (2.1) is

$$p_0(D) + p_1(D)E + p_2(D)E^2 + \cdots + p_n(D)E^n.$$

If it is factorised into linear factors in E, we shall have

$$p_n(D)\,[E - f_1(D)]\,[E - f_2(D)] \cdots [E - f_n(D)],$$

or

$$p_n(D)\,[e^D - f_1(D)]\,[e^D - f_2(D)] \cdots [e^D - f_n(D)] \tag{2.6}$$

since $E = e^D$ (See. 1), where $f_i(D)$ are known functions of D, and these factors are commutative, since they are differential operators with constant coefficients.

Now if a is a root of the characteristic equation (2.3), then one of the factors in (2.6) will furnish

$$[e^{D} - f_i(D)] \cdot e^{ax} = 0,$$
$$\therefore \quad e^{a} - f_i(a) = 0. \tag{2.7}$$

If two factors are identical the characteristic equation will be

$$[e^{a} - f_i(a)]^2 = 0,$$

and every root of (2.7) is a double root. In this case (2.4) will not represent the general solution. To find independent solutions, assume

$$u(x) = v(x)\, e^{ax} \tag{2.8}$$

and solve

$$[e^{D} - f_i(D)]^2 \cdot e^{ax}\, v(x) = 0, \tag{2.9}$$

where a is the double root.

Since (See. 2)

$$[e^{D} - f_1(D)]^2 e^{ax}\, v(x) = e^{ax}\, [e^{D+a} - f_i(D + a)]^2\, v(x),$$

(2.9) is satisfied if

$$[e^{D+a} - f_i(D + a)]^2\, v(x) = 0.$$

Combining this with (2.7) we can factorise the operator in the form

$$g(D) \cdot D^2 v(x) = 0,$$

where $g(D)$ is an infinite series in ascending integral powers of D, and $g(0) \neq 0$. Hence the equation (2.9) is satisfied by $D^2 v(x) = 0$, i. e. by

$$v(x) = C_0 + C_1 x,$$

and

$$u(x) = (C_0 + C_1 x) e^{ax},$$

then (2.5) follows

The complex roots of the characteristic equation occur in conjugate pairs, since the coefficients of the equation are real, and can be combined in the usual way to obtain real terms when x is real.

In all these cases the linear independence of the particular solutions are evident.

When k is infinite, the series in (2.4) must converge, and the generality of the solution follows if all the roots (2.3) are taken into account.

3. When $n=0$, $\nu \neq 0$ the equation (2.1) reduces to

$$p_0(D)\, u(x) = 0, \tag{3.1}$$

which is an ordinary differential equation with constant coefficients of order ν. Its solution is

$$u(x) = \sum_{r=1}^{\nu} C_r e^{a_r x}, \tag{3.2}$$

if all the roots of the characteristic equation

$$p_0(a) = 0 \tag{3.3}$$

are distinct, which is what the equation (2.3) in this case reduces to.

If the factor $p_n(D)$ in (2.6) is not a constant, particular solutions of the equation (2.1) are obtained from a similar ordinary differential equation $p_n(D)\, u(x) = 0$.

When $n \neq 0$, $\nu = 0$ the equation (2.1) reduces to a linear difference equation with constant coefficients, and (2.3) becomes an algebraic equation of degree n to determine n values of

$$q_r = e^{a_r} \qquad (r = 1, 2, \cdots, n).$$

Although q_r are definite, there is an indeterminacy in the values of $a_r = \log q_r$ when q_r is complex, since $\log q_r$ is a many valued function. If the principal value of $\log q_r$ is L_r, the general expression for a_r becomes

$$a_r = L_r + 2k\pi i, \tag{3.4}$$

where k is an arbitrary integer, and $L_r = \log \rho_r + i\theta_r$ if $q_r = \rho_r e^{(i\theta_r + 2k\pi)}$. Now, corresponding to this root we obtain the set of particular solutions

$$\sum_k C_r e^{L_r x} (\cos 2k\pi x + i \sin 2k\pi x) = \omega_r(x)\, e^{L_r x}, \tag{3.5}$$

where $\omega_r(x)$ is an arbitrary periodic function with period unity. For the conjugate of a_r, $L_r = \log \rho_r - i\theta_r$. The same expression (3.5) represents the terms corresponding to real values of q_r, since then $\theta_r = 0$, and $L_r = \log \rho_r$. Hence the general solution is

$$u(x) = \sum_{r=1}^{n} \omega_r(x) \cdot e^{L_r x}, \tag{3.6}$$

In the general case, when neither n nor ν is zero, the equation (2.3) determines certain values of a, to each of which corresponds a particular solution of the form (2.2).

4. When the equation is not homogeneous, namely when

$$\sum_{r=0}^{n} p_r(D)\, u(x+r) = \varphi(x), \tag{4.1}$$

the general solution is the sum of a particular solution of (4.1) and the general solution of (2.1). The form of the particular solution depends on the form of the function $\varphi(x)$. It may suggest a kind of substitution which determines the particular integral by the use of indeterminate coefficients. Or, standard methods may be used. We may, for instance, factorise the operator on the left hand side, or express the inverse operator as the sum of simple rational fractions and use the properties of D and E.

Example. Solve

$$D^2 u(x) - 2D\, u(x+1) + u(x+2) = b^x.$$

where b is a positive constant. The equation can be written in the form

$$(D - E)^2\, u(x) = b^x. \tag{4.2}$$

The particular integral is

$$\frac{1}{(D - E)^2}\, b^x = \frac{1}{(\log b - b)^2}\, b^x \tag{4.3}$$

since $b^x = e^{x \log b}$, see also (2, p. 34).

The characteristic equation for the homogeneous one is

$$(a - e^a)^2 = 0, \tag{4.4}$$

and every value of a which satisfies

$$\begin{aligned} a &= e^a \\ \therefore\ \log a &= a \end{aligned} \tag{4.5}$$

is a double root of the equation.

The equation (4.5) has no real roots, but it has an infinite number of complex roots. Let $a = \rho e^{i\theta}$, then (4.5) becomes

$$\log \rho + i\theta = \rho(\cos\theta + i\sin\theta)$$

$$\therefore \quad \left.\begin{aligned} \log \rho &= \rho\cos\theta \\ \theta &= \rho\sin\theta \end{aligned}\right\} . \tag{4.6}$$

Eliminating ρ, we obtain

$$\frac{\log(\theta/\sin\theta)}{\theta/\sin\theta} = \cos\theta. \tag{4.7}$$

Now $\theta/\sin\theta \geqq 1$ if $0 \leqq \theta < \pi/2$, and $\log(\theta/\sin\theta) \geqq 0$. Hence (4.7) shows that $0 \leqq \cos\theta < 1$, and there is some real value of θ between 0 and $\pi/2$ which satisfies (4.7).

One can solve (4.7) without much difficulty and find

$$\theta = 1{,}4574 \text{ radians } (83°{,}5) \text{ appr.} \tag{4.8}$$

and then (4.6) gives

$$\rho = 1{,}4664 \quad \text{(approximately)}. \tag{4.9}$$

Hence the solution of (4.5) are

$$a_k = \rho e^{i(\theta+2k\pi)} \qquad (k = 0,1,2,\ldots),$$

but since the equation (4.7) is even, namely it does not change if we replace θ by $-\theta$, we also have

$$\bar{a}_k = \rho e^{-i(\theta+2k\pi)} \qquad (k = 0, 1, 2,\ldots),$$

where $\bar{a}_k$ is the conjugate of a_k, and each root is a double root. Hence the general solution of the homogeneous equation is

$$\sum_k (c_{0k} + xc_{1k})\, e^{a_k x} + \sum_k (c_{2k} + xc_{3k})\, e^{\bar{a}_k x}$$

or, since $e^a = a_k$, it is

$$\sum_k \left\{ (c_{0k} + cx_{1k})\, \rho^x e^{i(\theta+2k\pi)x} + (c_{2k} + xc_{3k})\, \rho^x e^{-i(\theta+2k\pi)x} \right\}$$

$$= \rho^x \cdot \sum_k \left\{ (C_{0k} + xC_{1k}) \cos(\theta+2k\pi)x + (C_{2k} + xC_{3k}) \sin(\theta+2k\pi)x \right\}$$

where ρ and θ are given by (4.8) and (4.9), and C's are arbitrary constants of integration. Finally the general solution of (4.2) is obtained by adding the particular solution (4.3) to the expression above.

Referenees

[1]. Milne-Thomson L.M. The Calculus of Finite Differences, p. 83, 1933.
[2]. Piaggio H.T.H. An elementary treatise on differential equations and their applications. p. 32, 1950.

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