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On the Degree of Approximation to a Function by the Norlund Means of its Fourier Series

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1. Let f(x) be integrable in the sense of Lebesgue in $(-\pi, \pi)$ and be periodic with period 2π and let

$$f(x) \backsim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Let $\{P_n\}$ be a sequence of Positive real numbers. We write

$$P_n = \sum_{r=0}^{n} P_v, \ P(t) = P[t]$$

$$K_n(t) = \frac{1}{P_n} \sum_{r=o}^{n} P_{n-k} D_k(t), \text{ where}$$

$$D_n(t) = \tfrac{1}{2} + \sum_{1}^{n} \cos vt = \frac{\sin(n+\frac{1}{2})t}{2 \sin t/2}$$

$$F_n(t) = \operatorname{Im}\{e^{i(n+\frac{1}{2})t} \sum_{v=o}^{\infty} P_v e^{ivt}\},$$

$$t_n(t) = \frac{1}{P_n} \sum_{k=0}^{n} P_{n-k} s_k(x),$$

where $S_k(x)$ is the k-th partial sum of the Fourier series of f(x).

$$\Phi(t) = f(x+t) + f(x-t) - 2f(x).$$

We establish the following theorem which generalizes Theorem 1 of Flett (The quarterly journal of Mathematics, Oxford, 7(1956), 81–95).

Theorem: Let $\{P_n\}$ be a positive non-increasing sequence of real numbers such that $\int_t^{\xi} F_n(u)\, du = O\left(\frac{P(1/t)}{n}\right)$, $\frac{1}{n} \leqq t \leqq \xi \leqq \pi$. Also let $0 < \alpha < 1$, $0 < \delta \leqq \pi$. If x is a point such that,

$$\int_o^t . \mid d\, \Phi(u) \mid \leqq At^{\alpha}, \text{ when } 0 \leqq t \leqq \delta, \text{ then}$$

$$t_n(x) - f(x) = O(n^{-\alpha}) + O\left(\frac{1}{P_n}\right).$$

1. Let f(x) be integrable in the sense of Lebesgue in $(-\pi, \pi)$ and be periodic with period 2π and let

$$f(x) \backsim \tfrac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin x).$$

Let $\{p_n\}$ be a sequence of positive real numbers. We write

$$P_n = \sum_{r=o}^{n} p_v, \quad P(t) = P[t]$$

$$K_n(t) = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k} D_k(t), \quad \text{where}$$

$$D_n(t) = \tfrac{1}{2} + \sum_1^{n} \cos vt = \frac{\sin (n+\frac{1}{2})\, t}{2 \sin t/2}.$$

$$F_n(t) = \operatorname{Im} \{e^{i(n+\frac{1}{2})t} \sum_{v=o}^{\infty} p_v e^{-ivt}\},$$

$$t_n(x) = \frac{1}{P_n} \sum_{k=o}^{n} p_{n-k} s_k(x),$$

where $s_k(x)$ is the k-th partial sum of the Fourier series of f(x).

$$\Phi(t) = f(x+t) + f(x-t) - 2f(x).$$

$$E_n^k = \binom{n+k}{n}.$$

If $p_n = E_n^k$. $k > 0$, the Nörlund mean $t_n(x)$ becomes Cesaro mean

$$\sigma_n^k(x) = \frac{1}{E_n^k} \sum_{v=o}^{n} E_{n-v}^{k-1} s_v(x)$$

of the Fourier series of f(x).

2. Concerning the degree of approximation to a function by the Cesaro means of its Fourier series, Flett [1] proved a number of interesting theorems. Among others he proved the following theorem:

THEOREM A. Let $0 < \alpha < 1$, $0 < \delta \leqq \pi$. If x is a point such that

$$\int_0^t |d\,\Phi(u)| \leqq At^\alpha,$$

when $0 \leqq t \leqq \delta$, then

$$\sigma_n^\alpha(x) - f(x) = O(n^{-\alpha})$$

In the present note we shall examine the problem as to whether $\sigma_n^\alpha(x)$ in Theorem A can be replaced by Nörlund mean $t_n(x)$. Concerning this problem we prove the following theorem which includes Theorem A as a special case for $p_n = E_n^{\alpha-1}$, $0 < \alpha < 1$.

THEOREM: *Let* $\{p_n\}$ *be a positive non-increasing sequence of real numbers such that*

$$(2.1) \int_t^{\xi} F_n(u)\,du = O\left(\frac{P(1/t)}{n}\right), \quad \frac{1}{n} \leqq t \leqq \xi \leqq \pi.$$

Also let $0 < \alpha < 1$, $0 < \delta \leqq \pi$. *If* x *is a point such that*

$$\int_0^t |\, d\,\Phi(u)\,| < At^{\alpha},$$

when $0 \leqq t \leqq \delta$, then

$$t_n(x) - f(x) = O(n^{-\alpha}) + O\left(\frac{1}{P_n}\right).$$

3. The following lemmas are pertinent for the proof of this theorem:

Lemma 1. *we have*

$$K_n(t) = \begin{cases} O(n), & 0 \leqq t \leqq \pi \\ \dfrac{F_n(t)}{2P_n \sin \dfrac{t}{2}} + O\left(\dfrac{p_n}{P_n t^2}\right), & \dfrac{1}{n} \leqq t \leqq t \end{cases}$$

Lemma 2. *Let* $\Phi(t) \in L$, $0 < \alpha < 1$ and $0 < \leqq \pi$, then

$$\int_{\delta}^{\pi} \Phi(t)\, K_n(t)\, dt = O\left(\frac{1}{P_n}\right), \quad n \to \infty.$$

Lemma 3. *Under the hypothesis of the theorem we have*

$$\int_0^{\delta} \Phi(t)\, K_n(t)\, dt = O(n^{-\alpha}) + O\left(\frac{1}{n\,P_n}\right)$$

Proof of Lemma 1.

$$K_n(t) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, D_k'(t)$$

$$= O \frac{1}{P_n} \sum_{k=0}^{n} k\, p_{n-k} = O(n)$$

We write

$$K_n(t) = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-} \frac{\sin (v+1/2)\, t}{2 \sin t/2}$$

$$= \frac{1}{2P_n \sin\frac{t}{2}} \operatorname{Im} \left\{ \sum_{v=o}^{n} p_{n-v}\, e^{i(v+\frac{1}{2})\, t} \right\}$$

$$= \frac{1}{2P_n \sin\frac{t}{2}} \operatorname{Im} \left\{ \sum_{v=o}^{n} p_v . e^{i(n-v+\frac{1}{2})\, t} \right\}$$

$$= \frac{1}{2P_n \sin\frac{t}{2}} \operatorname{Im} \left\{ e^{i(n+\frac{1}{2})t} \sum_{v=o}^{n} p_v\, e^{-ivt} \right\}$$

$$= \frac{1}{2P_n \sin\frac{t}{2}} \operatorname{Im} \left\{ e^{i(n+\frac{1}{2})t} \left(\sum_{v=0}^{\infty} - \sum_{n+1}^{\infty}\right) \right\}$$

$$= \frac{F_n(t)}{2P_n \sin^t/_2} - \frac{1}{2P_n \sin^t/_2} \operatorname{Im} \left\{ e^{\,i(n+\frac{1}{2})\, t} \sum_{n+1}^{\infty} p_v\, e^{-ivt} \right\}$$

Since p_v is non-increasing, we have

$$\sum_{n+1}^{\infty} p_v\, e^{-ivt} \leqq 2\, p_{n+1} \operatorname{Max} \left| \sum_{v=0}^{\infty} e^{-ivt} \right|$$

$$= 2\ p_{n+1} \left| \frac{1}{1 - e^{-it}} \right|$$

$$= \frac{p_{n+1}}{\sin \frac{t}{2}}$$

As $\operatorname{Im}\{G(z)\} \leqq |G(z)|$, it follows that the second term is

$$= O\left(\frac{1}{P_n t} \cdot \frac{P_n}{t}\right) = O\left(\frac{p_n}{P_n t^2}\right)$$

This proves the Lemma 1.

Proof of Lemma 2. It is well known [2] that if $\{p_n\}$ is non-negative and non-increasing, then

$$\left| \sum_{k=0}^{\infty} p_k e^{-ikt} \right| \leqq P\left(\frac{1}{t}\right), 0, \leqq t \leqq \pi$$

and;

$$n^{-1} P_n \leqq t\, P\left(\frac{1}{t}\right) \text{ for } \frac{1}{n} \leqq t \leqq \pi$$

since $P_n \geqq (n+1)\, p_n$ it follows that

$$p_n \leqq \frac{P_n}{n} \leqq t\, P\left(\frac{1}{t}\right) \text{ so that } \frac{P_n}{t} \leqq P\left(\frac{1}{t}\right)$$

We have then for $\frac{1}{n} \leqq t \leqq \pi$

$$K_n(t) = O\left(\frac{|F_n(t)|}{tP_n}\right) + O\left(\frac{p_n}{P_{n+2}}\right)$$

$$= O\left(\frac{P(\frac{1}{t})}{t\,P_n}\right) + O\left(\frac{P(\frac{1}{t})}{tP_n}\right)$$

$$= O\left(\frac{P(\frac{1}{t})}{tP_n}\right)$$

Hence

$$\left| \int_{\delta}^{\pi} \Phi(t)\, K_n(t)\, dt \right| \leqq A \int_{\delta}^{\pi} |\, \Phi(t)\, | \cdot \frac{P(\frac{1}{t})}{t\,P_n}\, dt$$

$$\leqq A \frac{P(\frac{1}{\delta})}{\delta\, P_n} \int_{\delta}^{\pi} |\, \Phi(t)\, |\, dt.$$

$$= O\left(\frac{1}{P_n}\right)$$

Proof of Lemma 3.

$$\int_0^{\delta} \Phi(t)\, K_n(t)\, dt = \int_0^{1/n} + \int_{1/n}^{\delta}$$

$$= \int_0^{1/n} \Phi(t)\, K_n(t)\, dt + \int_{1/n}^{\delta} \frac{\Phi(t)\, F_n(t)}{2\, P_n \sin\frac{t}{2}}\, dt$$

$$+ O\left(\int_{1/n}^{\delta} |\, \Phi(t)\, | \frac{p_n}{P_n t^2}\, dt\right)$$

$$= Q_1 + Q_2 + Q_2, \text{ say.}$$

Since $\Phi(0) = 0$, $|\Phi(t)| = |\Phi(t) - \Phi(0)| =$

$$= \left| \int_0^t d\,\Phi(u) \right| \leqq \int_0^t |d\,\Phi(u)|$$

$$\leqq At^\alpha$$

We have

$$Q_1 = O\left(n \int_0^{1/n} |\Phi(t)|\,dt\right) = O\left(n \int_0^{1/n} t^\alpha\,dt\right)$$

$$= O(n^{-\alpha})$$

Also

$$Q_3 = O\left(\int_{1/n}^{\delta} \frac{p_n}{P_n} t^{\alpha-2}\,dt\right)$$

$$= O\left(\frac{p_n}{P_n} n^{-\alpha+1}\right) = O(n^{-\alpha}),$$

since $n\,p_n < P_n$

Next let us set

$$H(t) = \int_t^\pi \frac{F_n(u)\,du}{2P_n \sin\frac{u}{2}} = \frac{1}{2P_n \sin\frac{t}{2}} \int_t^\xi F_n(u)\,du, \quad t < \xi < \pi$$

$$= O\left(\frac{P\left(\frac{1}{t}\right)}{n\,P_n\,t}\right),$$

by the hypothesis. Then

$$Q_2 = \int_{1/n}^{\delta} \frac{F_n(t)\,\Phi(t)}{2\,P_n \sin\frac{t}{2}}\,dt = -\int_{1/n}^{\delta} \Phi\,H'(t)\,dt$$

$$= -\int_{1/n}^{\delta} \Phi \, dH(t) = -[\Phi\, H(t)] \int_{1/n}^{\delta} + \int_{1/n}^{\delta} H(t) d\Phi(t)$$

$$= O(\frac{1}{n\, P_n}) + O(n^{-\alpha}) + \int_{1/n}^{\delta} H(t)\, d\, \Phi\, (t).$$

Let Φ^* denote the total variation of $\Phi(t)$ in $(0,t)$, so that $\Phi^*(t) \leqq A\, t^{\alpha}$. We have

$$\left| \int_{1/n}^{\delta} H(t)\, d\Phi(t) \leqq \int_{1/n}^{\delta} |\, H(t) \,|\, d\Phi^*\right.$$

$$= O(\frac{1}{nP_n} \int_{1/n}^{\delta} \frac{P(\frac{1}{t})}{t}\, d\, \Phi^*)$$

$$= O(\frac{1}{n} \{[\frac{\Phi^*}{t}] \int_{1/n}^{\delta} - \int_{1/n}^{\delta} \frac{\Phi^*}{t^2}\, dt\})$$

$$= O(n^{-\alpha}) + O(\frac{1}{n} \int_{1/n}^{\delta} t^{\alpha-2}\, dt)$$

$$= O(n^{-\alpha}).$$

This completes the proof of Lemma 3.

4. *Proof of the Theorem:* We have

$$t_n(x) - f(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, s_k(x) - f(x)$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k}\, (s_k(x) - f(x))$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \frac{1}{\pi} \int_0^{\pi} \Phi(t)\, D_k(t)\, dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \Phi(t) \left(\frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} D_k(t)\right) dt$$

$$= \frac{1}{\pi} \int_0^{\pi} \Phi(t)\, K_n(t)\, dt$$

$$= \frac{1}{\pi} \left(\int_0^{\delta} + \int_{\delta}^{\pi} \right) = S_1 + S_2, \quad \text{say}$$

Applying lemma 3, we have

$$S_1 = O(n^{-\alpha}) + O\left(\frac{1}{P_n}\right)$$

and by virtue of lemma 2 we get

$$S_2 = O\left(\frac{1}{P_n}\right)$$

This proves the theorem.

I am thankful to Dr. S.M. Mazhar for his encouragement in the preparation of this paper.

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[1] Flett, T. M.: *On the degree of approximation to a function by the Cesaro means of its Fourier series*, *The Quarterly Journal of Mathematics*, Oxford Second series, 7 (1956) 81–95.

[2] McFadden, L.,: *Absolute Norlund Summability*, *Duke Mathematical Journal* 9 (1942) 168–207.

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ÖZET

1. f(x), $(-\pi, \pi)$ aralığında Lebesque anlamında integrallenebilir, 2π periyodlu bir fonksiyon olsun ve

$$f(x) \backsim \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

yazalım.

$\{Pn\}$ pozitif bir reel sayılar dizisi olsun.

$$D_n(t) = \tfrac{1}{2} + \sum_{1}^{n} \cos vt = \frac{\sin(n+\frac{1}{2})t}{2 \sin t/2}$$

olmak üzere

$$P_n = \sum_{r=o}^{n} P_v, \quad P(t) = P[t]$$

$$K_n(t) = \frac{1}{P_n} \sum_{r=o}^{n} P_{n-k} D_k(t)$$

yazalım.

Sk(x), f(x) in Fourier serisinin k yinci kısmı toplamı olmak üzere

$$F_n)t) = \operatorname{Im} \{ e^{i(n+\frac{1}{2})t} \sum_{v=o}^{\infty} P_v e^{ivt} \},$$

$$t_n(x) = \frac{1}{P_n} \sum_{k=o}^{n} P_{n-k} S_k(x)$$

yazalım.

$$\Phi(t) = f(x+t) + f(x-t) - 2 f(x)$$

Flett'in I. teoremini genelleştiren aşağıdaki teoremi ispat edebiliriz. (The quorterly journal of Mathematics, Oxford, 7 (1956) 81-95).

TEOREM:

$$\int_{t}^{\xi} F_n(u)\, du = O\left(\frac{P(1/t)}{n}\right), \quad \frac{1}{n} \leqq t \leqq \xi \leqq \pi$$

olacak şekilde reel sayılardan meydana gelen pozitif, artmıyan bir dizi $\{Pn\}$ olsun. Ayrıca $0<\alpha<1$, $0<\delta\leq\pi$ alalım. Eğer x, $0\leq t\leq\delta$ olduğunda;

$$\int_0^t \mid d\,\Phi(u)\mid \leq At^{\alpha}$$

olacak şekilde bir nokta ise,

$$t_n(x) - f(x) = O\,(n^{-\alpha}) + O\,(\frac{1}{P_n})$$

dir.

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